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Research Article

Study of an Approximation Process of Time Optimal Control for Fractional Evolution Systems in Banach Spaces

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The paper is devoted to the study of an approximation process of time optimal control for fractional evolution systems in Banach spaces. We firstly convert time optimal control problem into Meyer problem. By virtue of the properties of the family of solution operators given by us, the existence of optimal controls for Meyer problem is proved. Secondly, we construct a sequence of Meyer problems to successive approximation of the original time optimal control problem. Finally, a new approximation process is established to find the solution of time optimal control problem. Our method is different from the standard method.

1. Introduction

It has been shown that the accurate modelling in dynamics of many engineering, physics, and economy systems can be obtained by using fractional differential equations. Numerous applications can be found in viscoelasticity, electrochemistry, control, porous media, electromagnetic, and so forth. There has been a great deal of interest in the solutions of fractional differential equations in analytical and numerical sense. One can see the monographs of Kilbas et al. [1], Miller and Ross [2], Podlubny [3], and Lakshmikantham et al. [4]. The fractional evolution equations in infinite dimensional spaces attract many authors including us (see, for instance, [5–21] and the references therein).

When the fractional differential equations describe the performance index and system dynamics, a classical optimal control problem reduces to a fractional optimal control problem. The optimal control of a fractional dynamics system is a fractional optimal control with system dynamics defined with partial fractional differential equations.

There has been very little work in the area of fractional optimal control problems [18, 22], especially the time optimal control for fractional evolution equations [19]. Recalling that the research on time optimal control problems dates back to the 1960s, many problems such as existence and necessary conditions for optimality and controllability have been discussed, for example, see [23] for the finite dimensional case and [7, 24–37] for the infinite dimensional case. Since the cost functional for a time optimal control problem is the infimum of a number set, it is different with the Lagrange problem, the Bolza problem and the Meyer problem, which arise some new difficulties. As a result, we regard the time optimal control as another problem which is not the same as the above three problems.

Motivated by our previous work in [18–21, 38], we consider the time optimal control problem (P) of a fractional evolution system governed by

$$\begin{gathered} {}^{C}D_t^q z(t) = Az(t) + f(t, z(t), B(t)v(t)), \quad t \in (0, \tau),\ q \in (0,1), \\ z(0) = z_0 \in X, \quad v \in V_{\mathrm{ad}}, \end{gathered} \tag{1.1}$$

where ${}^{C}D_t^q$ is the Caputo fractional derivative of order q, $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$, V_{ad} is the admissible control set and $f : I_\tau := [0, \tau] \times X \times X \to X$ will be specified latter.

Let us mention, we do not study the time optimal control problem (P) of the above system by standard method used in our earlier work [19]. In the present paper, we will construct a sequences of Meyer problems $(\mathrm{P}_{\varepsilon_n})$ to successive approximation time optimal control problem (P). Therefore, we need introduce the following new fractional evolution system

$$\begin{gathered} {}^{C}D_s^q x(s) = k^q Ax(s) + k^q f(ks, x(s), B(ks)u(s)), \quad s \in (0,1], \\ x(0) = z_0 \in X, \qquad w = (u, k) \in W, \end{gathered} \tag{1.2}$$

whose controls are taken from a product space W will be specified latter.

By applying the family of solution operators $\mathcal{T}_k$ and $\mathcal{S}_k$ (see Lemma 3.7) associated with the family of C_0-semigroups with parameters and some probability density functions, the existence of optimal controls for Meyer problems (P_ε) is proved. Then, we show that there exists a subsequence of Meyer problems $(\mathrm{P}_{\varepsilon_n})$ whose corresponding sequence of optimal controls $\{w_{\varepsilon_n}\} \in W$ converges to a time optimal control of problem (P) in some sense. In other words, in a limiting process, the sequence $\{w_{\varepsilon_n}\} \in W$ can be used to find the solution of time optimal control problem (P). The existence of time optimal controls for problem (P) is proved by this constructive approach which provides a new method to solve the time optimal control.

The rest of the paper is organized as follows. In Section 2, some notations and preparation results are given. In Section 3, we formulate the time optimal control problem (P) and Meyer problem (P_ε). In Section 4, the existence of optimal controls for Meyer problems (P_ε) is proved. Finally, we display the Meyer approximation process of time optimal control and derive the main result of this paper.

2. Preliminaries

Throughout this paper, we denote by X a Banach space with the norm $\|\cdot\|$. For each $\tau < +\infty$, let $I_\tau \equiv [0,\tau]$ and $C(I_\tau, X)$ be the Banach space of continuous functions from I_τ to X with the usual supremum norm. Let $A : D(A) \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$. This means that there exists $M > 0$ such that $\sup_{t\in I_\tau}\|T(t)\| \leq M$. We will also use $\|f\|_{L^p(I_\tau,R^+)}$ to denote the $L^p(I_\tau, R^+)$ norm of f whenever $f \in L^p(I_\tau, R^+)$ for some p with $1 < p < \infty$.

