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Research Article

New Existence Results and Comparison Principles for Impulsive Integral Boundary Value Problem with Lower and Upper Solutions in Reversed Order

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This paper investigates the existence of the extremal solutions to the integral boundary value problem for first-order impulsive functional integrodifferential equations with deviating arguments under the assumption of existing upper and lower solutions in the reversed order. The sufficient conditions for the existence of solutions were obtained by establishing several new comparison principles and using the monotone iterative technique. At last, a concrete example is presented and solved to illustrate the obtained results.

1. Introduction

Impulsive differential equations arise naturally from a wide variety of applications, such as control theory, physics, chemistry, population dynamics, biotechnology, industrial robotic, and optimal control ([1–4]). Therefore, it is very important to develop a general theory for differential equations with impulses including some basic aspects of this theory.

In this paper, we consider the following integral boundary value problem for first-order impulsive functional integrodifferential equations with deviating arguments:

$$\begin{gathered}
u'(t) = f(t, u(t), u(\alpha(t)), Wu(t), Su(t)), \quad t \in J', \\
\Delta u(t_k) = I_k(u(t_k)), \quad k = 1, 2, \dots, m, \\
u(0) = ru(T) + \mu \int_0^T \omega(s, u(s))ds + d,
\end{gathered} \tag{1.1}$$

where $t \in J = [0,T]$ $(T > 0)$, $f \in C(J \times R^4, R)$, $I_k \in C(R,R)$, $\omega \in C(J \times R, R)$, $r, \mu, d \in R$, $\alpha \in C(J,J)$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k$ $(k = 1,2,\ldots,m)$, respectively, and

$$Wu(t) = \int_0^{\beta(t)} k(t,s)u(\gamma(s))ds, \qquad Su(t) = \int_0^T h(t,s)u(\delta(s))ds; \tag{1.2}$$

here $\beta, \gamma, \delta \in C(J,J)$, $k(t,s) \in C(D, R^+)$, $h(t,s) \in C(J \times J, R^+)$, $D = \{(t,s) \in R^2 \mid 0 \le s \le \beta(t), t \in J\}$, $R^+ = [0, +\infty)$. Let $PC(J,R) = \{u : J \to R \mid u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $u(t_k^+)$ exists, $k = 1,2,\ldots,m\}$ and $PC^1(J,R) = \{u \in PC(J,E) \mid u(t)$ is continuously differentiable at $t \neq t_k$, $u'(t_k^+)$ and $u'(t_k^-)$ exist, $k = 1,2,\ldots,m\}$. Evidently, $PC(J,R)$ and $PC^1(J,R)$ are Banach spaces with respective norms

$$\|u\|_{PC} = \sup_{t \in J}|u(t)|, \qquad \|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}. \tag{1.3}$$

In recent years, attention has been given to integral type of boundary conditions. The interest in the study of integral boundary conditions lies in the fact that it has various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, and population dynamics. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers ([5–12]) and the references therein.

The method of upper and lower solutions coupled with its associated monotone iteration scheme is an interesting and powerful mechanism that offers the theoretical as well constructive existence results for nonlinear problem in a closed set, generated by the lower and upper solutions (see [9–26]). In the above-mentioned papers, main results are formulated and proved under the assumption of existing upper and lower solutions in the usual order.

However in many cases, the lower and upper solutions occur in the reversed order. This is a fundamentally different situation. In 2009, Wang et al. [27] successfully investigated boundary value problem for functional differential equations without impulses under the assumption of existing upper and lower solutions in the reversed order. In our recent work [28], the monotone iterative technique, combining with the upper and lower solutions in the reversed order, has been successfully applied to obtain the existence of the extremal solutions for a class of nonlinear first-order impulsive functional differential equations. About other existence results for the nonordered case, see ([29–33]).

Motivated by the above-mentioned works, in this paper, we study the integral boundary value problem (1.1). As far as I am concerned, no paper has considered first-order impulsive functional integrodifferential equations with integral boundary conditions and deviating arguments (i.e., problem (1.1)) under the assumption of existing upper and lower solutions in the reverse order. This paper fills this gap in the literature.

