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Research Article

Existence Results for Nonlinear Fractional Difference Equation

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This paper is concerned with the initial value problem to a nonlinear fractional difference equation with the Caputo like difference operator. By means of some fixed point theorems, global and local existence results of solutions are obtained. An example is also provided to illustrate our main result.

1. Introduction

This paper deals with the existence of solutions for nonlinear fractional difference equations

$$\begin{gathered}\Delta_*^\alpha x(t) = f(t+\alpha-1, x(t+\alpha-1)), \quad t \in \mathbb{N}_{1-\alpha},\ 0<\alpha\le 1,\\ x(0)=x_0,\end{gathered} \tag{1.1}$$

where Δ_*^α is a Caputo like discrete fractional difference, $f : [0,+\infty)\times X \to X$ is continuous in t and X. $(X, \|\cdot\|)$ is a real Banach space with the norm $\|x\| = \sup\{\|x(t)\|, t\in N\}$, $\mathbb{N}_{1-\alpha} = \{1-\alpha, 2-\alpha, \dots\}$.

Fractional differential equation has received increasing attention during recent years since fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [1]. However, there are few literature to develop the theory of the analogues fractional finite difference equation [2–6]. Atici and Eloe [2] developed the commutativity properties of the fractional sum and the fractional difference operators, and discussed the uniqueness of a solution for a nonlinear fractional difference equation with the Riemann-Liouville like discrete fractional difference operator. To the best of our knowledge, this is a pioneering work on discussing initial value problems (IVP for short) in discrete fractional calculus. Anastassiou [4] defined a Caputo like discrete fractional difference and compared it to the Riemann-Liouville fractional discrete analog.

For convenience of numerical calculations, the fractional differential equation is generally discretized to corresponding difference one which makes that the research about fractional difference equations becomes important. Following the definition of Caputo like difference operator defined in [4], here we investigate the existence and uniqueness of solutions for the IVP (1.1). A merit of this IVP with Caputo like difference operator is that its initial condition is the same form as one of the integer-order difference equation.

2. Preliminaries and Lemmas

We start with some necessary definitions from discrete fractional calculus theory and preliminary results so that this paper is self-contained.

Definition 2.1 (see [2, 3]). Let $\nu > 0$. The νth fractional sum f is defined by

$$\Delta^{-\nu} f(t,a) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s). \tag{2.1}$$

Here f is defined for $s = a \bmod (1)$ and $\Delta^{-\nu} f$ is defined for $t = (a+\nu) \bmod (1)$; in particular, $\Delta^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$, where $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$. In addition, $t^{(\nu)} = \Gamma(t+1)/\Gamma(t-\nu+1)$. Atici and Eloe [2] pointed out that this definition of the νth fractional sum is the development of the theory of the fractional calculus on time scales [7].

Definition 2.2 (see [4]). Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$, $\lceil \cdot \rceil$ ceiling of number. Set $\nu = m - \mu$. The μth fractional Caputo like difference is defined as

$$\Delta_*^{\mu} f(t) = \Delta^{-\nu}\left(\Delta^m f(t)\right) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} \left(\Delta^m f\right)(s), \quad \forall t \in N_{a+\nu}. \tag{2.2}$$

Here Δ^m is the mth order forward difference operator

$$\left(\Delta^m f\right)(s) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(s+k). \tag{2.3}$$

Theorem 2.3 (see [4]). *For $\mu > 0$, μ noninteger, $m = \lceil \mu \rceil$, $\nu = m - \mu$, it holds*

$$f(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) + \frac{1}{\Gamma(\mu)} \sum_{s=a+\nu}^{t-\mu} (t-s-1)^{(\mu-1)} \Delta_*^{\mu} f(s), \tag{2.4}$$

where f is defined on $\mathbb{N}_a$ with $a \in N$.

