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Research Article

Nonlocal Conditions for Lower Semicontinuous Parabolic Inclusions

Abdelkader Boucherif

Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, P. O. Box 5046 Dhahran, 31261, Saudi Arabia

Correspondence should be addressed to Abdelkader Boucherif, aboucher@kfupm.edu.sa

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We discuss conditions for the existence of at least one solution of a discontinuous parabolic equation with lower semicontinuous right hand side and a nonlocal initial condition of integral type. Our technique is based on fixed point theorems for multivalued maps.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. We denote the norm (usually the Euclidean norm) of $x \in \Omega$ by $\|x\|$. Let T be a positive real number. Set $Q_T = \Omega \times (0,T)$ and $\Gamma_T = \partial\Omega \times [0,T]$. For $u : D \to \mathbb{R}$ we denote its partial derivatives (when they exist) by $u_t = \partial u/\partial t$, $u_{x_i} = \partial u/\partial x_i$, $u_{x_i x_j} = \partial^2 u/\partial x_i \partial x_j$, $i, j = 1, \ldots, N$.

Let $X = C(Q_T)$ denote the Banach space of continuous functions $u : Q_T \to \mathbb{R}$, endowed with the norm

$$|u|_0 = \sup\{|u(x,t)|; (x,t) \in Q_T\}$$
$$u \in C^{2,1}(Q_T) \quad \text{if } u(\cdot,t) \in C^2(\Omega), \quad t \in (0,T), \quad u(x,\cdot) \in C^1(0,T), \quad x \in \Omega. \tag{1.1}$$

For $1 \leq p < +\infty$, we say that $u : Q_T \to \mathbb{R}$ is in $L^p(Q_T)$ if u is measurable and $\int_{Q_T} |u(x,t)|^p dxdt < +\infty$, in which case we define its norm by

$$|u|_{L^p} = \left(\int_{QT} |u(x,t)|^p dxdt \right)^{1/p}. \tag{1.2}$$

Consider the linear nonhomogeneous problem

$$u_t + Lu = f(x,t), \quad (x,t) \in Q_T, \tag{1.3}$$

$$u(x,t) = 0, \quad (x,t) \in \Gamma_T, \tag{1.4}$$

with the following nonlocal initial condition:

$$u(x,0) = \int_0^T k(x,t,u(x,t))dt, \quad x \in \Omega. \tag{1.5}$$

Here, L is an elliptic operator given by

$$Lu = -\sum_{i,j=1}^{N} a_{ij}(x,t)u_{x_i x_j} + c(x,t)u. \tag{1.6}$$

We will assume throughout this paper that the functions $a_{ij}, c : Q_T \to \mathbb{R}$ are Hölder continuous, $a_{ij} = a_{ji}$, and moreover, there exist positive numbers λ_0, λ_1 such that

$$\lambda_0 \|\xi\|^2 \le \sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \le \lambda_1 \|\xi\|^2, \quad \forall \xi \in \mathbb{R}^N, \forall (x,t) \in Q_T. \tag{1.7}$$

Let $u_0 : \Omega \to \mathbb{R}$ be continuous. For the problem (1.3), (1.4) together with initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.8}$$

we have the following classical result.

Lemma 1.1 (see [1–4]). *Assume that the function f is Hölder continuous on Q_T and u_0 is continuous on Ω. Then problem* (1.3), (1.4), (1.8) *has a unique solution $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$, which for each $(x,t) \in Q_T$, is given by*

$$u(x,t) = \int_\Omega G(x,t;y,0)u_0(y)dy + \int_0^t \int_\Omega G(x,t;y,s)f(y,s)dyds, \tag{1.9}$$

where $G(x,t;y,s)$, is the Green's function corresponding to the linear homogeneous problem. This function has the following properties (see [1, 4]).

(i) $D_t G + LG = \delta(t-s)\delta(x-y)$, $s < t$, $x, y \in \Omega$.

(ii) $G(x,t;y,s) = 0$, $s > t$, $x, y \in \Omega$.

(iii) $G(x,t;y,s) = 0$, $(x,t), (y,s) \in \Gamma_T$.

(iv) $G(x,t;y,s) > 0$ for $(x,t) \in Q_T$.

(v) G, G_t, G_x, G_{xx} are continuous functions of (x,t), $(y,s) \in Q_T$, $t - s > 0$.

In addition to the above, $G(x,t;y,s)$ satisfies the following important estimate.

(vi) $|G(x,t;y,s)| \leq C(t-s)^{-N/2}\exp((-a\|x-y\|^2)/(t-s))$, for some positive constants C, a (see [2]).

Since $u \in C^{2,1}(Q_T) \cap C(\overline{Q_T})$, it is clear that the functions $(x,t) \to \int_\Omega G(x,t;y,0)dy$ and $(x,t) \to \int_0^t \int_\Omega G(x,t;y,s)dyds$ are continuous. Let $d_0 := \max_{(x,t)\in Q_T} \int_\Omega G(x,t;y,0)dy$ and let $\delta := \max_{(x,t)\in\overline{Q_T}} \int_0^t \int_\Omega G(x,t;y,s)dyds$. Also, property (vi) above shows that $G \in L^2(Q_T \times Q_T)$.