The rest of the paper is organized as follows. In Section 2, we establish several new comparison principles, which play an important role in the proof of main results. Further, to study the nonlinear problem (1.1), we consider the associated linear problem and obtain the uniqueness of the solutions to the associated linear problem. In Section 3, the main theorems are formulated and proved. In Section 4, we give an example about integral boundary value problem for impulsive functional integrodifferential equations of mixed type (1.1).

2. Several Comparison Principles and Linear Problem

Lemma 2.1 (comparison result). *Assume that $u \in PC^1(J,R)$ satisfies*

$$\begin{gathered}u'(t) \geq M(t)u(t) + K(t)u(\alpha(t)) + N(t)(Wu)(t) + L(t)(Su)(t), \quad t \in J', \\ \Delta u(t_k) \geq L_k u(t_k), \quad k = 1,2,\ldots,m, \\ u(0) \geq ru(T),\end{gathered} \tag{2.1}$$

where $M \in C(J,R)$, $K,N,L \in C(J,R^+)$, $L_k \geq 0$, $r > 0$ satisfy

(i) $r[1 + \int_0^T q(t)dt + \sum_{0<t_k<T} L_k]e^{\int_0^T M(\tau)d\tau} > 1$,

(ii) $[\int_0^T q(t)dt + \sum_{0<t_k<T} L_k](1 + re^{\int_0^T M(\tau)d\tau}) \leq 1$;

here

$$q(t) = \left[K(t)e^{\int_0^{\alpha(t)} M(\tau)d\tau} + N(t)\int_0^{\beta(t)} k(t,s)e^{\int_0^{\gamma(s)} M(\tau)d\tau}ds + L(t)\int_0^T h(t,s)e^{\int_0^{\delta(s)} M(\tau)d\tau}ds\right]e^{-\int_0^t M(\tau)d\tau}. \tag{2.2}$$

Then $u(t) \leq 0$, $t \in J$.

Proof. Supposing that contrary (i.e., $u(t) > 0$ for some $t \in J$), we consider the following two possible cases:

(1) $u(t) \geq 0$ for all $t \in J$;

(2) there exist $t^*, t_* \in J$ such that $u(t^*) > 0$ and $u(t_*) < 0$.

Let $v(t) = u(t)e^{-\int_0^t M(\tau)d\tau}$; we have

$$\begin{aligned}v'(t) &\geq \Big[K(t)v(\alpha(t))e^{\int_0^{\alpha(t)} M(\tau)d\tau} + N(t)\int_0^{\beta(t)} k(t,s)v(\gamma(s))e^{\int_0^{\gamma(s)} M(\tau)d\tau}ds \\ &\quad + L(t)\int_0^T h(t,s)v(\delta(s))e^{\int_0^{\delta(s)} M(\tau)d\tau}ds\Big]e^{-\int_0^t M(\tau)d\tau}, \quad t \in J', \\ &\Delta v(t_k) \geq L_k v(t_k), \quad k = 1,2,\ldots,m, \\ &v(0) \geq rv(T)e^{\int_0^T M(\tau)d\tau}.\end{aligned} \tag{2.3}$$

Case 1. Equation (2.3) implies that $v'(t) \geq 0$ for $t \neq t_k$ and $\Delta v(t_k) \geq 0$ $(k = 1,2,\ldots,m)$, hence, $v(t)$ is nondecreasing on J. By (2.3), we can get

$$v'(t) \geq \left[K(t)e^{\int_0^{\alpha(t)} M(\tau)d\tau} + N(t)\int_0^{\beta(t)} k(t,s)e^{\int_0^{\gamma(s)} M(\tau)d\tau}ds + L(t)\int_0^T h(t,s)e^{\int_0^{\delta(s)} M(\tau)d\tau}ds\right] \times e^{-\int_0^t M(\tau)d\tau}v(0). \tag{2.4}$$