In particular, when $0<\mu<1$ and $a=0$, we have

$$f(t)=f(0)+\frac{1}{\Gamma(\mu)}\sum_{s=1-\mu}^{t-\mu}(t-s-1)^{(\mu-1)}\Delta_*^{\mu}f(s). \tag{2.5}$$

Lemma 2.4. *A solution $x(t):N\to X$ is a solution of the IVP* (1.1) *if and only if $x(t)$ is a solution of the the following fractional Taylor's difference formula:*

$$\begin{gathered}x(t)=x_0+\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s+\alpha-1,x(s+\alpha-1)),\quad 0<\alpha\leq 1,\\ x(0)=x_0.\end{gathered} \tag{2.6}$$

Proof. Suppose that $x(t)$ for $t\in N$ is a solution of (1.1), that is $\Delta_*^{\alpha}x(t)=f(t+\alpha-1,x(t+\alpha-1))$ for $t\in\mathbb{N}_{1-\alpha}$, then we can obtain (2.6) according to Theorem 2.3.

Conversely, we assume that $x(t)$ is a solution of (2.6), then

$$x(t)=x(0)+\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s+\alpha-1,x(s+\alpha-1)). \tag{2.7}$$

On the other hand, Theorem 2.3 yields that

$$x(t)=x(0)+\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\Delta_*^{\alpha}x(s). \tag{2.8}$$

Comparing with the above two equations, it is obtained that

$$\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\left[\Delta_*^{\alpha}x(s)-f(s+\alpha-1,x(s+\alpha-1))\right]=0. \tag{2.9}$$

Let $t=1,2,\ldots$, respectively, we have that $\Delta_*^{\alpha}x(t)=f(t+\alpha-1,x(t+\alpha-1))$ for $t\in\mathbb{N}_{1-\alpha}$, which implies that $x(t)$ is a solution of (1.1). □

Lemma 2.5. *One has*

$$\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}=\frac{\Gamma(t+\alpha)}{\alpha\Gamma(t)}. \tag{2.10}$$

Proof. For $x>k$, $x,k\in R$, $k>-1$, $x>-1$, we have

$$\frac{\Gamma(x+1)}{\Gamma(k+1)\Gamma(x-k+1)}=\frac{\Gamma(x+2)}{\Gamma(k+2)\Gamma(x-k+1)}-\frac{\Gamma(x+1)}{\Gamma(k+2)\Gamma(x-k)}, \tag{2.11}$$

that is,

$$\frac{\Gamma(x+1)}{\Gamma(x-k+1)} = \frac{1}{k+1}\left[\frac{\Gamma(x+2)}{\Gamma(x-k+1)} - \frac{\Gamma(x+1)}{\Gamma(x-k)}\right]. \tag{2.12}$$

Then

$$\begin{aligned}
\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)} &= \sum_{s=1-\alpha}^{t-\alpha}\frac{\Gamma(t-s)}{\Gamma(t-s-\alpha+1)} \\
&= \sum_{s=1-\alpha}^{t-\alpha-1}\frac{\Gamma(t-s)}{\Gamma(t-s-\alpha+1)} + \Gamma(\alpha) \\
&= \sum_{s=1-\alpha}^{t-\alpha-1}\frac{1}{\alpha}\left[\frac{\Gamma(t-s+1)}{\Gamma(t-s-\alpha+1)} - \frac{\Gamma(t-s)}{\Gamma(t-s-\alpha)}\right] + \Gamma(\alpha) \\
&= \frac{1}{\alpha}\left[\frac{\Gamma(t+\alpha)}{\Gamma(t)} - \frac{\Gamma(\alpha+1)}{\Gamma(1)}\right] + \Gamma(\alpha) \\
&= \frac{\Gamma(t+\alpha)}{\alpha\Gamma(t)}.
\end{aligned} \tag{2.13}$$

□

Lemma 2.6 (see [2]). *Let $\nu \neq 1$ and assume $\mu+\nu+1$ is not a nonpositive integer. Then*

$$\Delta^{-\nu}t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}t^{(\mu+\nu)}. \tag{2.14}$$

In particular, $\Delta^{-\nu}a = a\Delta^{-\nu}(t+\alpha-1)^{(0)} = (a/\Gamma(\nu+1))(t+\alpha-1)^{(\nu)}$, where a is a constant.

The following fixed point theorems will be used in the text.