In this paper, we consider a nonlocal problem for a class of nonlinear parabolic equations with a lower semicontinuous multivalued right hand side. More specifically, we consider the following problem,

$$\begin{gathered} u_t + Lu \in F(x,t,u), \quad (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \in \Gamma_T, \\ u(x,0) = \int_0^T k(x,t,u(x,t))dt, \quad x \in \Omega. \end{gathered} \tag{1.10}$$

Parabolic problems with discontinuous nonlinearities arise as simplified models in the description of porous medium combustion [5], chemical reactor theory [6]. Also, best response dynamics arising in game theory can be modeled by a parabolic equation with a discontinuous right hand side [7, 8]. Parabolic problems with discontinuous nonlinearities have been also investigated in the papers [9–13]. On the other hand, parabolic problems with integral boundary conditions appear in the modeling of concrete problems, such as heat conduction [14, 15] and thermoelasticity [16]. Also, the importance of nonlocal conditions and their applications in different field has been discussed in [17, 18]. Several papers have been devoted to the study of parabolic problems with integral conditions [19, 20]. Next, we state some important facts about multivalued functions and results that will be used in the remainder of the paper.

A subset $\Sigma \subset Q_T \times \mathbb{R}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if Σ belongs to the σ-algebra generated by all sets of the form $\mathfrak{D} \times \mathcal{Q}$ where $\mathfrak{D}$ is Lebesgue measurable in Q_T and $\mathcal{Q}$ is Borel measurable in $\mathbb{R}$. Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be Banach spaces. $\wp(Y)$ denotes the set of all nonempty subsets of Y. The domain of a multivalued map $\mathfrak{R} : X \to \wp(Y)$ is the set $\mathrm{Dom}(\mathfrak{R}) = \{u \in X; \mathfrak{R}(u) \neq \emptyset\}$. $\mathfrak{R}$ has closed values if $\mathfrak{R}(u)$ is a closed subset of Y for each $u \in X$ and we write $\mathfrak{R}(u) \in \wp_c(Y)$. Also, $\wp_{cc}(Y)$ denotes the set of all nonempty closed and convex subsets of Y. $\mathfrak{R}$ is bounded if $\sup\{|y|; y \in \mathfrak{R}(u)\} < +\infty$. $\mathfrak{R}$ is called lower semicontinuous (lsc) on X if $\mathfrak{R}^{-1}(B)$ is open in X whenever B is open in Y, or the set $\{u \in X; \mathfrak{R}(u) \subset B\}$ is closed in X whenever B is closed in Y. For more details on multivalued maps, we refer the interested reader to the books [21–24].

Let β denote the Kuratowski measure of noncompactness. See [25] for definitions and details.

Theorem 1.2 (see [26, Theorem 3.1]). *Let E be a separable Banach space. Assume the following conditions hold. There exists $M > 0$, independent of λ, with $|u|_{L^p} \neq M$ for any solution $u \in L^2([0,T],E)$ to $u \in \lambda Fu$* a.e. *on $[0,T]$ for each $\lambda \in (0,1)$, $F : X = \{u \in L^2([0,T],E); |u|_{L^p} \leq M\} \to \wp_{cc}(L^2([0,T],E))$ is a closed map, $F(X)$ is a bounded subset of $L^2([0,T],E)$, and $\beta(F(V)) \leq \beta(V)$ for all $V \subseteq X$ with strict inequality if $\beta(V) \neq 0$. Then the inclusion $u \in Fu$ has a solution $u \in X$.*

2. Main Result

By a solution of problem (1.10), (7), (8) we mean a function $u \in L^2(Q_T)$ such that there exists a function $f \in L^2(Q_T)$ with $f(x,t) \in F(x,t,u(x,t))$ for each $(x,t) \in Q_T$ and (1.3), (1.4), (1.5) hold.

Theorem 2.1. *Assume that the following conditions are satisfied.*

(HF) $F : Q_T \times \mathbb{R} \to \wp_{cc}(\mathbb{R})$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,* $u \mapsto F(x,t,u)$ *is* lsc *for* a.e.$(x,t) \in Q_T$*, there exist* $a > 0, b > 0$ *such that* $|F(x,t,u)| \le a + b|u|$ *with* $2\mathrm{Vol}(Q_T)(b|G|_{L^2(Q_T\times Q_T)})^2 < 1$ *and there exists* $\ell_0 \in L^2(Q_T)$ *such that* $\beta(F(x,t,B)) \le \ell_0(x,t)\beta(B)$ *for any bounded set* $B \subset \mathbb{R}$*,*

(Hk) $k : Q_T \times \mathbb{R} \to \mathbb{R}$ *is continuous, bounded and there exists* $\ell_1 \in C(Q_T)$ *such that* $\beta(k(x,t,B)) \le \ell_1(x,t)\beta(B)$*.*