Integrating the above inequality from 0 to t, we have

$$\begin{aligned}
v(t) &= v(0) + \int_0^t v'(r)dr + \sum_{0<t_k<t}\left[v(t_k^+) - v(t_k)\right] \\
&\geq v(0) + \int_0^t \left[K(r)e^{\int_0^{\alpha(r)} M(\tau)d\tau} + N(r)\int_0^{\beta(r)} k(r,s)e^{\int_0^{\gamma(s)} M(\tau)d\tau}ds \right.\\
&\qquad \left. + L(r)\int_0^T h(r,s)e^{\int_0^{\delta(s)} M(\tau)d\tau}ds\right]e^{-\int_0^r M(\tau)d\tau}v(0)dr + \sum_{0<t_k<t} L_k v(t_k) \\
&\geq \left\{1+\int_0^t \left[K(r)e^{\int_0^{\alpha(r)} M(\tau)d\tau}+N(r)\int_0^{\beta(r)} k(r,s)e^{\int_0^{\gamma(s)} M(\tau)d\tau}ds+L(r)\int_0^T h(r,s)e^{\int_0^{\delta(s)} M(\tau)d\tau}ds\right]\right. \\
&\qquad \left. \times e^{-\int_0^r M(\tau)d\tau}dr + \sum_{0<t_k<t} L_k\right\}v(0) \\
&= \left\{1 + \int_0^t q(r)dr + \sum_{0<t_k<t} L_k\right\}v(0).
\end{aligned} \tag{2.5}$$

Thus,

$$v(0) \geq rv(T)e^{\int_0^T M(\tau)d\tau} \geq r\left[1 + \int_0^T q(r)dr + \sum_{0<t_k<t} L_k\right]v(0)e^{\int_0^T M(\tau)d\tau}. \tag{2.6}$$

Noting condition (i), we have $v(0) = 0$. Besides, $rv(T)e^{\int_0^T M(\tau)d\tau} \leq v(0) = 0$, that is, $v(T) \leq 0$. Since $v(t)$ is nondecreasing on J, then we have $v(t) \equiv 0$, for all $t \in J$. That is, $u(t) \equiv 0$, for all $t \in J$.

Case 2. Firstly, we consider (2.3). Let $\inf_{t\in J} v(t) = -\lambda$, then $\lambda > 0$, and for some $i \in \{1,2,\ldots,m\}$, there exists a $t_* \in (t_i, t_{i+1}]$, such that $v(t_*) = -\lambda$ or $v(t_i^+) = -\lambda$. We only consider $v(t_*) = -\lambda$, for the case $v(t_i^+) = -\lambda$, and the proof is similar.

By (2.3), we have

$$\begin{aligned}
v(t) &= v(0) + \int_0^t v'(s)ds + \sum_{0<t_k<t}\left[v(t_k^+) - v(t_k)\right] \\
&\geq v(0) - \lambda\left[\int_0^t q(s)ds + \sum_{0<t_k<t} L_k\right].
\end{aligned} \tag{2.7}$$

Let $t = t_*$ in (2.7); we have

$$-\lambda \geq v(0) - \lambda\left[\int_0^t q(s)ds + \sum_{0<t_k<t} L_k\right]. \tag{2.8}$$

So,

$$v(0) \leq -\lambda + \lambda\left[\int_0^T q(s)ds + \sum_{0<t_k<T} L_k\right]. \tag{2.9}$$

On the other hand,

$$v(t) = v(T) - \int_t^T v'(s)ds - \sum_{t\leq t_k<T}\left[v\left(t_k^+\right) - v(t_k)\right]. \tag{2.10}$$

Let $t = t^*$ in (2.10), then

$$0 < u(t^*)e^{-\int_0^{t^*} M(\tau)d\tau} = v(t^*) = v(T) - \int_{t^*}^T v'(s)ds - \sum_{t^*\leq t_k<T}\left[v\left(t_k^+\right) - v(t_k)\right]. \tag{2.11}$$

That is,

$$v(T) > \int_{t^*}^T v'(s)ds + \sum_{t^*\leq t_k<T}\left[v\left(t_k^+\right) - v(t_k)\right]. \tag{2.12}$$

By (2.3), we have

$$v(T) > -\lambda\left[\int_0^T q(s)ds + \sum_{0<t_k<T} L_k\right]. \tag{2.13}$$

Thus, by (2.9), (2.13), and $v(0) \geq rv(T)e^{\int_0^T M(\tau)d\tau}$, we obtain

$$-\lambda\left[\int_0^T q(s)ds + \sum_{0<t_k<T} L_k\right]re^{\int_0^T M(\tau)d\tau} < -\lambda + \lambda\left[\int_0^T q(s)ds + \sum_{0<t_k<T} L_k\right]. \tag{2.14}$$

So, $[\int_0^T q(t)dt + \sum_{0<t_k<T} L_k](1 + re^{\int_0^T M(\tau)d\tau}) > 1$, which contradicts condition (ii). Hence, $u(t) \leq 0$ on J.

