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Research Article

On a Nonlinear Integral Equation with Contractive Perturbation

Huan Zhu

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China

Correspondence should be addressed to Huan Zhu, mathzhuhuan@gmail.com

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We get an existence result for solutions to a nonlinear integral equation with contractive perturbation by means of Krasnoselskii's fixed point theorem and especially the theory of measure of weak noncompactness.

1. Introduction

The integral equations have many applications in mechanics, physics, engineering, biology, economics, and so on. It is worthwhile mentioning that some problems considered in the theory of abstract differential equations also lead us to integral equations in Banach space, and some foundational work has been done in [1–8].

In this paper we want to study the following integral equation:

$$x(t) = g(t, x(t), x(\lambda(t))) + f_1\left(t, \int_0^t k(t,s) f_2(s, x(s)) ds\right), \quad t \in \mathbb{R}_+ \tag{1.1}$$

in the Banach space X.

This equation has been studied when $X = \mathbb{R}$ in [9] with $g \equiv 0$ and [10] with a perturbation term g. Our paper extends their work to more general spaces and some modifications are also given on an error of [10].

Our paper is organized as follows.

In Section 2, some notations and auxiliary results will be given. We will introduce the main tools measure of weak noncompactness and Krasnoselskii's fixed point theorem in Section 3 and Section 4. The main theorem in our paper will be established in Section 5.

2. Preliminaries

First of all, we give out some notations to appear in the following.

$\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. Suppose that X is a separable locally compact Banach space with norm $\|\cdot\|_X$ in the whole paper. (Remark: the locally compactness of X will be used in Lemma 2.2). Let A be a Lebesgue measurable subset of $\mathbb{R}$ and $m(A)$ denote the Lebesgue measure of A.

Let $L^1(A, X)$ denote the space of all real Lebesgue measurable functions defined on A to X. $L^1(A, X)$ forms a Banach space under the norm

$$\|x\|_{L^1(A,X)} = \int_A \|x(t)\|_X dt \tag{2.1}$$

for $x \in L^1(A, X)$.

Definition 2.1. A function $f(t, x) : \mathbb{R}_+ \times X \to X$ is said to satisfy Carathéodory conditions if

(i) f is measurable in t for any $x \in X$;

(ii) f is continuous in x for almost all $t \in \mathbb{R}_+$.

The following lemma which we will use in the proof of our main theorem explains the structure of functions satisfying Carathéodory conditions with the assumption that the space X is separable and locally compact (see [11]).

Lemma 2.2. *Let I be a bounded interval and $f(t, x) : I \times X \to X$be a function satisfying Carathéodory conditions. Then it possesses the Scorza-Dragoni property. That is each $\varepsilon > 0$, there exists a closed subset D_ε of I such that $m(I \setminus D_\varepsilon) \le \varepsilon$ and $f|_{D_\varepsilon \times X}$ is continuous.*

The operator $(Fx)(t) = f(t, x(t))$ is called superposition operator or Nemytskij operator associated to f. The following lemma on superposition operator is important in our theorem (see [12] and also in [13]).

Lemma 2.3. *The superposition operator F generated by the function $f(t, x)$ maps continuously the space $L^1(I, X)$ into itself (I may be unbounded interval) if and only if there exist $a(t) \in L^1(I)$ and a nonnegative constant b such that*

$$\|f(x, t)\|_X \le a(t) + b\|x\|_X \tag{2.2}$$

for all $t \in I$ and $x \in X$.

The Volterra operator which is defined by $(Kx)(t) = \int_0^t k(t, s)x(s)ds$ for $x \in L^1(\mathbb{R}_+, X)$ where $k(t, s)$ is measurable with respect to both variables. If K transforms $L^1(\mathbb{R}_+, X)$ into itself it is then a bounded operator with norm $\|K\|$ which is majorized by the number

$$\operatorname{ess\,sup}_{s\ge 0} \int_s^\infty |k(t, s)| dt < \infty. \tag{2.3}$$

3. Measure of Weak Noncompactness

In this section we will recall the concept of measure of weak noncompactness which is the key point to complete our proof of main theorem in Section 5.