Let us recall the following definitions in [1].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}ds, \quad t > 0,\ \gamma > 0, \tag{2.1}$$

provided the right side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0,\infty) \to R$ can be written as

$${}^L D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}ds, \quad t > 0,\ n-1 < \gamma < n. \tag{2.2}$$

Definition 2.3. The Caputo derivative of order γ for a function $f : [0,\infty) \to R$ can be written as

$${}^C D^\gamma f(t) = {}^L D^\gamma\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t > 0,\ n-1 < \gamma < n. \tag{2.3}$$

Remark 2.4. (i) If $f(t) \in C^n[0,\infty)$, then

$${}^C D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}}ds = I^{n-\gamma}f^{(n)}(t), \quad t > 0,\ n-1 < \gamma < n. \tag{2.4}$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Lemma 2.5 (see [38, Lemma 3.1]). *If the assumption [A] holds, then*

(1) *for given $k \in [0,\widehat{T}]$, kA is the infinitesimal generator of C_0-semigroup $\{T_k(t), t \geq 0\}$ on X,*

(2) *there exist constants $C \geq 1$ and $\omega \in (-\infty,+\infty)$ such that*

$$\|T_k(t)\| \leq Ce^{\omega kt}, \quad \forall t \geq 0, \tag{2.5}$$

(3) *if* $k_n \to k_\varepsilon$ *in* $[0,\widehat{T}]$ *as* $n \to \infty$*, then for arbitrary* $x \in X$ *and* $t \geq 0$*,*

$$T_{k_n}(t) \xrightarrow{s} T_{k_\varepsilon}(t), \quad \text{as } n \longrightarrow \infty \tag{2.6}$$

uniformly in t *on some closed interval of* $[0,\widehat{T}]$ *in the strong operator topology sense.*

3. System Description and Problem Formulation

Consider the following fractional nonlinear controlled system

$$\begin{aligned} {}^{C}D_t^q z(t) &= Az(t) + f(t, z(t), B(t)v(t)), \quad t \in (0,\tau), \\ z(0) &= z_0 \in X, \quad v \in V_{\text{ad}}. \end{aligned} \tag{3.1}$$

We make the following assumptions.

[A] : A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on X with domain $D(A)$.

[F] : $f : I_\tau \times X \times X \to X$ is measurable in t on I_τ and for each $\rho > 0$, there exists a constant $L(\rho) > 0$ such that for almost all $t \in I_\tau$ and all $z_1, z_2, y_1, y_2 \in X$, satisfying $\|z_1\|, \|z_2\|, \|y_1\|, \|y_2\| \leq \rho$, we have

$$\|f(t, z_1, y_1) - f(t, z_2, y_2)\| \leq L(\rho)(\|z_1 - z_2\| + \|y_1 - y_2\|). \tag{3.2}$$

For arbitrary $(t, z, y) \in I_\tau \times X \times X$, there exists a positive constant $M > 0$ such that

$$\|f(t, z, y)\| \leq M(1 + \|z\| + \|y\|). \tag{3.3}$$

[B] : Let E be a separable reflexive Banach space, $B \in L_\infty(I_\tau, L(E, X))$, $\|B\|_\infty$ stands for the norm of operator B on Banach space $L_\infty(I_\tau, L(E, X))$. $B : L^p(I_\tau, E) \to L^p(I_\tau, X)(1 < p < +\infty)$ is strongly continuous.

[U] : Multivalued maps $\mathcal{V}(\cdot) : I_\tau \to 2^E \setminus \{\emptyset\}$ has closed, convex and bounded values. $\mathcal{V}(\cdot)$ is graph measurable and $\mathcal{V}(\cdot) \subseteq \Omega$ where Ω is a bounded set of E.

Set

$$V_{\text{ad}} = \{v(\cdot) \mid I_\tau \longrightarrow E \text{ measurable, } v(t) \in \mathcal{V}(t) \text{ a.e.}\}. \tag{3.4}$$

Obviously, $V_{\text{ad}} \neq \emptyset$ (see [39, Theorem 2.1]) and $V_{\text{ad}} \subset L^p(I_\tau, E)$ $(1 < p < +\infty)$ is bounded, closed and convex.

Based on our previous work [21, Lemma 3.1 and Definition 3.1], we use the following definition of mild solutions for our problem.

