Research Article

Solvability of Nonautonomous Fractional Integrodifferential Equations with Infinite Delay

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We study the existence and uniqueness of mild solution of a class of nonlinear nonautonomous fractional integrodifferential equations with infinite delay in a Banach space X. The existence of mild solution is obtained by using the theory of the measure of noncompactness and Sadovskii's fixed point theorem. An application of the abstract results is also given.

1. Introduction

The Cauchy problem for various delay equations in Banach spaces has been receiving more and more attention during the past decades (cf., e.g., [1–15]). This paper is concerned with existence results for nonautonomous fractional integrodifferential equations with infinite delay in a Banach space X

$$\frac{d^q u(t)}{dt^q} = -A(t)u(t) + f\left(t, \int_0^t K(t,s)u(s)ds, u_t\right), \quad t \in [0,T],$$

$$u(t) = \phi(t), \quad t \in (-\infty, 0],$$

(1.1)

where $T > 0$, $0 < q < 1$, $\{A(t)\}_{t \in [0,T]}$ is a family of linear operators in X, $K \in C(D, \mathbf{R}^+)$ with $D = \{(t,s) \in \mathbf{R}^2 : 0 \le s \le t \le T\}$ and

$$\sup_{t \in [0,T]} \int_0^t K(t,s)ds < \infty,$$

(1.2)

$f : [0,T] \times X \times \mathcal{P} \to X$, $u_t : (-\infty,0] \to X$ defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in (-\infty,0]$, ϕ belongs to the phase space $\mathcal{P}$, and $\phi(0) = 0$. The fractional derivative is understood here in the Riemann-Liouville sense.

In recent years, the fractional differential equations have been proved to be good tools in the investigation of many phenomena in engineering, physics, economy, chemistry, aerodynamics, electrodynamics of complex medium and they have been studied by many researchers (cf., e.g., [13, 14, 16, 17] and references therein). Moreover, many phenomena cannot be described through classical differential equations but the integral and integrodifferential equations in abstract spaces in fields like electronic, fluid dynamics, biological models, and chemical kinetics. So many significant works on this topic have been appeared (cf., e.g., [10, 15, 18–25] and references therein).

In this paper, we study the existence of mild solution of (1.1) and obtain the existence theorem based on the measures of noncompactness without the assumptions that the nonlinearity f satisfies a Lipschitz type condition and the semigroup $\{\exp(-tA(s))\}$ generated by $-A(s)$ ($s \in [0,T]$) is compact (see Theorem 3.1). An example is given to show an application of the abstract results.

2. Preliminaries

Throughout this paper, we set $J = [0,T]$, a compact interval in $\mathbf{R}$. We denote by X a Banach space, $L(X)$ the Banach space of all linear and bounded operators on X, and $C(J,X)$ the space of all X-valued continuous functions on J. We set

$$Gu(t) := \int_0^t K(t,s)u(s)ds, \qquad G^* := \sup_{t \in J} \int_0^t K(t,s)ds < \infty. \tag{2.1}$$

Next, we recall the definition of the Riemann-Liouville integral.

Definition 2.1 (see [26]). The fractional (arbitrary) order integral of the function $g \in L^1(\mathbf{R}^+, \mathbf{R})$ of order $\nu > 0$ is defined by

$$I^\nu g(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} g(s)ds, \tag{2.2}$$

where Γ is the Gamma function. Moreover, $I^{\nu_1} I^{\nu_2} = I^{\nu_1+\nu_2}$, for all $\nu_1, \nu_2 > 0$.

Remark 2.2. (1) $I^\nu : L^1[0,T] \to L^1[0,T]$ (see [26]),
(2) obviously, for $g \in L^1(J,\mathbf{R})$, it follows from Definition 2.1 that

$$\int_0^t \int_0^\eta (t-\eta)^{q-1} (\eta-s)^{\gamma-1} g(s)ds\,d\eta = B(q,\gamma) \int_0^t (t-s)^{q+\gamma-1} g(s)ds, \tag{2.3}$$

where $B(q,\gamma)$ is a beta function.

See the following definition about phase space according to Hale and Kato [27].

Definition 2.3. A linear space $\mathcal{P}$ consisting of functions from $\mathbf{R}^-$ into X, with seminorm $\|\cdot\|_{\mathcal{P}}$, is called an admissible phase space if $\mathcal{P}$ has the following properties.

(1) If $x : (-\infty, T] \to X$ is continuous on J and $x_0 \in \mathcal{P}$, then $x_t \in \mathcal{P}$ and x_t is continuous in $t \in J$, and

$$\|x(t)\| \leq L\|x_t\|_{\mathcal{P}}, \tag{2.4}$$

where $L \geq 0$ is a constant.