Let H be a Banach space. $\mathcal{B}(H)$ and $\mathcal{W}(H)$ denote the collections of all nonempty bounded subsets and relatively weak compact subsets, respectively.

Definition 3.1. A function $\mu : \mathcal{B}(H) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:

(1) the family $\operatorname{Ker}\mu = \{E \in \mathcal{B}(H) : \mu(E) = 0\}$ is nonempty and $\operatorname{Ker}\mu \subset \mathcal{W}(H)$;

(2) if $E \subset F$, we have $\mu(E) \le \mu(F)$;

(3) $\mu(\operatorname{Conv}(E)) = \mu(E)$, where $\operatorname{Conv}(E)$ denotes the convex closed hull of E;

(4) $\mu(\lambda E + (1-\lambda)F) \le \lambda\mu(E) + (1-\lambda)\mu(F)$ for $\lambda \in [0,1]$;

(5) If $\{E_n\} \subset \mathcal{B}(H)$ is a decreasing sequence, that is, $E_{n+1} \subset E_n$, every E_n is weakly closed,

and $\lim_{n\to\infty}\mu(E_n) = 0$, then $E_\infty = \bigcap_{n=1}^{\infty} E_n$ is nonempty.

From [14], we see the following measure of weak noncompact:

$$c(E) = \inf\{r > 0, \exists K \in \mathcal{W}(H) : E \subseteq K + B_r\}, \tag{3.1}$$

where B_r denotes the closed ball in H centered at 0 with radius $r > 0$.

In [15], Appel and De Pascale gave to c the following simple form in $L^1(\mathbb{R}_+, X)$ space:

$$c(E) = \limsup_{\varepsilon\to 0}\left\{\sup\nolimits_{x\in E}\left[\left[\int_D \|x(t)\|_X dt : D \subset \mathbb{R}_+, m(D) \le \varepsilon\right]\right]\right\} \tag{3.2}$$

for a nonempty and bounded subset E of space $L^1(\mathbb{R}_+, X)$.

Let

$$d(E) = \limsup_{T\to\infty}\left\{\sup\nolimits_{x\in E}\int_T^{\infty} \|x(t)\|_X dt\right\},$$
$$\mu(E) = c(E) + d(E) \tag{3.3}$$

for a nonempty and bounded subset E of space $L^1(\mathbb{R}_+, X)$.

It is easy to know that μ is a measure of weak noncompactness in space $L^1(\mathbb{R}_+, X)$ following the verification in [16].

4. Krasnoselskii's Fixed Point Theorem

The following is the Krasnoselskii's fixed point theorem which will be utilized to obtain the existence of solutions in the next section.

Theorem 4.1. *Let K be a closed convex and nonempty subset of a Banach space E. Let P, Q be two operators such that*

(i) $P(K) + Q(K) \subseteq K$;

(ii) *P is a contraction mapping;*

(iii) *$Q(K)$ is relatively compact and Q is continuous.*

Then there exists $z \in K$ such that $Pz + Qz = z$.

Remark 4.2. In [9], they proved the existence of solutions by means of Schauder fixed point theorem. With the presence of the Perturbation term $g(t, x(t))$ in the integral equation, the Schauder fixed point theorem is invalid. To overcome this difficulty we will use the Kransnoselskii's fixed point theorem to obtain the existence of solutions.

Remark 4.3. We will see in the following section that the important step is the construction of K by means of measure of weak noncompactness. This is the biggest difference between our paper from [10].

Remark 4.4. The Krasnoselskii's fixed point theorem was extended to general case in [17] (see also in [13]). In [10], they used the general Krasnoselskii's fixed point theorem to obtain the existence result. It can be seen in the next section of our paper that the classical Krasnoselskii's fixed point theorem is enough unless we need more general conditions on the perturbation term g.