The proof of Lemma 2.1 is complete. □

Corollary 2.2. *Assume that $M \in C(J,R)$, $K,N,L \in C(J,R^+)$, $L_k \geq 0$, $re^{\int_0^T M(\tau)d\tau} > 1$, and condition (ii) in Lemma 2.1 hold. Let $u \in PC^1(J,R)$ satisfy* (2.1). *Then $u(t) \leq 0$, $t \in J$.*

Proof. The proof of Corollary 2.2 is easy, so we omit it. □

Lemma 2.3 (comparison result). *Let $u \in PC^1(J,R)$ satisfy* (2.1). *Assume that $M,K,N,L \in C(J,[0,+\infty))$, $L_k \geq 0$, $r \geq 0$ and condition (i) in Lemma 2.1 hold. In addition assume that*

(iii) $\int_0^T [M(t) + K(t) + N(t) \int_0^{\beta(t)} k(t,s)ds + L(t) \int_0^T h(t,s)ds]dt + \sum_{0<t_k<T} L_k \leq 1/(r+1)$.

Then $u(t) \leq 0,\ t \in J$.

Proof. The proof is similar to the proof of Lemma 2.1 [28], so we omit it. □

Corollary 2.4. *Assume that* $M, K, N, L \in C(J, [0, +\infty))$, $\int_0^T M(t)dt > 0$, $L_k \geq 0$, $r \geq 1$, *and condition (iii) in Lemma 2.3 hold. Let* $u \in PC^1(J, R)$ *satisfy* (2.1). *Then* $u(t) \leq 0,\ t \in J$.

Proof. The proof of Corollary 2.4 is easy, so we omit it. □

Remark 2.5. Corollary 2.4 holds for $r > 1$ if we delete $\int_0^T M(t)dt > 0$.

Remark 2.6. In the special case where (2.1) does not contain the operators $Wu(t) = \int_0^{\beta(t)} k(t,s)u(\gamma(s))ds$ and $Su(t) = \int_0^T h(t,s)u(\delta(s))ds$, Lemmas 2.1 and 2.3 develop Lemma 2.1 [28], and Corollaries 2.2 and 2.4 develop Corollary 2.1 [28]. Moreover, the condition $M \in C(J,R)$ in Lemma 2.1 and Corollary 2.2 is more extensive than the corresponding condition in [28], and if we let $N(t) = L(t) = 0$ in Lemma 2.3 and Corollary 2.4, we can obtain Lemma 2.1 and Corollary 2.1 in [28], respectively. Therefore, our comparison results in this paper develop and generalize the corresponding results in [28].

To study the nonlinear problem (1.1), we first consider the associated linear problem

$$\begin{gathered} u'(t) = \sigma(t) + M(t)u(t) + K(t)u(\alpha(t)) + N(t)(Wu)(t) + L(t)(Su)(t), \quad t \in J', \\ \Delta u(t_k) = \gamma_k + L_k u(t_k), \quad k = 1, 2, \ldots, m, \\ u(0) = ru(T) + b, \end{gathered} \tag{2.15}$$

where $\sigma \in PC(J,R)$, $\gamma_k, b \in R$.

Definition 2.7. One says $u \in PC^1(J,R)$ is a solution of (2.15) if it satisfies (2.15).

Definition 2.8. One says that $u \in PC^1(J,R)$ is called a lower solution of (2.15) if

$$\begin{gathered} u'(t) \leq \sigma(t) + M(t)u(t) + K(t)u(\alpha(t)) + N(t)(Wu)(t) + L(t)(Su)(t), \quad t \in J', \\ \Delta u(t_k) \leq \gamma_k + L_k u(t_k), \quad k = 1, 2, \ldots, m, \\ u(0) \leq ru(T) + b, \end{gathered} \tag{2.16}$$

and it is an upper solution of (2.15) if the above inequalities are reversed.