Definition 3.1. By the mild solution of system (3.1), we mean that the function $x \in C(I_\tau, X)$ which satisfies

$$z(t) = \mathcal{T}(t)z_0 + \int_0^t (t-\theta)^{q-1}\mathcal{S}(t-\theta)f(\theta, z(\theta), B(\theta)v(\theta))d\theta, \quad t \in I_\tau, \tag{3.5}$$

where

$$\mathcal{T}(t) = \int_0^\infty \xi_q(\theta)T(t^q\theta)d\theta, \qquad \mathcal{S}(t) = q\int_0^\infty \theta\xi_q(\theta)T(t^q\theta)d\theta,$$
$$\xi_q(\theta) = \frac{1}{q}\theta^{-1-1/q}\varpi_q\left(\theta^{-1/q}\right) \geq 0, \tag{3.6}$$
$$\varpi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty),$$

ξ_q is a probability density function defined on $(0,\infty)$, that is

$$\xi_q(\theta) \geq 0, \quad \theta \in (0,\infty), \quad \int_0^\infty \xi_q(\theta)d\theta = 1. \tag{3.7}$$

Remark 3.2. (i) It is not difficult to verify that for $v \in [0,1]$

$$\int_0^\infty \theta^v\xi_q(\theta)d\theta = \int_0^\infty \theta^{-qv}\varpi_q(\theta)d\theta = \frac{\Gamma(1+v)}{\Gamma(1+qv)}. \tag{3.8}$$

(ii) For another suitable definition of mild solutions for fractional differential equations, the reader can refer to [13].

Lemma 3.3 (see [21, Lemmas 3.2-3.3]). *The operators $\mathcal{T}$ and $\mathcal{S}$ have the following properties.*

(i) *For any fixed $t \geq 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are linear and bounded operators; that is, for any $x \in X$,*

$$\|\mathcal{T}(t)x\| \leq M\|x\|, \qquad \|\mathcal{S}(t)x\| \leq \frac{qM}{\Gamma(1+q)}\|x\|. \tag{3.9}$$

(ii) $\{\mathcal{T}(t), t \geq 0\}$ *and* $\{\mathcal{S}(t), t \geq 0\}$ *are strongly continuous.*

We present the following existence and uniqueness of mild solutions for system (3.1).

Theorem 3.4. *Under the assumptions [A], [B], [F] and [U], for every $v \in V_{ad}$ and $pq > 1$, system* (3.1) *has a unique mild solution $z \in C(I_\tau, X)$ which satisfies the following integral equation*

$$z(t) = \mathcal{T}(t)z_0 + \int_0^t (t-\theta)^{q-1}\mathcal{S}(t-\theta)f(\theta, z(\theta), B(\theta)v(\theta))d\theta. \tag{3.10}$$

Proof. Consider the ball given by $\mathcal{B} = \{x \in C([0,T_1],X) \mid \|x(t) - x_0\| \leq 1, 0 \leq t \leq T_1\}$, where T_1 would be chosen, and $\|x(t)\| \leq 1 + \|x_0\| = \rho$, $0 \leq t \leq T_1$, $\mathcal{B} \subseteq C([0,T_1],X)$ is a closed convex set. Define a map $\mathscr{H}$ on $\mathcal{B}$ given by

$$(\mathscr{H}z)(t) = \mathcal{T}(t)z_0 + \int_0^t (t-\theta)^{q-1}\mathcal{S}(t-\theta)f(\theta, z(\theta), B(\theta)v(\theta))d\theta. \tag{3.11}$$

Note that by the properties of $\mathcal{T}$ and $\mathcal{S}$, assumptions [A], [F], [B], and [U], by standard process (see [19, Theorem 3.2]), one can verify that $\mathscr{H}$ is a contraction map on $\mathcal{B}$ with $T_1 > 0$. This means that system (3.1) has a unique mild solution on $[0,T_1]$. Again, using the singular version Gronwall inequality, we can obtain the a prior estimate of the mild solutions of system (3.1) and present the global existence of the mild solutions. □

Definition 3.5 (admissible trajectory). Take two points z_0, z_1 in the state space X. Let z_0 be the initial state and let z_1 be the desired terminal state with $z_0 \neq z_1$, denote $z(v) \equiv \{z(t,v) \in X \mid t \geq 0\}$ be the state trajectory corresponding to the control $v \in V_{\text{ad}}$. A trajectory $z(v)$ is said to be admissible if $z(0,v) = z_0$ and $z(t,v) = z_1$ for some finite $t > 0$.

Set $V_0 = \{v \in V_{\text{ad}} \mid z(v)$ is an admissible trajectory$\} \subset V_{\text{ad}}$. For given z_0, $z_1 \in X$ and $z_0 \neq z_1$, if $V_0 \neq \emptyset$ (i.e., there exists at least one control from the admissible class that takes the system from the given initial state z_0 to the desired target state z_1 in the finite time.), we say the system (3.1) can be controlled.

Let $\tau(v) \equiv \inf\{t \geq 0 \mid z(t,v) = z_1\}$ denote the transition time corresponding to the control $v \in V_0 \neq \emptyset$ and define $\tau^* = \inf\{\tau(v) \geq 0 \mid v \in V_0\}$.

Then, the time optimal control problem can be stated as follows.