5. Main Theorem and Proof

Our main theorem in this paper is stated as follows.

Theorem 5.1. *Suppose that the following assumptions are satisfied.*

(H1) The functions $f_i : \mathbb{R}_+ \times X \to X$ satisfy Carathéodory conditions, and there exist constants $b_i > 0$ and functions $a_i \in L^1(\mathbb{R}_+)$ such that

$$\|f_i(t, x)\|_X \leq a_i(t) + b_i \|x\|_X \tag{5.1}$$

for $t \in \mathbb{R}_+$ and $x \in X (i = 1, 2)$.

(H2) Then function $k(t, s) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditons, and the linear Volterra integral operator K defined by

$$(Kx)(t) = \int_0^t k(t, s) x(s) ds \tag{5.2}$$

transforms the space $L^1(\mathbb{R}_+, X)$ into itself.

(H3) The function $g(t, x, y) : \mathbb{R}_+ \times X \times X \to X$ is measurable in t and continuous in x and y for almost all t. And there exist two positive constants β_1, β_2 and a function $\alpha \in L^1(\mathbb{R}_+)$ such that

$$\|g(t, x, y)\|_X \leq \alpha(t) + \beta_1 \|x\|_X + \beta_2 \|y\|_X \tag{5.3}$$

for $t \in \mathbb{R}_+$ and $x, y \in X$. Additionally, the function g satisfies the following Lipschitz condition for almost all t:

$$\left\|g(t, x_1, y_1) - g(t, x_2, y_2)\right\|_X \le C_1\|x_1 - x_2\|_X + C_2\left\|y_1 - y_2\right\|_X. \tag{5.4}$$

(H4) The function $\lambda(t) \in C^1(\mathbb{R}_+, \mathbb{R})$ such that $\lambda(D) \subset D$ where D is an arbitrary subset of $\mathbb{R}_+$, and $1/|\lambda'(t)|$ is bounded by M_0 for all $t \in [0, \infty)$.

(H5) $q = \beta_1 + M_0\beta_2 + b_1b_2\|K\| < 1$, where $\|K\|$ denotes the norm of the linear Volterra operator K.

(H6) $p = C_1 + M_0C_2 < 1$.

Then the integral equation (1.1) *has at least one solution $x \in L^1(\mathbb{R}_+, X)$.*

Proof. Equation (1.1) may be written in the following form:

$$\begin{gathered} x = Px + Qx, \\ Px = g(t, x(t), x(\lambda(t))), \\ Qx = f_1\left(t, \int_0^t k(t, s) f_2(s, x(s)) ds\right) = F_1 K F_2 x, \end{gathered} \tag{5.5}$$

where K is the linear Volterra integral operator and F_i is the superposition operator generated by the function $f_i(t, x)$ $(i = 1, 2)$.

The proof will be given in six steps.

Step 1. There exists $r > 0$ such that $P(B_r) + Q(B_r) \subseteq B_r$, where B_r is a ball centered zero element with radius r in $L^1(\mathbb{R}_+, X)$.

Let x and y be arbitrary functions in $B_r \subset L^1(\mathbb{R}_+, X)$ with r to be determined later. In view of our assumptions we get