Lemma 2.9. *Let all assumptions of Lemma 2.1 hold. In addition assume that* $u_0, v_0 \in PC^1(J,R)$ *are lower and upper solutions of* (2.15), *respectively, and* $u_0(t) \geq v_0(t)$, *for all* $t \in J$. *Then the problem* (2.15) *has a unique solution* $w \in PC^1(J,R)$.

Proof. The proof is similar to the proof of Lemma 2.2 [28], so we omit it. □

Remark 2.10. In Lemma 2.9, if we replace "Lemma 2.1" by any of "Corollary 2.2", "Lemma 2.3", or "Corollary 2.4", then the conclusion of Lemma 2.9 holds.

3. Nonlinear Problem

Definition 3.1. One says $u \in PC^1(J,R)$ is a solution of (1.1) if it satisfies (1.1).

Definition 3.2. One says that $u \in PC^1(J,R)$ is called a lower solution of (1.1) if

$$\begin{gathered} u'(t) \leq f(t,u(t),u(\alpha(t)),Wu(t),Su(t)), \quad t \in J', \\ \Delta u(t_k) \leq I_k(u(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) \leq ru(T) + \mu \int_0^T \omega(s,u(s))ds + d, \end{gathered} \tag{3.1}$$

and it is an upper solution of (1.1) if the above inequalities are reversed.

Theorem 3.3. *Let all assumptions of Lemma 2.1 hold. In addition assume that*

(H_1) $u_0, v_0 \in PC^1(J,R)$ *are lower and upper solutions of* (1.1), *respectively, and* $u_0(t) \geq v_0(t)$, *for all* $t \in J$;

(H_2) *the function* $f \in C(J \times R^4, R)$ *satisfies*

$$f(t,u,v,w,z) - f(t,\overline{u},\overline{v},\overline{w},\overline{z}) \leq M(t)(u-\overline{u}) + K(t)(v-\overline{v}) + N(t)(w-\overline{w}) + L(t)(z-\overline{z}), \tag{3.2}$$

for $v_0(t) \leq \overline{u} \leq u \leq u_0(t)$, $v_0(\alpha(t)) \leq \overline{v} \leq v \leq u_0(\alpha(t))$, $Wv_0(t) \leq \overline{w} \leq w \leq Wu_0(t)$, $Sv_0(t) \leq \overline{z} \leq z \leq Su_0(t)$, *for all* $t \in J$;

(H_3) *the function* $I_k \in C(R,R)$ *satisfies*

$$I_k(u) - I_k(\overline{u}) \leq L_k(u-\overline{u}), \tag{3.3}$$

for $v_0(t_k) \leq \overline{u} \leq u \leq u_0(t_k)$, $k = 1,2,\ldots,m$;

(H_4) *there exists* $a(t) \in C(J,R^+)$ *such that*

$$\mu \int_0^T [\omega(t,v) - \omega(t,\overline{v})]dt \leq -a(t)(v-\overline{v}) \tag{3.4}$$

if $v_0(t) \leq \overline{v} \leq v \leq u_0(t)$.

Then there exist monotone iterative sequences $\{u_n\}$, $\{v_n\}$, *which converge uniformly on* J *to the extremal solutions of* (1.1) *in* $[v_0,u_0] = \{u \in PC^1(J,R) : v_0(t) \leq u(t) \leq u_0(t)\}$.

Proof. For any $\eta \in [v_0,u_0]$, we consider the problem

$$\begin{gathered} u'(t) = \sigma_\eta(t) + M(t)u(t) + K(t)u(\alpha(t)) + N(t)(Wu)(t) + L(t)(Su)(t), \quad t \in J', \\ \Delta u(t_k) = \gamma_k + L_k u(t_k), \quad k = 1,2,\ldots,m, \\ u(0) = ru(T) + b, \end{gathered} \tag{3.5}$$

where

$$\begin{aligned}\sigma_\eta(t) &= f\big(t,\eta(t),\eta(\alpha(t)),W\eta(t),S\eta(t)\big)\\&\quad -M(t)\eta(t)-K(t)\eta(\alpha(t))-N(t)\big(W\eta\big)(t)-L(t)\big(S\eta\big)(t),\\ \gamma_k &= I_k\big(\eta(t_k)\big)-L_k\eta(t_k),\qquad b=\mu\int_0^T\omega\big(s,\eta(s)\big)ds+d.\end{aligned}\tag{3.6}$$