(2) There exist a continuous function $C_1(t) > 0$ and a locally bounded function $C_2(t) \geq 0$ in $t \geq 0$, such that

$$\|x_t\|_{\mathcal{P}} \leq C_1(t) \sup_{s \in [0,t]} \|x(s)\| + C_2(t)\|x_0\|_{\mathcal{P}}, \tag{2.5}$$

for $t \in [0, T]$ and x as in (1).

(3) The space $\mathcal{P}$ is complete.

Remark 2.4. Equation (2.4) in (1) is equivalent to $\|\phi(0)\| \leq L\|\phi\|_{\mathcal{P}}$, for all $\phi \in \mathcal{P}$.

Next, we consider the properties of Kuratowski's measure of noncompactness.

Definition 2.5. Let B be a bounded subset of a seminormed linear space Y. The Kuratowski's measure of noncompactness(for brevity, α-measure) of B is defined as

$$\alpha(B) = \inf\{d > 0 : B \text{ has a finite cover by sets of diameter} \leq d\}. \tag{2.6}$$

From the definition we can get some properties of α-measure immediately, see [28].

Lemma 2.6 (see [28]). *Let A and B be bounded subsets of X, then*

(1) $\alpha(A) \leq \alpha(B)$, *if* $A \subseteq B$;

(2) $\alpha(A) = \alpha(\overline{A})$, *where* $\overline{A}$ *denotes the closure of* A;

(3) $\alpha(A) = 0$ *if and only if A is precompact*;

(4) $\alpha(\lambda A) = |\lambda|\alpha(A), \lambda \in \mathbf{R}$;

(5) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;

(6) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, *where* $A + B = \{x + y : x \in A, y \in B\}$;

(7) $\alpha(A + x) = \alpha(A)$, *for any* $x \in X$.

For $H \subset C(J, X)$, we define

$$\int_0^t H(s)ds = \left\{\int_0^t u(s)ds : u \in H\right\}, \quad \text{for } t \in J, \tag{2.7}$$

where $H(s) = \{u(s) \in X : u \in H\}$.

The following lemma will be needed.

Lemma 2.7. *If $H \subset C(J, X)$ is a bounded, equicontinuous set, then*

(i) $\alpha(H) = \sup_{t \in J} \alpha(H(t))$,

(ii) $\alpha(\int_0^t H(s)ds) \leq \int_0^t \alpha(H(s))ds$, *for* $t \in J$.

For a proof refer to [28].

Lemma 2.8 (see [29]). *If $\{u_n\}_{n=1}^\infty \subset L^1(J, X)$ and there exists an $m \in L^1(J, \mathbf{R}^+)$ such that $\|u_n(t)\| \leq m(t)$, a.e. $t \in J$, then $\alpha(\{u_n(t)\}_{n=1}^\infty)$ is integrable and*

$$\alpha\left(\left\{\int_0^t u_n(s)ds\right\}_{n=1}^\infty\right) \leq 2 \int_0^t \alpha(\{u_n(s)\}_{n=1}^\infty)ds. \tag{2.8}$$

We need to use the following Sadovskii's fixed point theorem here, see [30].

Definition 2.9. Let P be an operator in Banach space X. If P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) < \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$, then P is said to be a condensing operator on X.

Lemma 2.10 (Sadovskii's fixed point theorem [30]). *Let P be a condensing operator on Banach space X. If $P(H) \subseteq H$ for a convex, closed, and bounded set H of X, then P has a fixed point in H.*

In this paper, we denote that C is a positive constant, and assume that a family of closed linear operators $\{A(t)\}_{t \in J}$ satisfying the following.

(A1) The domain $D(A)$ of $\{A(t)\}_{t \in J}$ is dense in the Banach space X and independent of t.

(A2) The operator $[A(t) + \lambda]^{-1}$ exists in $L(X)$ for any λ with $\text{Re}\,\lambda \leq 0$ and

$$\left\|[A(t) + \lambda]^{-1}\right\| \leq \frac{C}{|\lambda| + 1}, \quad t \in J. \tag{2.9}$$

(A3) There exist constants $\gamma \in (0, 1]$ and C such that

$$\left\|[A(t_1) - A(t_2)]A^{-1}(s)\right\| \leq C|t_1 - t_2|^\gamma, \quad t_1, t_2, s \in J. \tag{2.10}$$

Under condition (A2), each operator $-A(s)$, $s \in J$, generates an analytic semigroup $\exp(-tA(s))$, $t > 0$, and there exists a constant C such that

$$\left\|A^n(s)\exp(-tA(s))\right\| \leq \frac{C}{t^n}, \tag{2.11}$$

where $n = 0, 1$, $t > 0$, $s \in J$ (see [31]).