$$\begin{aligned} &\left\|Px + Qy\right\|_{L^1(\mathbb{R}_+, X)} \\ &\quad = \int_0^\infty \left\|g(t, x(t), x(\lambda(t))) + f_1\left(t, \int_0^t k(t, s) f_2(s, y(s)) ds\right)\right\|_X dt \\ &\quad \le \int_0^\infty \left(\alpha(t) + \beta_1\|x(t)\|_X + \beta_2\|x(\lambda(t))\|_X + a_1(t) + b_1\left\|\int_0^t k(t, s) f_2(s, y(s)) ds\right\|_X\right) dt \\ &\quad \le \|\alpha\|_{L^1(\mathbb{R}_+)} + \beta_1\|x\|_{L^1(\mathbb{R}_+, X)} + \beta_2 M_0\|x\|_{L^1(\mathbb{R}_+, X)} + \|a_1\|_{L^1(\mathbb{R}_+)} + b_1\left\|KF_2y\right\|_{L^1(\mathbb{R}_+, X)} \\ &\quad \le \|\alpha\|_{L^1(\mathbb{R}_+)} + \beta_1\|x\|_{L^1(\mathbb{R}_+, X)} + \beta_2 M_0\|x\|_{L^1(\mathbb{R}_+, X)} + \|a_1\|_{L^1(\mathbb{R}_+)} \\ &\qquad + b_1\|K\| \int_0^\infty \left\|f_2(t, y(t))\right\|_X dt \le \|\alpha\|_{L^1(\mathbb{R}_+)} + \beta_1\|x\|_{L^1(\mathbb{R}_+, X)} + \beta_2 M_0\|x\|_{L^1(\mathbb{R}_+, X)} \end{aligned}$$

$$
\begin{aligned}
&+\|a_1\|_{L^1(\mathbb{R}_+)}+b_1\|K\|\int_0^{\infty}\left(a_2(t)+b_2\|y(t)\|_X\right)dt\le\|\alpha\|_{L^1(\mathbb{R}_+)}\\
&+\beta_1\|x\|_{L^1(\mathbb{R}_+,X)}+\beta_2M_0\|x\|_{L^1(\mathbb{R}_+,X)}+\|a_1\|_{L^1(\mathbb{R}_+)}+b_1\|K\|\|a_2\|_{L^1(\mathbb{R}_+)}\\
&+b_1b_2\|K\|\|y\|_{L^1(\mathbb{R}_+,X)}\le\|\alpha\|_{L^1(\mathbb{R}_+)}+\beta_1r+\beta_2M_0r+\|a_1\|_{L^1(\mathbb{R}_+)}\\
&+b_1\|K\|\|a_2\|_{L^1(\mathbb{R}_+)}+b_1b_2\|\mathrm{k}\|r\le r.
\end{aligned}\tag{5.6}
$$

We then derive that $P(B_r)+Q(B_r)\subseteq B_r$ by taking

$$
r=\frac{\|\alpha\|_{L^1(\mathbb{R}_+)}+\|\alpha_1\|_{L^1(\mathbb{R}_+)}+b_1\|K\|\|a_2\|_{L^1(\mathbb{R}_+)}}{1-q}>0,\tag{5.7}
$$

where $q=\beta_1+\beta_2M_0+b_1b_2\|K\|<1$ by assumption *(H5)*.

Step 2. $\mu(P(M)+Q(M))\le q\mu(M)$ for all bounded subset M of $L^1(\mathbb{R}_+,X)$.

Take a arbitrary numbers $\varepsilon>0$ and $D\subset\mathbb{R}_+$ such that $m(D)\le\varepsilon$.

For any $x,y\in M$, we have

$$
\begin{aligned}
\int_D\|Px+Qy\|_Xdt&\le\int_D\|Px\|_Xdt+\int_D\|Qy\|_Xdt\\
&\le\int_D\alpha(t)dt+\beta_1\int_D\|x\|_Xdt+\beta_2\int_D\|x(\lambda(t))\|_Xdt\\
&\quad+\int_D a_1(t)dt+b_1\int_D\|KF_2y\|_Xdt\\
&\le\int_D\alpha(t)dt+\int_D a_1(t)dt+b_1\|K\|\int_D a_2(t)dt\\
&\quad+\beta_1\int_D\|x\|_Xdt+\beta_2M_0\int_D\|x\|_Xdt+b_1b_2\|K\|\int_D\|y(t)\|_Xdt.
\end{aligned}\tag{5.8}
$$

It follows that $c(P(M)+Q(M))\le(\beta_1+M_0\beta_2+b_1b_2\|K\|)c(M)=qc(M)$ by definition (3.2).