Firstly, we verify that u_0, v_0 are lower and upper solutions in the reversed order of (3.5). By $(H_1)\sim(H_4)$, we obtain, for $t\neq t_k$,

$$\begin{aligned}u_0'(t)&\leq f(t,u_0(t),u_0(\alpha(t)),Wu_0(t),Su_0(t))\\&\leq f\big(t,\eta(t),\eta(\alpha(t)),W\eta(t),S\eta(t)\big)-M(t)\eta(t)-K(t)\eta(\alpha(t))-N(t)\big(W\eta\big)(t)\\&\quad -L(t)\big(S\eta\big)(t)+M(t)u_0(t)+K(t)u_0(\alpha(t))+N(t)(Wu_0)(t)+L(t)(Su_0)(t)\\&=\sigma_\eta(t)+M(t)u_0(t)+K(t)u_0(\alpha(t))+N(t)(Wu_0)(t)+L(t)(Su_0)(t),\end{aligned}\tag{3.7}$$

and, analogously,

$$v_0'(t)\geq\sigma_\eta(t)+M(t)v_0(t)+K(t)v_0(\alpha(t))+N(t)(Wv_0)(t)+L(t)(Sv_0)(t).\tag{3.8}$$

Besides, for $t=t_k$,

$$\begin{gathered}\Delta u_0(t_k)\leq I_k(u_0(t_k))\leq I_k\big(\eta(t_k)\big)-L_k\eta(t_k)+L_ku_0(t_k)=\gamma_k+L_ku_0(t_k),\\ \Delta v_0(t_k)\geq\gamma_k+L_kv_0(t_k).\end{gathered}\tag{3.9}$$

In addition,

$$\begin{gathered}u_0(0)\leq ru_0(T)+\mu\int_0^T\omega(s,u_0(s))ds+d\leq ru_0(T)+\mu\int_0^T\omega\big(s,\eta(s)\big)ds+d=ru_0(T)+b,\\ v_0(0)\geq rv_0(T)+b.\end{gathered}\tag{3.10}$$

Therefore, u_0, v_0 are lower and upper solutions in the reversed order of (3.5). By Lemma 2.9, we know that (3.5) has a unique solution $w\in PC^1(J,R)$.

Now, we prove that $w\in[v_0,u_0]$. Let $p=w-u_0$; we can get

$$\begin{gathered}p'(t)\geq M(t)p(t)+K(t)p(\alpha(t))+N(t)\big(Wp\big)(t)+L(t)\big(Sp\big)(t),\quad t\in J',\\ \Delta p(t_k)\geq L_kp(t_k),\quad k=1,2,\ldots,m,\\ p(0)\geq rp(T).\end{gathered}\tag{3.11}$$

By Lemma 2.1, we have that $p(t)\leq 0$, for all $t\in J$. That is, $w\leq u_0$. Similarly, we can show that $v_0\leq w$. Therefore, we have $w\in[v_0,u_0]$.

Next, we denote an operator $A : [v_0, u_0] \to [v_0, u_0]$ by $u = A\eta$. We prove that A is nondecreasing. Let $\eta_1, \eta_2 \in [v_0, u_0]$ such that $\eta_1 \le \eta_2$. Setting $p = u_1 - u_2,\ u_1 = A\eta_1,\ u_2 = A\eta_2$, by $(H_2) \sim (H_4)$, we have