Theorem 2.7 (Leray-Schauder alternative theorem [8]). *Let E be a Banach space with $C \subseteq E$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $A : \overline{U} \to C$ is a continuous, compact map. Then either*

(1) *A has a fixed point in $\overline{U}$; or*

(2) *there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda u$.*

Theorem 2.8 (Schauder fixed point theorem [9]). *If U is a closed, bounded convex subset of a Banach space X and $T : U \to U$ is completely continuous, then T has a fixed point in U.*

Theorem 2.9 (Ascoli-Arzela theorem [10]). *Let X be a Banach space, and $S = \{s(t)\}$ is a function family of continuous mappings $s : [a,b] \to X$. If S is uniformly bounded and equicontinuous, and for any $t^* \in [a,b]$, the set $\{s(t^*)\}$ is relatively compact, then there exists a uniformly convergent function sequence $\{s_n(t)\}$ $(n = 1,2,\ldots, t \in [a,b])$ in S.*

Lemma 2.10 (Mazur Lemma [11]). *If $\widetilde{S}$ is a compact subset of Banach space X, then its convex closure $\overline{\text{conv}}\widetilde{S}$ is compact.*

3. Local Existence and Uniqueness

Set $N_K = \{0, 1, \ldots, K\}$, where $K \in N$.

Theorem 3.1. *Assume $f : [0, K] \times X \to X$ is locally Lipschitz continuous (with constant L) on X, then the IVP* (1.1) *has a unique solution $x(t)$ on $t \in N$ provided that*

$$\frac{L\Gamma(K+\alpha)}{\Gamma(\alpha+1)\Gamma(K)} < 1. \tag{3.1}$$

Proof. Define a mapping $T : X|_{t \in N_K} \to X|_{t \in N_K}$ by

$$(Tx)(t) = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha-1, x(s+\alpha-1)) \tag{3.2}$$

for $t \in N_K$. Now we show that T is contraction. For any $x, y \in X|_{t \in N_K}$ it follows that

$$\begin{aligned}
&\|(Tx)(t) - (Ty)(t)\| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \|f(s+\alpha-1, x(s+\alpha-1)) - f(s+\alpha-1, y(s+\alpha-1))\| \\
&\quad \le \frac{L}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \|x-y\| \\
&\quad \le \frac{L\Gamma(t+\alpha)}{\alpha\Gamma(\alpha)\Gamma(t)} \|x-y\| \\
&\quad \le \frac{L(K+\alpha)\cdots(t+\alpha)\Gamma(t+\alpha)}{\alpha\Gamma(\alpha)(K-1)\cdots t\Gamma(t)} \|x-y\| \\
&\quad \le \frac{L\Gamma(K+\alpha)}{\Gamma(\alpha+1)\Gamma(K)} \|x-y\|.
\end{aligned} \tag{3.3}$$

By applying Banach contraction principle, T has a fixed point $x^*(t)$ which is a unique solution of the IVP (1.1). □

Theorem 3.2. *Assume that there exist $L_1, L_2 > 0$ such that $\|f(t,x)\| \le L_1\|x\| + L_2$ for $x \in X$, and the set $\widetilde{S} = \{(t-s-1)^{(\alpha-1)} f(s+\alpha-1, x(s+\alpha-1)) : x \in X, s \in \{1-\alpha, \ldots, t-\alpha\}\}$ is relatively compact for every $t \in N_K$, then there exists at least one solution $x(t)$ of the IVP* (1.1) *on $t \in N_K$ provided that*

$$\frac{L_1\Gamma(K+\alpha)}{\Gamma(\alpha+1)\Gamma(K)} < 1. \tag{3.4}$$

Proof. Let T be the operator defined by (3.2), we define the set E as follows:

$$E = \{x(t) : \|x(t)\| \le M+1, t \in N_K\}, \tag{3.5}$$

where

$$M = \frac{\Gamma(\alpha+1)\Gamma(K)\|x_0\| + L_2\Gamma(K+\alpha)}{\Gamma(\alpha+1)\Gamma(K) - L_1\Gamma(K+\alpha)}. \tag{3.6}$$