Problem (Problem (P)). Take two points z_0, z_1 in the state space X. Let z_0 be the initial state and let z_1 be the desired terminal state with $z_0 \neq z_1$. Suppose that there exists at least one control from the admissible class that takes the system from the given initial state z_0 to the desired target state z_1 in the finite time. The time optimal control problem is to find a control $v^* \in V_0$ such that

$$\tau(v^*) = \tau^* = \inf\{\tau(v) \geq 0 \mid v \in V_0\}. \tag{3.12}$$

For fixed $\hat{v} \in V_{\text{ad}}$, $\widehat{T} = \tau(\hat{v}) > 0$. Now, we introduce the following linear transformation

$$t = ks, \quad 0 \leq s \leq 1, \; k \in \left[0, \widehat{T}\right]. \tag{3.13}$$

Through this transformation, system (3.1) can be replaced by

$$\begin{gathered} {}^C D_s^q x(s) = k^q A x(s) + k^q f(ks, x(s), B(ks)u(s)), \quad s \in (0,1], \\ x(0) = z(0) = z_0 \in X, \qquad w = (u,k) \in W, \end{gathered} \tag{3.14}$$

where $x(\cdot) = z(k\cdot)$, $u(\cdot) = v(k\cdot)$, and define

$$W = \left\{(u,k) \mid u(s) = v(ks),\ 0 \le s \le 1,\ v \in V_{\mathrm{ad}},\ k \in \left[0,\widehat{T}\right]\right\}. \tag{3.15}$$

By Theorem 3.4, one can obtain the following existence result.

Theorem 3.6. *Under the assumptions of Theorem 3.4, for every $w \in W$ and $pq > 1$, system* (3.14) *has a unique mild solution $x \in C([0,1],X)$ which satisfies the following integral equation*

$$x(s) = \mathcal{T}_k(s)z_0 + \int_0^s (s-\theta)^{q-1}\mathcal{S}_k(s-\theta)kf(k\theta, x(\theta), B(k\theta)u(\theta))d\theta, \tag{3.16}$$

where

$$\mathcal{T}_k(s) = \int_0^\infty \xi_q(\theta)T_{k^q}(s^q\theta)d\theta, \qquad \mathcal{S}_k(s) = q\int_0^\infty \theta\xi_q(\theta)T_{k^q}(s^q\theta)d\theta, \tag{3.17}$$

and $\{T_{k^q}(t), t \ge 0\}$ is a C_0-semigroup generated by the infinitesimal generator k^qA.

By Lemmas 2.5 and 3.3, it is not difficult to verify the following result.

Lemma 3.7. *The family of solution operators $\mathcal{T}_k$ and $\mathcal{S}_k$ given by* (3.17) *has the following properties.*

(i) *For any $x \in X$, $t \ge 0$, there exists a constant $C_{k^q} > 0$ such that*

$$\|\mathcal{T}_k(t)x\| \le C_{k^q}\|x\|, \qquad \|\mathcal{S}_k(t)x\| \le \frac{qC_{k^q}}{\Gamma(1+q)}\|x\|. \tag{3.18}$$

(ii) *$\{\mathcal{T}_k(t), t \ge 0\}$ and $\{\mathcal{S}_k(t), t \ge 0\}$ are also strongly continuous.*

(iii) *If $k_n^q \to k_\varepsilon^q$ in $[0,\widehat{T}]$ as $n \to \infty$, then for arbitrary $x \in X$ and $t \ge 0$*

$$\begin{aligned} \mathcal{T}_{k_n^q}(t) &\xrightarrow{s} \mathcal{T}_{k_\varepsilon^q}(t), \quad \text{as } n \longrightarrow \infty, \\ \mathcal{S}_{k_n^q}(t) &\xrightarrow{s} \mathcal{S}_{k_\varepsilon^q}(t), \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{3.19}$$

uniformly in t on some closed interval of $[0,\widehat{T}]$ in the strong operator topology sense.

For system (3.14), we turn to consider the following Meyer problem.

Meyer Problem (P_ε)

Minimize the cost functional given by

$$J_\varepsilon(w) = \frac{1}{2\varepsilon}\|x(w)(1) - z_1\|^2 + k \tag{3.20}$$

over W, where $x(w)$ is the mild solution of (3.14) corresponding to control w, that is, find a control $w_\varepsilon = (u_\varepsilon, k_\varepsilon)$ such that the cost functional $J_\varepsilon(w)$ attains its minimum on W at w_ε.

4. Existence of Optimal Controls for Meyer Problem $(\mathbf{P}_\varepsilon)$

In this section, we discuss the existence of optimal controls for Meyer problem (P_ε).

We show that Meyer problem (P_ε) has a solution $w_\varepsilon = (u_\varepsilon, k_\varepsilon)$ for fixed $\varepsilon > 0$.