$$
\begin{aligned}
p'(t) &= f\big(t, \eta_1(t), \eta_1(\alpha(t)), W\eta_1(t), S\eta_1(t)\big) - M(t)\eta_1(t) - K(t)\eta_1(\alpha(t)) - N(t)\big(W\eta_1\big)(t) \\
&\quad - L(t)\big(S\eta_1\big)(t) + M(t)u_1(t) + K(t)u_1(\alpha(t)) + N(t)(Wu_1)(t) + L(t)(Su_1)(t) \\
&\quad - f\big(t, \eta_2(t), \eta_2(\alpha(t)), W\eta_2(t), S\eta_2(t)\big) + M(t)\eta_2(t) + K(t)\eta_2(\alpha(t)) + N(t)\big(W\eta_2\big)(t) \\
&\quad + L(t)\big(S\eta_2\big)(t) - M(t)u_2(t) - K(t)u_2(\alpha(t)) - N(t)(Wu_2)(t) - L(t)(Su_2)(t) \\
&\ge M(t)p(t) + K(t)p(\alpha(t)) + N(t)\big(Wp\big)(t) + L(t)\big(Sp\big)(t), \quad t \in J', \\
\Delta p(t_k) &= I_k\big(\eta_1(t_k)\big) - L_k\eta_1(t_k) + L_k u_1(t_k) - I_k\big(\eta_2(t_k)\big) + L_k\eta_2(t_k) - L_k u_2(t_k) \\
&\ge L_k p(t_k), \quad k = 1, 2, \ldots, m, \\
p(0) &= ru_1(T) + \mu \int_0^T \omega\big(s, \eta_1(s)\big)ds + d - ru_2(T) - \mu \int_0^T \omega\big(s, \eta_2(s)\big)ds - d \\
&\ge rp(T).
\end{aligned}
\tag{3.12}
$$

By Lemma 2.1, we know $p(t) \le 0$ on J, that is, A is nondecreasing.

Now, let $u_n = Au_{n-1},\ v_n = Av_{n-1},\ n = 1, 2, \ldots$, then we have

$$
v_0 \le v_1 \le \cdots \le v_n \le \cdots \le u_n \le \cdots \le u_1 \le u_0, \quad n = 1, 2, \ldots. \tag{3.13}
$$

Obviously, $u_n, v_n\ (n = 1, 2, \ldots)$ satisfy

$$
\begin{aligned}
u_n'(t) &= F(u_{n-1}(t), u_n(t)), \quad t \in J', \\
\Delta u_n(t_k) &= I_k(u_{n-1}(t_k)) + L_k(u_n - u_{n-1})(t_k), \quad k = 1, 2, \ldots, m, \\
u_n(0) &= ru_n(T) + \mu \int_0^T \omega(s, u_{n-1}(s))ds + d, \\
v_n'(t) &= F(v_{n-1}(t), v_n(t)), \quad t \in J', \\
\Delta v_n(t_k) &= I_k(v_{n-1}(t_k)) + L_k(v_n - v_{n-1})(t_k), \quad k = 1, 2, \ldots, m, \\
v_n(0) &= rv_n(T) + \mu \int_0^T \omega(s, v_{n-1}(s))ds + d,
\end{aligned}
\tag{3.14}
$$

with F defined by

$$
\begin{aligned}
F\big(x(t), y(t)\big) &= f\big(t, x(t), x(\alpha(t)), Wx(t), Sx(t)\big) + M(t)\big(y(t) - x(t)\big) + K(t)\big(y(\alpha(t)) - x(\alpha(t))\big) \\
&\quad + N(t)\big((Wy)(t) - (Wx)(t)\big) + L(t)\big((Sy)(t) - (Sx)(t)\big).
\end{aligned}
\tag{3.15}
$$

Therefore, there exist u^*, v^* such that

$$\lim_{n\to\infty} u_n(t) = u^*(t), \qquad \lim_{n\to\infty} v_n(t) = v^*(t) \tag{3.16}$$

uniformly on J, and the limit functions u^*, v^* satisfy (1.1). Moreover, $u^*, v^* \in [v_0, u_0]$.

Finally, we prove that u^*, v^* are the extremal solutions of (1.1) in $[v_0, u_0]$. Let $w \in [v_0, u_0]$ be any solution of (1.1), then $Aw = w$. By $v_0 \le w \le u_0$ and the properties of A, we have

$$v_n \le w \le u_n, \quad n = 1, 2, \ldots. \tag{3.17}$$

Thus, taking limit in (3.17) as $n \to \infty$, we have $v^* \le w \le u^*$. That is, u^*, v^* are the extremal solutions of (1.1) in $[v_0, u_0]$.