For $T>0$ and any $x,y\in M$, we have

$$
\begin{aligned}
\int_T^{\infty}\|Px+Qy\|_Xdt\le&\int_T^{\infty}\alpha(t)dt+\int_T^{\infty}a_1(t)dt+b_1\|K\|\int_T^{\infty}a_2dt\\
&+\beta_1\int_T^{\infty}\|x\|_Xdt+\beta_2M_0\int_T^{\infty}\|x\|_Xdt+b_1b_2\|K\|\int_T^{\infty}\|y(t)\|_Xdt,
\end{aligned}\tag{5.9}
$$

and then $d(P(M)+Q(M))\le(\beta_1+M_0\beta_2+b_1b_2\|K\|)d(M)=qd(M)$ by definition (3.3).

From above, we then obtain $\mu(P(M)+Q(M))\le q\mu(M)$ for all bounded subset M of $L^1(\mathbb{R}_+,X)$.

Step 3. We will construct a nonempty closed convex weakly compact set in on which we will apply fixed point theorem to prove the existence of solutions.

Let $B_r^1 = \operatorname{Conv}(P(B_r) + Q(B_r))$ where B_r is defined in Step 1, $B_r^2 = \operatorname{Conv}(P(B_r^1) + Q(B_r^1))$ and so on. We then get a decreasing sequence $\{B_r^n\}$, that is, $B_r^{n+1} \subset B_r^n$ for $n = 1, 2, \ldots$. Obviously all sets belonging to this sequence are closed and convex, so weakly closed. By the fact proved in Step 2 that $\mu(P(M) + Q(M)) \le q\mu(M)$ for all bounded subset M of $L^1(\mathbb{R}_+, X)$, we have

$$\mu(B_r^n) \le q^n \mu(B_r), \tag{5.10}$$

which yields that $\lim_{n\to\infty}\mu(B_r^n) = 0$.

Denote $K = \bigcap_{n=1}^{\infty} B_r^n$, and then $\mu(K) = 0$. By the definition of measure of weak noncompact we know that K is nonempty. Moreover, $Q(K) \subset K$.

K is just nonempty closed convex weakly compact set which we need in the following steps.

Step 4. $Q(K)$ is relatively compact in $L^1(\mathbb{R}_+, X)$, where K is just the set constructed in Step 3.

Let $\{x_n\} \subset K$ be arbitrary sequence. Since $\mu(K) = 0$, $\exists T$, $\forall n$, the following inequality is satisfied:

$$\int_T^{\infty} \|x_n\|_X dt \le \frac{\varepsilon}{4}. \tag{5.11}$$

Considering the function $f_i(t, x)$ on $[0, T]$ and $k(t, s)$ on $[0, T] \times [0, T]$, we can find a closed subset D_ε of interval $[0, T]$, such that $m([0, T] \setminus D_\varepsilon) \le \varepsilon$, and such that $f_i|_{D_\varepsilon \times X}$ $(i = 1, 2)$ and $k|_{D_\varepsilon \times [0,T]}$ is continuous. Especially $k|_{D_\varepsilon \times [0,T]}$ is uniformly continuous.

Let us take arbitrarily $t_1, t_2 \in D_\varepsilon$ and assume $t_1 < t_2$ without loss of generality. For an arbitrary fixed n and denoting $\varphi_n(t) = (KF_2 x_n)(t)$ we obtain:

$$\begin{aligned}
\|\varphi_n(t_2) - \varphi_n(t_1)\|_X &= \left\| \int_0^{t_2} k(t_2, s) f_2(s, x_n(s)) ds - \int_0^{t_1} k(t_1, s) f_2(s, x_n(s)) ds \right\|_X \\
&\le \left\| \int_0^{t_1} k(t_2, s) f_2(s, x_n(s)) ds - \int_0^{t_1} k(t_1, s) f_2(s, x_n(s)) ds \right\|_X \\
&\quad + \left\| \int_{t_1}^{t_2} k(t_2, s) f_2(s, x_n(s)) ds \right\|_X \\
&\le \int_0^{t_1} |k(t_2, s) - k(t_1, s)| (a_2(s) + b_2 \|x_n(s)\|_X) ds \\
&\quad + \int_{t_1}^{t_2} |k(t_2, s)| (a_2(s) + b_2 \|x_n(s)\|_X) ds
\end{aligned}$$