Assume that there exist $x \in E$ and $\lambda \in (0,1)$ such that $x = \lambda Tx$. We claim that $\|x\| \neq M+1$. In fact,

$$x(t) = \lambda x_0 + \frac{\lambda}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s+\alpha-1, x(s+\alpha-1)), \tag{3.7}$$

then

$$\begin{aligned}
\|x(t)\| &\leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} \left\| f(s+\alpha-1, x(s+\alpha-1)) \right\| \\
&\leq \|x_0\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} (L_1\|x\| + L_2) \\
&\leq \|x_0\| + \frac{L_1\Gamma(K+\alpha)}{\Gamma(\alpha+1)\Gamma(K)}\|x\| + \frac{L_2\Gamma(K+\alpha)}{\Gamma(\alpha+1)\Gamma(K)}.
\end{aligned} \tag{3.8}$$

We have

$$\|x\| \leq \|x_0\| + \frac{L_1\Gamma(K+\alpha)}{\Gamma(\alpha+1)\Gamma(K)}\|x\| + \frac{L_2\Gamma(K+\alpha)}{\Gamma(\alpha+1)\Gamma(K)}. \tag{3.9}$$

that is,

$$\|x\| \leq \frac{\Gamma(\alpha+1)\Gamma(K)\|x_0\| + L_2\Gamma(K+\alpha)}{\Gamma(\alpha+1)\Gamma(K) - L_1\Gamma(K+\alpha)} = M, \tag{3.10}$$

which implies that $\|x\| \neq M+1$.

The operator T is continuous because that f is continuous. In the following, we prove that the operator T is also completely continuous in E. For any $\varepsilon > 0$, there exist $t_1, t_2 \in N_K$ $(t_1 > t_2)$ such that

$$\left| \frac{(t_1+\alpha-1)\cdots(t_2+\alpha)}{(t_1-1)\cdots t_2} - 1 \right| < \frac{\Gamma(K)\Gamma(\alpha)}{(L_1M+L_2)\Gamma(K+\alpha)}\varepsilon, \tag{3.11}$$

then we have

$$
\begin{aligned}
&\|(Tx)(t_1)-(Tx)(t_2)\| \\
&\quad= \frac{1}{\Gamma(\alpha)}\left\|\sum_{s=1-\alpha}^{t_1-\alpha}(t_1-s-1)^{(\alpha-1)}f(s+\alpha-1,x(s+\alpha-1))\right.\\
&\qquad\qquad \left.-\sum_{s=1-\alpha}^{t_2-\alpha}(t_2-s-1)^{(\alpha-1)}f(s+\alpha-1,x(s+\alpha-1))\right\|\\
&\quad\le \frac{1}{\Gamma(\alpha)}\left\|\sum_{s=1-\alpha}^{t_2-\alpha}\left[(t_1-s-1)^{(\alpha-1)}-(t_2-s-1)^{(\alpha-1)}\right]f(s+\alpha-1,x(s+\alpha-1))\right\|\\
&\qquad+\frac{1}{\Gamma(\alpha)}\left\|\sum_{s=t_2-\alpha+1}^{t_1-\alpha}(t_1-s-1)^{(\alpha-1)}f(s+\alpha-1,x(s+\alpha-1))\right\|\\
&\quad\le \frac{L_1M+L_2}{\Gamma(\alpha)}\left[\sum_{s=1-\alpha}^{t_2-\alpha}(t_1-s-1)^{(\alpha-1)}-\sum_{s=1-\alpha}^{t_2-\alpha}(t_2-s-1)^{(\alpha-1)}\right]\\
&\qquad+\frac{L_1M+L_2}{\Gamma(\alpha)}\sum_{s=t_2-\alpha+1}^{t_1-\alpha}(t_1-s-1)^{(\alpha-1)}\\
&\quad= \frac{L_1M+L_2}{\alpha\Gamma(\alpha)}\left[\frac{\Gamma(t_1+\alpha)}{\Gamma(t_1)}-\frac{\Gamma(t_1-t_2+\alpha)}{\Gamma(t_1-t_2)}-\frac{\Gamma(t_2+\alpha)}{\Gamma(t_2)}+\frac{\Gamma(t_1-t_2+\alpha)}{\Gamma(t_1-t_2)}\right]\\
&\quad= \frac{L_1M+L_2}{\alpha\Gamma(\alpha)}\left[\frac{\Gamma(t_1+\alpha)}{\Gamma(t_1)}-\frac{\Gamma(t_2+\alpha)}{\Gamma(t_2)}\right]\\
&\quad= \frac{L_1M+L_2}{\alpha\Gamma(\alpha)}\frac{\Gamma(t_2+\alpha)}{\Gamma(t_2)}\left[\frac{\Gamma(t_1+\alpha)\Gamma(t_2)}{\Gamma(t_1)\Gamma(t_2+\alpha)}-1\right]\\
&\quad\le \frac{L_1M+L_2}{\alpha\Gamma(\alpha)}\frac{\Gamma(K+\alpha)}{\Gamma(K)}\left[\frac{(t_1+\alpha-1)\cdots(t_2+\alpha)}{(t_1-1)\cdots t_2}-1\right]\\
&\quad<\varepsilon,
\end{aligned}
\tag{3.12}
$$