The proof of Theorem 3.3 is complete. □

Theorem 3.4. *Let conditions $(H_1) \sim (H_4)$ and all assumptions of any of Corollary 2.2, Lemma 2.3, or Corollary 2.4 satisfy, then the conclusion of Theorem 3.3 hold.*

Proof. The proof is similar to the proof of Theorem 3.3, so we omit it. □

4. Example

Consider the integral boundary value problem

$$\begin{aligned} u'(t) &= \frac{1}{2}t^4\left[t+u^2(t)\right] - \frac{1}{300}t^2\left[t-u(t^2)\right]^3 - \frac{1}{500}t\left[t^3 - \int_0^t tsu\left(s^3\right)ds\right]^5 \\ &\quad -\frac{1}{700}t^2\left[t^2 - \int_0^1 t^2 su(\sqrt{s})ds\right]^7, \quad t \in J = [0,1], \; t \ne t_1, \\ &\qquad \Delta u(t_1) = b\sin u(t_1), \quad 0 \le b \le \frac{26}{75}, \\ u(0) &= \frac{3}{2}u(1) + \mu\int_0^1 \left(8s - s^2u^3(s)\right)ds + d, \quad \mu \in R^+, \; d \in R, \end{aligned} \tag{4.1}$$

where $m = 1,\ 0 < t_1 < 1,\ r = 3/2,\ \alpha(t) = t^2,\ \beta(t) = t,\ \gamma(t) = t^3,\ \delta(t) = \sqrt{t}$, for all $t \in J$.

Obviously, $u_0 = 0,\ v_0 = -1$ are lower and upper solutions of (4.1), respectively, and $v_0 \le u_0$.

Note that $f(t,u,v,w,z) = (1/2)t^4(t+u^2) - (1/300)t^2(t-v)^3 - (1/500)t(t^3-w)^5 - (1/700)t^2(t^2-z)^7,\ I_1(u) = b\sin u$, and $\omega(t,u) = 8t - t^2u^3$.

We have

$$f(t,u,v,w,z) - f(t,\overline{u},\overline{v},\overline{w},\overline{z}) \le \frac{1}{25}t^2(v-\overline{v}) + \frac{4}{25}t(w-\overline{w}) + \frac{16}{25}t^{14}(z-\overline{z}), \tag{4.2}$$

where $v_0(t) \le \overline{u} \le u \le u_0(t),\ v_0(\alpha(t)) \le \overline{v} \le v \le u_0(\alpha(t)),\ Wv_0(t) \le \overline{w} \le w \le Wu_0(t),\ Sv_0(t) \le \overline{z} \le z \le Su_0(t)$, for all $t \in J$.

For $M(t) = a(t) = 0,\ L_1 = b, K(t) = (1/25)t^2,\ N(t) = (4/25)t,\ L(t) = (16/25)t^{14},\ r = 3/2$, it is easy to verify that all conditions of Theorem 3.3 hold. Therefore, by Theorem 3.3, there exist monotone iterative sequences $\{u_n\},\ \{v_n\}$, which converge uniformly on J to the extremal solutions of (4.1) in $[v_0, u_0]$.

Remark 4.1. For appropriate and suitable choices of b, μ, k, and t_1, we see that problem (4.1) has a very general form. For example, we can take $b = 1/3,\ \mu = 100,\ k = -50$, and $t_1 = 2/3$.

5. Conclusions

In this paper, we have discussed the integral boundary value problem for first-order impulsive functional integrodifferential equations with deviating arguments under the assumption of existing upper and lower solutions in the reversed order. The main results (Theorems 3.3 and 3.4) are new and the following results appear as its special cases.

(i) If we take $\alpha(t) = t$ in (1.1), we obtain the first-order impulsive ordinary integrodifferential equations with integral boundary conditions.

(ii) By taking $r = 1$ and $\mu = d = 0$ in (1.1), our result corresponds to periodic boundary value problem for first-order impulsive functional integrodifferential equations with deviating arguments.

(iii) For $I_k(u(t_k)) = 0,\ k = 1, 2, \ldots, m$, in (1.1), we get the integral boundary value problem for first-order mixed type integrodifferential equations with deviating arguments.

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