Theorem 4.1. *Under the assumptions of Theorem 3.6. Meyer problem (P_ε) has a solution.*

Proof. Let $\varepsilon > 0$ be fixed. Since $J_\varepsilon(w) \geq 0$, there exists $\inf\{J_\varepsilon(w), w \in W\}$. Denote $m_\varepsilon \equiv \inf\{J_\varepsilon(w), w \in W\}$ and choose $\{w_n\} \subseteq W$ such that $J_\varepsilon(w_n) \to m_\varepsilon$ where $w_n = (u_n, k_n) \in W = V_{\mathrm{ad}} \times [0, \widehat{T}]$. By assumption [U], there exists a subsequence $\{u_n\} \subseteq V_{\mathrm{ad}}$ such that $u_n \overset{w}{\to} u_\varepsilon$ in V_{ad} as $n \to \infty$, and V_{ad} is closed and convex, thanks to Mazur Lemma, $u_\varepsilon \in V_{\mathrm{ad}}$. By assumption [B], we have

$$Bu_n \overset{s}{\longrightarrow} Bu_\varepsilon, \quad \text{in } L^p([0,1], \mathrm{X}), \text{ as } n \longrightarrow \infty. \tag{4.1}$$

Since $k_n(k_n^q)$ is bounded and $k_n(k_n^q) > 0$, there also exists a subsequence $\{k_n\}(\{k_n^q\})$ denoted by $\{k_n\}(\{k_n^q\}) \subseteq [0, \widehat{T}]$ again, such that

$$k_n\left(k_n^q\right) \longrightarrow k_\varepsilon\left(k_\varepsilon^q\right), \quad \text{in } \left[0, \widehat{T}\right], \text{ as } n \longrightarrow \infty. \tag{4.2}$$

Let x_n and x_ε be the mild solutions of system (3.14) corresponding to $w_n = (u_n, k_n) \in W$ and $w_\varepsilon = (u_\varepsilon, k_\varepsilon) \in W$, respectively. Then, we have

$$\begin{aligned} x_n(s) &= \mathcal{T}_n(s)z_0 + \int_0^s (s-\theta)^{q-1}\mathcal{S}_n(s-\theta)k_n^q F_n(\theta)d\theta, \\ x_\varepsilon(s) &= \mathcal{T}_\varepsilon(s)z_0 + \int_0^s (s-\theta)^{q-1}\mathcal{S}_\varepsilon(s-\theta)k_\varepsilon^q F_\varepsilon(\theta)d\theta, \end{aligned} \tag{4.3}$$

where

$$
\begin{aligned}
\mathcal{T}_n(\cdot) &\equiv \int_0^\infty \xi_q(\theta) T_{k_n^q}(\cdot^q\theta)\,d\theta, \\
\mathcal{S}_n(\cdot) &\equiv q\int_0^\infty \theta\xi_q(\theta) T_{k_n^q}(\cdot^q\theta)\,d\theta, \\
F_n(\cdot) &\equiv f(k_n\cdot, x_n(\cdot), B(k_n\cdot)u_n(\cdot)), \\
\mathcal{T}_\varepsilon(\cdot) &\equiv \int_0^\infty \xi_q(\theta) T_{k_\varepsilon^q}(\cdot^q\theta)\,d\theta, \\
\mathcal{S}_\varepsilon(\cdot) &\equiv q\int_0^\infty \theta\xi_q(\theta) T_{k_\varepsilon^q}(\cdot^q\theta)\,d\theta, \\
F_\varepsilon(\cdot) &\equiv f(k_\varepsilon\cdot, x_\varepsilon(\cdot), B(k_\varepsilon\cdot)u_\varepsilon(\cdot)).
\end{aligned}
\tag{4.4}
$$

By Lemma 3.7, assumptions [F], [B], [U], and singular version Gronwall Lemma, it is easy to verify that there exists a constant $\rho > 0$ such that

$$
\|x_\varepsilon\|_{C([0,1],X)} \le \rho, \qquad \|x_n\|_{C([0,1],X)} \le \rho. \tag{4.5}
$$

Further, there exists a constant $M_\varepsilon > 0$ such that

$$
\|F_\varepsilon\|_{C([0,1],X)} \le M_\varepsilon\left(1 + \rho + \|B\|_\infty \max_{t\in[0,1]}\{\|u(t)\|\}\right). \tag{4.6}
$$

Denote

$$
\begin{aligned}
R_1 &= \|\mathcal{T}_n(s)z_0 - \mathcal{T}_\varepsilon(s)z_0\|, \\
R_2 &= \left\|\int_0^s (s-\theta)^{q-1}\mathcal{S}_n(s-\theta)k_n^q F_n(\theta)\,d\theta - \int_0^s (s-\theta)^{q-1}\mathcal{S}_n(s-\theta)k_n^q F_n^\varepsilon(\theta)\,d\theta\right\|, \\
R_3 &= \left\|\int_0^s (s-\theta)^{q-1}\mathcal{S}_n(s-\theta)k_n^q F_n^\varepsilon(\theta)\,d\theta - \int_0^s (s-\theta)^{q-1}\mathcal{S}_\varepsilon(s-\theta)k_\varepsilon^q F_\varepsilon(\theta)\,d\theta\right\|,
\end{aligned}
\tag{4.7}
$$

where

$$
F_n^\varepsilon(\theta) \equiv f(k_n\theta, x_\varepsilon(\theta), B(k_\varepsilon\theta)u_\varepsilon(\theta)). \tag{4.8}
$$