which means that the set TE is an equicontinuous set.

In view of Lemma 2.10 and the condition that $\widetilde{S}$ is relatively compact, we know that $\overline{\mathrm{conv}\widetilde{S}}$ is compact. For any $t^* \in N_K$,

$$
\begin{aligned}
(Tx_n)(t^*) &= x_0+\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t^*-\alpha}(t^*-s-1)^{(\alpha-1)}f(s+\alpha-1,x_n(s+\alpha-1))\\
&= x_0+\frac{1}{\Gamma(\alpha)}\xi_n,
\end{aligned}
\tag{3.13}
$$

where

$$
\xi_n=\sum_{s=1-\alpha}^{t^*-\alpha}(t^*-s-1)^{(\alpha-1)}f(s+\alpha-1,x_n(s+\alpha-1)).
\tag{3.14}
$$

Since $\overline{\text{conv}}\widetilde{S}$ is convex and compact, we know that $\xi_n \in \overline{\text{conv}}\widetilde{S}$. Hence, for any $t^* \in N_K$, the set $\{(Tx_n)(t^*)\}$ $(n = 1, 2, \ldots)$ is relatively compact. From Theorem 2.9, every $\{(Tx_n)(t)\}$ contains a uniformly convergent subsequence $\{(Tx_{n_k})(t)\}$ $(k = 1, 2, \ldots)$ on N_K which means that the set TE is relatively compact. Since TE is a bounded, equicontinuous and relatively compact set, we have that T is completely continuous.

Therefore, the Leray-Schauder fixed point theorem guarantees that T has a fixed point, which means that there exists at least one solution of the IVP (1.1) on $t \in N_K$. □

Corollary 3.3. *Assume that there exist $M > 0$ such that $\|f(t,x)\| \le M$ for any $t \in [0, K]$ and $x \in X$, and the set $\widetilde{S} = \{(t-s-1)^{(\alpha-1)} f(s+\alpha-1, x(s+\alpha-1)) : x \in X, s \in \{1-\alpha, \ldots, t-\alpha\}\}$ is relatively compact for every $t \in N_K$, then there exists at least one solution of the IVP* (1.1) *on $t \in N_K$.*

Proof. Let $L_1 = 0$, $L_2 = M$, we directly obtain the result by applying Theorem 3.2. □

Corollary 3.4. *Assume that the function f satisfies $\lim_{\|x\|\to 0}\|f(t,x)\|/\|x\| = 0$, and the set $\widetilde{S} = \{(t-s-1)^{(\alpha-1)} f(s+\alpha-1, x(s+\alpha-1)) : x \in X, s \in \{1-\alpha, \ldots, t-\alpha\}\}$ is relatively compact for every $t \in N_K$, then there exists at least one solution of the IVP* (1.1) *on $t \in N_K$.*

Proof. According to $\lim_{\|x\|\to 0}\|f(t,x)\|/\|x\| = 0$, for any $\varepsilon > 0$, there exists $P > 0$ such that $\|f(t,x)\| \le \varepsilon P$ for any $\|x\| \le P$. Let $M = \varepsilon P$, then Corollary 3.4 holds by Corollary 3.3. □