$$\begin{aligned}&\leq \omega^T(k,|t_2-t_1|)\int_0^T(a_2(s)+b_1\|x_n(s)\|_X)ds+\widetilde{k}\int_{t_1}^{t_2}(a_2(s)+b_2\|x_n(s)\|_X)ds\\&\leq \omega^T(k,|t_2-t_1|)\Big(\|a_2\|_{L^1(\mathbb{R})}+b_2r\Big)+\widetilde{k}\int_{t_1}^{t_2}a_2(s)ds+b_2\widetilde{k}\int_{t_1}^{t_2}\|x_n(s)\|_Xds\end{aligned}\tag{5.12}$$

where $\omega^T(k,\cdot)$ denotes the modulus of continuity of the function k on the set $D_\varepsilon \times [0,T]$ and $\widetilde{k} = \max\{|k(t,s)| : (t,s) \in D_\varepsilon \times [0,T]\}$. The last inequality of (5.12) is obtained since $K \subset B_r$, where r is just the one in the Step 1.

Taking into account the fact that the $\mu(\{x_n\}) \leq \mu(K) = 0$, we infer that the terms of the numerical sequence $\{\int_{t_1}^{t_2}\|x_n(s)\|_X)ds\}$ are arbitrarily small provided that the number $t_2 - t_1$ is small enough.

Since $\int_{t_1}^{t_2} a_2(s)ds$ is also arbitrarily small provided that the number $t_2 - t_1$ is small enough, the right of (5.12) then tends to zero independent of x_n as $t_2 - t_1$ tends to zero. We then have $\{\varphi_n\}$ is equicontinuous in the space $C(D_\varepsilon, X)$.

On the other hand,

$$\begin{aligned}\|\varphi_n(t)\|_X&=\left\|\int_0^t k(t,s)f_2(s,x_n)ds\right\|_X\\&\leq\int_0^t|k(t,s)|(a_2(s)+b_2\|x_n(s)\|_X)ds\\&\leq\widetilde{k}\left(\int_0^t a_2(s)ds+b_2\int_0^t\|x_n(s)\|_Xds\right)\\&\leq\widetilde{k}\Big(\|a_2\|_{L^1(\mathbb{R}_+)}+b_2\|x_n\|_{L^1(\mathbb{R}_+,X)}\Big)\\&\leq\widetilde{k}\Big(\|a_2\|_{L^1(\mathbb{R}_+)}+b_2r\Big).\end{aligned}\tag{5.13}$$

From above, we then obtain that $\{\varphi_n\}$ is equibounded in the space $C(D_\varepsilon, X)$.

By assumption *(H1)*,we have the operator F_1 is continuous. So $\{Q(x_n)\} = \{F_1\varphi_n\}$ forms a relatively compact set in the space $C(D_\varepsilon, X)$.

Further observe that the above result does not depend on the choice of ε. Thus we can construct a sequence D_l of closed subsets of the interval $[0,T]$ such that $m([0,T] \setminus D_l) \to 0$ as $l \to 0$ and such that the sequence $\{Q(x_n)\}$ is relatively compact in every space $C(D_l, X)$. Passing to subsequence if necessary we can assume that $\{Q(x_n)\}$ is a cauchy sequence in each space $C(D_l, X)$.

Observe the fact $Q(K) \subset K$, then $\mu(Q(K)) = 0$. By the definition (3.2), let us choose a number $\delta > 0$ such that for each closed subset D of the interval $[0,T]$ provided that $m([0,T] \setminus D) \leq \delta$ we have

$$\int_{D'}\|Qx\|_Xdt\leq\frac{\varepsilon}{4}\tag{5.14}$$

for any $x \in K$, where $D' = [0,T] \setminus D$.