By assumption [F],

$$\begin{aligned}
R_2 &\le \frac{qC_{k_n^q}k_n^q}{\Gamma(1+q)}\int_0^s (s-\theta)^{q-1}\|F_n(\theta)-F_n^{\varepsilon}(\theta)\|d\theta \\
&\le \frac{qC_{k_n^q}k_n^q L(\rho)}{\Gamma(1+q)}\int_0^s (s-\theta)^{q-1}\|x_n(\theta)-x_{\varepsilon}(\theta)\|d\theta \\
&\quad + \frac{qC_{k_n^q}k_n^q L(\rho)}{\Gamma(1+q)}\int_0^s (s-\theta)^{q-1}\|B(k_n\theta)u_n(\theta)-B(k_{\varepsilon}\theta)u_{\varepsilon}(\theta)\|d\theta \\
&\le R_{21}+R_{22}+R_{23},
\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}
M_{k_n^q} &\equiv \frac{qC_{k_n^q}k_n^q L(\rho)}{\Gamma(1+q)}, \\
R_{21} &\equiv M_{k_n^q}\int_0^s (s-\theta)^{q-1}\|x_n(\theta)-x_{\varepsilon}(\theta)\|d\theta, \\
R_{22} &\equiv M_{k_n^q}\int_0^s (s-\theta)^{q-1}\|B(k_n\theta)u_{\varepsilon}(\theta)-B(k_{\varepsilon}\theta)u_{\varepsilon}(\theta)\|d\theta, \\
R_{23} &\equiv M_{k_n^q}\int_0^s (s-\theta)^{q-1}\|B(k_n\theta)u_n(\theta)-B(k_n\theta)u_{\varepsilon}(\theta)\|d\theta, \\
R_3 &\le \int_0^s (s-\theta)^{q-1}\left\|k_n^q\mathcal{S}_n(s-\theta)F_n^{\varepsilon}(\theta)-k_{\varepsilon}^q\mathcal{S}_n(s-\theta)F_{\varepsilon}(\theta)\right\|d\theta \\
&\quad + k_{\varepsilon}^q\int_0^s (s-\theta)^{q-1}\|\mathcal{S}_n(s-\theta)F_{\varepsilon}(\theta)-\mathcal{S}_{\varepsilon}(s-\theta)F_{\varepsilon}(\theta)\|d\theta \\
&\le R_{31}+R_{32}+R_{33},
\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
R_{31} &\equiv M_{k_n^q}k_n^q\int_0^s (s-\theta)^{q-1}\|F_n^{\varepsilon}(\theta)-F_{\varepsilon}(\theta)\|d\theta, \\
R_{32} &\equiv M_{k_n^q}\int_0^s (s-\theta)^{q-1}\left|k_n^q-k_{\varepsilon}^q\right|\|F_{\varepsilon}(\theta)\|d\theta, \\
R_{33} &\equiv k_{\varepsilon}^q M_{\varepsilon}(1+\rho)\int_0^s (s-\theta)^{q-1}\|\mathcal{S}_n(s-\theta)-\mathcal{S}_{\varepsilon}(s-\theta)\|d\theta.
\end{aligned} \tag{4.11}$$

Note that Lemma 3.7 and (4.1), combining Hölder inequality with Lebesgue dominated convergence theorem, one can verify $R_1 \to 0$, $R_{23} \to 0$, $R_{31} \to 0$ and $R_{33} \to 0$ as $n \to \infty$ immediately. Since $k_n(k_n^q) \to k_\varepsilon(k_\varepsilon^q)$ as $n \to \infty$, $\|F_\varepsilon\|_{C([0,1],X)}$ and $\|u_\varepsilon(t)\|_E$ are bounded, $R_{22} \to 0$, $R_{32} \to 0$ as $n \to \infty$.