Corollary 3.5. *Assume the function $F : R^+ \to R^+$ is nondecreasing continuous and there exist L_3, $L_4 > 0$ such that*

$$\|f(t,x)\| \le L_3 F(\|x\|) + L_4, \quad t \in [0, K], \tag{3.15}$$

$$\lim_{u\to+\infty} F(u) < +\infty, \tag{3.16}$$

and the set $\widetilde{S} = \{(t-s-1)^{(\alpha-1)} f(s+\alpha-1, x(s+\alpha-1)) : x \in X, s \in \{1-\alpha, \ldots, t-\alpha\}\}$ is relatively compact for every $t \in N_K$, then there exists at least one solution of the IVP (1.1) *on $t \in N_K$.*

Proof. By inequity (3.16), there exist positive constants R_1, d_1, such that $F(u) \le R_1$, for all $u \ge d_1$. Let $R_2 = \sup_{0\le u\le d_1} F(u)$. Then we have $F(u) \le R_1 + R_2$, for all $u \ge 0$. Let $M = L_3(R_1+R_2)+L_4$, then Corollary 3.5 holds by Corollary 3.3. □

4. Global Uniqueness

Theorem 4.1. *Assume $f : [0, +\infty) \times X \to X$ is globally Lipschitz continuous (with constant L) on X, then the IVP* (1.1) *has a unique solution $x(t)$ provided that $0 < L < 1/(1+\alpha)$.*

Proof. For $t \in \{0, 1\}$, let $T : X \to X$ be the operator defined by (3.2). For any $x, y \in X$ it follows that

$$\begin{aligned}
&\|(Tx)(t)-(Ty)(t)\| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\|f(s+\alpha-1,x(s+\alpha-1))-f(s+\alpha-1,y(s+\alpha-1))\| \\
&\quad\le \frac{L}{\Gamma(\alpha)}(\alpha-1)^{(\alpha-1)}\|x-y\| \\
&\quad= \frac{L}{\Gamma(\alpha)}\Gamma(\alpha)\|x-y\| \\
&\quad= L\|x-y\|.
\end{aligned} \tag{4.1}$$

Since $L<1/(1+\alpha)<1$, by applying Banach contraction principle, T has a fixed point $x_1(t)$ which is a unique solution of the IVP (1.1) on $t\in\{0,1\}$.

Since $x_1(1)$ exists, for $t\in\{1,2\}$, we may define the following mapping $T_1 : X \to X$:

$$(T_1x)(t)=x_1(1)+\frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s+\alpha-1,x(s+\alpha-1)). \tag{4.2}$$

For any $x,y\in X$, $t\in\{1,2\}$, we have

$$\begin{aligned}
&\|(T_1x)(t)-(T_1y)(t)\| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\|f(s+\alpha-1,x(s+\alpha-1))-f(s+\alpha-1,x(s+\alpha-1))\| \\
&\quad\le \frac{L}{\Gamma(\alpha)}\sum_{s=1-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}\|x-y\| \\
&\quad\le \frac{L}{\Gamma(\alpha)}\left[\sum_{s=1-\alpha}^{2-\alpha}(2-s-1)^{(\alpha-1)}\right]\|x-y\| \\
&\quad= \frac{L}{\Gamma(\alpha)}\left[\alpha^{(\alpha-1)}+(\alpha-1)^{(\alpha-1)}\right]\|x-y\| \\
&\quad= \frac{L}{\Gamma(\alpha)}\left[\frac{\Gamma(\alpha+1)}{\Gamma(2)}+\frac{\Gamma(\alpha)}{\Gamma(1)}\right]\|x-y\| \\
&\quad= L(1+\alpha)\|x-y\|.
\end{aligned} \tag{4.3}$$

Since $L(1+\alpha)<1$, by applying Banach contraction principle, T_1 has a fixed point $x_2(t)$ which is a unique solution of the IVP (1.1) on $t\in\{1,2\}$.