By the fact that $\{Qx_n\}$ is a cauchy sequence in each space $C(D_l, X)$, we can choose a natural number l_0 such that $m([0,T] \setminus D_{l_0}) \leq \delta$ and $m(D_{l_0}) > 0$, and for arbitrary natural number $n, m \geq l_0$ the following inequality holds:

$$\|(Qx_n)(t) - (Qx_m)(t)\|_X \leq \frac{\varepsilon}{4m(D_{l_0})} \tag{5.15}$$

for any $t \in D_{l_0}$.

Combining (5.11), (5.14) and (5.15), we get

$$\begin{aligned}
\|Qx_n - Qx_m\|_{L^1(\mathbb{R}_+,X)} &= \int_0^\infty \|(Qx_n)(t) - (Qx_m)(t)\|_X dt \\
&= \int_T^\infty \|(Qx_n)(t) - (Qx_m)(t)\|_X dt + \int_{D_{l_0}} \|(Qx_n)(t) - (Qx_m)(t)\|_X dt \\
&\quad + \int_{[0,T]\setminus D_{l_0}} \|(Qx_n)(t) - (Qx_m)(t)\|_X dt \leq \varepsilon
\end{aligned} \tag{5.16}$$

which means that $\{Qx_n\}$ is a cauchy sequence in the space $L^1(\mathbb{R}, X)$. Hence we conclude that $Q(K)$ is relatively compact in $L^1(\mathbb{R}, X)$.

Step 5. The operator P is a contraction mapping:

$$\begin{aligned}
\|Px_1 - Px_2\|_{L^1(\mathbb{R}_+,X)} &= \left\|g(t, x_1(t), x_1(\lambda(t))) - g(t, x_2(t), x_2(\lambda(t)))\right\|_{L^1(\mathbb{R}_+,X)} \\
&\leq C_1\|x_1(t) - x_2(t)\|_{L^1(\mathbb{R}_+,X)} + C_2\|x_1(\lambda(t)) - x_2(\lambda(t))\|_{L^1(\mathbb{R}_+,X)} \\
&\leq C_1\|x_1(t) - x_2(t)\|_{L^1(\mathbb{R}_+,X)} + C_2\int_0^\infty \|x_1(\lambda(t)) - x_2(\lambda(t))\|_X dt \\
&\leq C_1\|x_1(t) - x_2(t)\|_{L^1(\mathbb{R}_+,X)} + C_2 M_0\int_0^\infty \|x_1(s) - x_2(s)\|_X ds \\
&= (C_1 + M_0C_2)\|x_1(t) - x_2(t)\|_{L^1(\mathbb{R}_+,X)} \\
&= p\|x_1(t) - x_2(t)\|_{L^1(\mathbb{R}_+,X)},
\end{aligned} \tag{5.17}$$

where we have made a transformation $s = \lambda(t)$ in the above process. Since $p < 1$ by assumption *(H6)*, we then get the fact that the operator P is a contraction mapping.

Step 6. We now check out that the conditions needed in Krasnoselskii's fixed point theorem are fulfilled.

(1) From Step 3, we know that $P(K) + Q(K) \subseteq K$, where K is the set constructed in Step 3.

(2) From Step 5, we know that P is a contraction mapping.

(3) From the Step 4 and assumptions *(H1)*, *(H2)*, $Q(K)$ is relatively compact and Q is continuous.

We apply Theorem 4.1, and then obtain that (1.1) has at least one solution in $L^1(\mathbb{R}^+, X)$. □

Remark 5.2. When $X = \mathbb{R}$, in [10] they said Q is weakly sequence compact in their Step 1 of main proof. From our proof, we know that their proof is not precise, since in Step 4, one of the crucial conditions to prove the relatively compactness of $Q(K)$ is that $Q(K)$ is weakly compact. We can only obtain that Q is weakly sequence compact as a mapping from K to K which is the weakly compact set defined in Step 3. The construction of set K overcomes the fault in [10], and we obtain the existence result finally.

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