Then, we obtain that

$$\begin{aligned} \|x_n(s) - x_\varepsilon(s)\| &\le R_1 + R_2 + R_3 \\ &\le \sigma_\varepsilon + M_{k_n^q} \int_0^s (s-\theta)^{q-1} \|x_n(\theta) - x_\varepsilon(\theta)\| d\theta, \end{aligned} \tag{4.12}$$

where

$$\sigma_\varepsilon = R_1 + R_{22} + R_{23} + R_{31} + R_{32} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{4.13}$$

By singular version Gronwall Lemma again, we obtain

$$x_n \xrightarrow{s} x_\varepsilon, \quad \text{in } C([0,1], X), \text{ as } n \longrightarrow \infty. \tag{4.14}$$

Thus, there exists a unique control $w_\varepsilon = (u_\varepsilon, k_\varepsilon) \in W$ such that

$$m_\varepsilon = \lim_{n \to \infty} J_\varepsilon(w_n) = J_\varepsilon(w_\varepsilon) \ge m_\varepsilon. \tag{4.15}$$

This shows that $J_\varepsilon(w)$ attains its minimum at $w_\varepsilon \in W$, and hence x_ε is the solution of system (3.14) corresponding to control w_ε. □

5. Meyer Approximation Process of Time Optimal Control

In this section, we display the Meyer approximation process of the time optimal control problem (P).

For the sake of convenience, we subdivide the approximation process into several steps.

Step 1. By Theorem 4.1, there exists a $w_\varepsilon = (u_\varepsilon, k_\varepsilon) \in W$ such that $J_\varepsilon(w)$ attains its minimum at $w_\varepsilon \in W$, that is,

$$J_\varepsilon(w_\varepsilon) = \frac{1}{2\varepsilon} \|x(w_\varepsilon)(1) - z_1\|^2 + k_\varepsilon = \inf_{w \in W} J_\varepsilon(w). \tag{5.1}$$

By controllability of problem (P), $V_0 \neq \emptyset$. Take $\tilde{v} \in V_0$ and let $\tau(\tilde{v}) = \tilde{\tau} < +\infty$ then $z(\tilde{v})(\tilde{\tau}) = z_1$. Define $\tilde{u}(s) = \tilde{v}(\tilde{\tau}s)$, $0 \le s \le 1$ and $\tilde{w} = (\tilde{u}, \tilde{\tau}) \in W$. Then $\tilde{x}(\cdot) = z(\tilde{v})(\tilde{\tau}\cdot)$ is the mild solution of system (3.14) corresponding to control $\tilde{w} = (\tilde{u}, \tilde{\tau}) \in W$. Of course, we have $\tilde{x}(1) = z_1$.

For any $\varepsilon > 0$, submitting $\widetilde{w}$ to J_ε, we have

$$J_\varepsilon(\widetilde{w}) = \widetilde{\tau} \geq J_\varepsilon(w_\varepsilon) = \frac{1}{2\varepsilon}\|x(w_\varepsilon)(1) - z_1\|^2 + k_\varepsilon. \tag{5.2}$$

This inequality implies that

$$\begin{gathered} 0 \leq k_\varepsilon \leq \widetilde{\tau}, \\ \|x(w_\varepsilon)(1) - z_1\|^2 \leq 2\varepsilon\widetilde{\tau}, \quad \text{hold for all } \varepsilon > 0. \end{gathered} \tag{5.3}$$

We can choose a subsequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\begin{gathered} k_{\varepsilon_n}^q \longrightarrow (k^q)^0, \quad \text{in } \left[0, \widehat{T}\right], \\ k_{\varepsilon_n} \longrightarrow k^0, \quad \text{in } \left[0, \widehat{T}\right], \\ x(w_{\varepsilon_n})(1) \equiv x_{\varepsilon_n}(1) \longrightarrow z_1, \quad \text{in X, as } n \longrightarrow \infty, \\ u_{\varepsilon_n} \xrightarrow{w} u^0, \quad \text{in } V_{\text{ad}}, \qquad w_{\varepsilon_n} = (u_{\varepsilon_n}, k_{\varepsilon_n}) \in W. \end{gathered} \tag{5.4}$$

Since V_{ad} is closed and convex, thanks to Mazur Lemma again, $u^0 \in V_{\text{ad}}$.

Further, by assumption [B], we obtain

$$\begin{gathered} k_{\varepsilon_n}^q \longrightarrow (k^q)^0, \quad \text{in } \left[0, \widehat{T}\right], \\ k_{\varepsilon_n} \longrightarrow k^0, \quad \text{in } \left[0, \widehat{T}\right], \\ x(w_{\varepsilon_n})(1) \equiv x_{\varepsilon_n}(1) \longrightarrow z_1, \quad \text{in X, as } n \longrightarrow \infty, \\ Bu_{\varepsilon_n} \xrightarrow{s} Bu^0, \quad \text{in } L^p([0,1], X). \end{gathered} \tag{5.5}$$