In general, since $x_m(m)$ exists, we may define the operator T_m as follows

$$(T_mx)(t)=x_m(m)+\frac{1}{\Gamma(\alpha)}\sum_{s=m-\alpha}^{t-\alpha}(t-s-1)^{(\alpha-1)}f(s+\alpha-1,x(s+\alpha-1)) \tag{4.4}$$

for $t \in \{m, m+1\}$. Similar to the deduction of (4.3), we may obtain that the IVP (1.1) has a unique solution $x_{m+1}(t)$ on $t \in \{m, m+1\}$, then $x_{m+1}(m+1)$ exists.

Define $x(t)$ as follows

$$x(t) = \begin{cases} x_0, & t = 0, \\ x_1(t), & t = 1, \\ \vdots & \\ x_m(t), & t = m, \\ \vdots & \end{cases} \tag{4.5}$$

then $x(t)$ is the unique solution of (1.1) on $t \in N$. □

5. Example

Example 5.1. Consider the fractional difference equation

$$\begin{gathered} \Delta_*^{\alpha} x(t) = \lambda x(t+\alpha-1), \quad t \in \mathbb{N}_{1-\alpha}, \ 0 < \alpha \leq 1, \\ x(0) = x_0. \end{gathered} \tag{5.1}$$

According to Theorem 4.1, the IVP (5.1) has a unique solution $x(t)$ provided that $\lambda < 1/(1+\alpha)$. In fact, we can employ the method of successive approximations to obtain the solution of (5.1).

Set

$$\begin{aligned} x_0(t) &= x_0, \\ x_m(t) &= x_0(t) + \frac{\lambda}{\Gamma(\alpha)} \sum_{s=1-\alpha}^{t-\alpha} (t-s-1)^{(\alpha-1)} x_{m-1}(s+\alpha-1) \\ &= x_0(t) + \lambda \Delta^{-\alpha} x_{m-1}(t+\alpha-1), \quad m = 1, 2, \ldots. \end{aligned} \tag{5.2}$$

Applying Lemma 2.6, we have

$$\begin{aligned} x_1(t) &= x_0(t) + \lambda \Delta^{-\alpha} x_0(t+\alpha-1) \\ &= x_0 + \lambda \Delta^{-\alpha} x_0 \\ &= x_0 \left[1 + \frac{\lambda}{\Gamma(\alpha+1)} (t+\alpha-1)^{(\alpha)} \right]. \end{aligned} \tag{5.3}$$

By induction, it follows that

$$x_m(t) = x_0 \sum_{i=0}^{m} \frac{\lambda^i}{\Gamma(i\alpha+1)} (t+i(\alpha-1))^{(i\alpha)}, \quad m = 1, 2, \ldots. \tag{5.4}$$

Taking the limit $m \to \infty$, we obtain

$$x(t) = x_0 \sum_{i=0}^{\infty} \frac{\lambda^i}{\Gamma(i\alpha+1)} (t+i(\alpha-1))^{(i\alpha)} \tag{5.5}$$

which is the unique solution of (5.1). In particular, when $\alpha = 1$, the IVP (5.1) becomes the following integer-order IVP

$$\begin{gathered} \Delta x(t) = \lambda x(t), \quad t \in N, \\ x(0) = x_0, \end{gathered} \tag{5.6}$$

which has the unique solution $x(t) = (1+\lambda)^t x_0$. At the same time, (5.5) becomes that

$$x(t) = x_0 \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} t^{(i)} = (1+\lambda)^t x_0. \tag{5.7}$$

Equation (5.7) implies that, when $\alpha = 1$, the result of the IVP (5.5) is the same as one of the corresponding integer-order IVP (5.6).

Remark 5.2. Example 5.1 is similar to Example 3.1 in [2] in which the difference operator is in the Riemann-Liouville like discrete sense. Compared with the solution of Example 3.1 in [2] defined on $\mathbb{N}_{\alpha-1}$, where $\mathbb{N}_{\alpha-1} = \{\alpha-1, \alpha, \alpha+1, \ldots\}$, the solution of Example 5.1 in this paper is defined on N. This difference makes that fractional difference equation with the Caputo like difference operator is more similar to classical integer-order difference equation.

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