Then problem (1.10)*, (7), (8) has a solution provided that* $d_0|\ell_1|_0 + |\ell_0|_{L^2(Q_T)}|G|_{L^2(Q_T\times Q_T)} < 1$*.*

Proof. We shall follow the ideas developed in [27]. It follows from the integral representation (1.9) that any solution $u \in L^2(Q_T)$ of (1.10), (7), (8) is a solution of the operator inclusion

$$u \in F(u), \tag{2.1}$$

for $\lambda = 1$, where

$$F(u) = \mathbf{k}(u) + GN_F(u), \tag{2.2}$$

where $\mathbf{k}$ is given by

$$\mathbf{k}(u) = \int_0^T \int_\Omega G(x,t;y,0)k(y,s,u(y,s))\,dy\,ds, \tag{2.3}$$

while $GN_F(u)$ is given by

$$GN_F(u)(x,t) = \int_0^t \int_\Omega G(x,t;y,s)N_F(u(y,s))\,dy\,ds, \quad (x,t) \in Q_T. \tag{2.4}$$

First, we show that solutions of (2.1) are a priori bounded. We have

$$u(x,t) = \lambda \int_0^T \int_\Omega G(x,t;y,0)k(y,s,u(y,s))\,dy\,ds + \lambda \int_0^t \int_\Omega G(x,t;y,s)f(y,s)\,dy\,ds, \tag{2.5}$$

where $f \in N_F(u)$, that is $f(x,t) \in F(x,t,u)$ for each $(x,t) \in Q_T$. Since k is bounded there exists $C_k > 0$ such that $|k(y,s,u(y,s))| \le C_k$. It follows from the properties of the Green's function and the assumption (HF) that

$$|u(x,t)| \le TC_k d_0 + \int_0^t \int_\Omega G(x,t;y,s)\big(a + b\big|u(y,s)\big|\big)\,dy\,ds. \tag{2.6}$$

Hence

$$|u(x,t)| \le TC_k d_0 + a\delta + b|G|_{L^2(Q_T\times Q_T)}|u|_{L^2(Q_T)}. \tag{2.7}$$

Equation (2.7) implies that

$$|u(x,t)|^2 \le 2(TC_k d_0 + a\delta)^2 + 2\Big(b|G|_{L^2(Q_T\times Q_T)}|u|_{L^2(Q_T)}\Big)^2, \tag{2.8}$$

or

$$|u|^2_{L^2(Q_T)} \le \frac{2\mathrm{Vol}(Q_T)(TC_k d_0 + a\delta)^2}{1-2\mathrm{Vol}(Q_T)\Big(b|G|_{L^2(Q_T\times Q_T)}\Big)^2}. \tag{2.9}$$

Therefore, there exists $M > 0$, independent of λ, but depending on Q_T, a, b, C_k and the Green's function such that any possible solution of (2.1) satisfies

$$|u|_{L^2(Q_T)} \le M. \tag{2.10}$$

Let $U = \{u \in L^2(Q_T); |u|_{L^2(Q_T)} \le M\}$. Then U is nonempty, closed, and bounded subset of $L^2(Q_T)$.

Since the multifunction F has nonempty, closed and convex values, it follows that N_F has nonempty, closed, and convex values. Since $\mathbf{k}$ is a continuous single valued operator, it is clear that F has nonempty, closed, and convex values. Next, we can easily show that $F : U \to \wp_{cc}(L^2(Q_T))$ is a closed map (i.e., has a closed graph) and $F(U)$ is a bounded subset of $L^2(Q_T)$.

Finally, we show that $\beta(F(B)) \le \beta(B)$ for any bounded subset $B \subset U$. So, let $u \in B$. Then, since $F(B) = \{F(u); u \in B\}$, we have

$$F(B) = \mathbf{k}(B) + \mathrm{GN}_F(B) = \{\mathbf{k}(u) + \mathrm{GN}_F(u);\ u \in B\}. \tag{2.11}$$

Hence

$$\beta(F(B)) = \beta(\{\mathbf{k}(u) + \mathrm{GN}_F(u);\ u \in B\}). \tag{2.12}$$

It follows from the assumption that

$$\begin{aligned}
\beta(F(B)) &\le \int_0^T\int_\Omega G(x,t;y,0)\ell_1(y,s)\beta(B)\,dy\,ds + \int_0^t\int_\Omega G(x,t;y,s)\ell_0(y,s)\beta(B)\,dy\,ds \\
&\le \left(\int_0^T\int_\Omega G(x,t;y,0)\ell_1(y,s)\,dy\,ds + \int_0^t\int_\Omega G(x,t;y,s)\ell_0(y,s)\,dy\,ds\right)\beta(B) \\
&\le \Big(d_0|\ell_1|_0 + |\ell_0|_{L^2(Q_T)}|G|_{L^2(Q_T\times Q_T)}\Big)\beta(B) \\
&< \beta(B).
\end{aligned} \tag{2.13}$$

This shows that F is a condensing multivalued map.

By Theorem 3.1 in [26], F has a fixed point in U, which is a solution of problem (1.10), (7), (8). This completes the proof of the main result. □

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