Step 2. Let x_{ε_n} and x^0 be the mild solutions of system (3.14) corresponding to $w_{\varepsilon_n} = (u_{\varepsilon_n}, k_{\varepsilon_n}) \in W$ and $w^0 = (u^0, k^0) \in W$, respectively. Then, we have

$$\begin{gathered} x_{\varepsilon_n}(s) = \mathcal{T}_{\varepsilon_n}(s)z_0 + \int_0^s (s-\theta)^{q-1}\mathcal{S}_{\varepsilon_n}(s-\theta)k_{\varepsilon_n}^q F_{\varepsilon_n}(\theta)d\theta, \\ x^0(s) = \mathcal{T}_0(s)z_0 + \int_0^s (s-\theta)^{q-1}\mathcal{S}_0(s-\theta)(k^q)^0 F^0(\theta)d\theta, \end{gathered} \tag{5.6}$$

where

$$
\begin{aligned}
\mathcal{T}_{\varepsilon_n}(\cdot) &\equiv \int_0^\infty \xi_q(\theta) T_{k_{\varepsilon_n}^q}(\cdot^q\theta)\,d\theta,\\
\mathcal{S}_{\varepsilon_n}(\cdot) &\equiv q\int_0^\infty \theta\xi_q(\theta) T_{k_{\varepsilon_n}^q}(\cdot^q\theta)\,d\theta,\\
F_{\varepsilon_n}(\cdot) &\equiv f(k_{\varepsilon_n}\cdot, x_{\varepsilon_n}(\cdot), B(k_{\varepsilon_n}\cdot)u_{\varepsilon_n}(\cdot)),\\
\mathcal{T}_0(\cdot) &\equiv \int_0^\infty \xi_q(\theta) T_{(k^0)^q}(\cdot^q\theta)\,d\theta,\\
\mathcal{S}_0(\cdot) &\equiv q\int_0^\infty \theta\xi_q(\theta) T_{(k^0)^q}(\cdot^q\theta)\,d\theta,\\
F^0(\cdot) &\equiv f\Big(k^0\cdot, x^0(\cdot), B\Big(k^0\cdot\Big)u^0(\cdot)\Big).
\end{aligned}
\tag{5.7}
$$

Recalling (5.5) and the process in Theorem 4.1, after some calculation, using the singular version Gronwall Lemma again, we also obtain

$$
x_{\varepsilon_n} \overset{s}{\longrightarrow} x^0, \quad \text{in } C([0,1],X), \text{ as } n \longrightarrow \infty. \tag{5.8}
$$

Step 3. It follows from Steps 1 and 2,

$$
\begin{gathered}
\|x_{\varepsilon_n}(1) - z_1\| \le \sqrt{2\varepsilon_n\tilde{\tau}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,\\
\Big\|x_{\varepsilon_n}(1) - x^0(1)\Big\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,\\
\Big\|x^0(1) - z_1\Big\| \le \|x_{\varepsilon_n}(1) - z_1\| + \Big\|x_{\varepsilon_n}(1) - x^0(1)\Big\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,
\end{gathered}
\tag{5.9}
$$

that $x^0(1) = z_1$. It is very clear that $k^0 \neq 0$ unless $z_0 = z_1$. This implies that $k^0 > 0$.

Define $v^0(\cdot) = u^0(\cdot/k^0)$. In fact, $z^0(\cdot) = x^0(\cdot/k^0)$ is the mild solution of system (3.1) corresponding to control $v^0 \in V_0$, then $z^0(k^0) = x^0(1) = z_1$ and $\tau(v^0) = k^0 > 0$. By the definition of $\tau^* = \inf\{\tau(v) \ge 0 \mid v \in V_0\}$, we have $k^0 \ge \tau^*$.

For any $v \in V_0$,

$$
\tau(v) \ge J_\varepsilon(w_\varepsilon) = \frac{1}{2\varepsilon}\|x(w_\varepsilon)(1) - z_1\|^2 + k_\varepsilon. \tag{5.10}
$$

Thus, $\tau(v) \ge k_\varepsilon$. Further, $\tau(v) \ge k_{\varepsilon_n}$ for all $\varepsilon_n > 0$.

Since k^0 is the limit of k_{ε_n} as $n \to \infty$, $\tau(v) \ge \tau(v^0) = k^0$ for all $v \in V_0$. Hence, $k^0 \le \tau^*$. Thus, $0 < \tau(v^0) = k^0 = \tau^*$. This implies that v^0 is an optimal control of Problem (P) and $k^0 > 0$ is just optimal time.

Remark 5.1. Under the above assumptions, there exists a sequence of Meyer problems $(\mathrm{P}_{\varepsilon_n})$ whose corresponding sequence of optimal controls $\{w_{\varepsilon_n}\} \in W$ can successive approximation the time optimal control problem (P) in some sense. In other words, by limiting process, the sequence of the optimal controls $\{w_{\varepsilon_n}\} \in W$ can be used to find the solution of time optimal control problem (P).

As a result, we obtain the existence result of time optimal control for system (3.1) directly.

Theorem 5.2. *Under the assumptions of Theorem 4.1. The time optimal control problem* (P) *has a solution, that is, there exists an optimal control* $v^* \in V_0 \subset V_{ad}$ *such that*

$$\tau(v^*) = \tau^* = \inf\{\tau(v) \geq 0 \mid v \in V_0\}. \tag{5.11}$$

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