Let Ω be set defined by

$$\Omega = \left\{ u : (-\infty, T] \longrightarrow X \text{ such that } u|_{(-\infty,0]} \in \mathcal{P} \text{ and } u|_J \in C(J, X) \right\}. \tag{2.12}$$

According to [16], a mild solution of (1.1) can be defined as follows.

Definition 2.11. A function $u \in \Omega$ satisfying the equation

$$u(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\[2mm] \displaystyle\int_0^t \psi(t - \eta, \eta) f(\eta, Gu(\eta), u_\eta) d\eta \\[2mm] \quad + \displaystyle\int_0^t \int_0^\eta \psi(t - \eta, \eta)\varphi(\eta, s) f(s, Gu(s), u_s) ds\, d\eta, & t \in J \end{cases} \tag{2.13}$$

is called a mild solution of (1.1), where

$$\psi(t, s) = q \int_0^\infty \theta t^{q-1} \xi_q(\theta) \exp(-t^q \theta A(s)) d\theta, \tag{2.14}$$

and ξ_q is a probability density function defined on $(0, \infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\sigma x} \xi_q(\sigma) d\sigma = \sum_{j=0}^\infty \frac{(-x)^j}{\Gamma(1 + qj)}, \quad 0 < q \leq 1, \ x > 0,$$

$$\varphi(t, \tau) = \sum_{k=1}^\infty \varphi_k(t, \tau), \tag{2.15}$$

where

$$\varphi_1(t, \tau) = [A(t) - A(\tau)]\psi(t - \tau, \tau),$$

$$\varphi_{k+1}(t, \tau) = \int_\tau^t \varphi_k(t, s)\varphi_1(s, \tau) ds, \quad k = 1, 2, \dots. \tag{2.16}$$

To our purpose the following conclusions will be needed. For the proofs refer to [16].

Lemma 2.12 (see [16]). *The operator-valued functions $\psi(t-\eta, \eta)$ and $A(t)\psi(t-\eta, \eta)$ are continuous in uniform topology in the variables t, η, where $0 \leq \eta \leq t - \varepsilon$, $0 \leq t \leq T$, for any $\varepsilon > 0$. Clearly,*

$$\|\psi(t - \eta, \eta)\| \leq C(t - \eta)^{q-1}. \tag{2.17}$$

Moreover, we have

$$\|\varphi(t, \eta)\| \leq C(t - \eta)^{\gamma-1}. \tag{2.18}$$

3. Main Results

We need the hypotheses as follows:

(H1) $f : J \times X \times \mathcal{P} \to X$ satisfies $f(\cdot, v, w) : J \to X$ is measurable for all $(v, w) \in X \times \mathcal{P}$ and $f(t, \cdot, \cdot) : X \times \mathcal{P} \to X$ is continuous for a.e. $t \in J$, and there exist a positive function $\mu(\cdot) \in L^p(J, \mathbf{R}^+)(p > 1/q > 1)$ and a continuous nondecreasing function $\mathcal{W} : [0, \infty) \to [0, \infty)$, such that

$$\|f(t, v, w)\| \leq \mu(t)\mathcal{W}(\|v\| + \|w\|_p), \quad (t, v, w) \in J \times X \times \mathcal{P}, \tag{3.1}$$

and set $T_{p,q} := T^{q-1/p}$,

(H2) for any bounded sets $D_1 \subset X$, $D_2 \subset \mathcal{P}$, and $0 \leq \tau \leq s \leq t \leq T$,

$$\alpha\big(\psi(t - s, s)f(s, D_1, D_2)\big) \leq \beta_1(t, s)\alpha(D_1) + \beta_2(t, s) \sup_{-\infty < \theta \leq 0} \alpha(D_2(\theta)),$$

$$\alpha\big(\psi(t - s, s)\varphi(s, \tau)f(\tau, D_1, D_2)\big) \leq \beta_3(t, s, \tau)\alpha(D_1) + \beta_4(t, s, \tau) \sup_{-\infty < \theta \leq 0} \alpha(D_2(\theta)), \tag{3.2}$$

where $\sup_{t \in J} \int_0^t \beta_i(t, s)ds := \beta_i < \infty$ $(i = 1, 2)$, $\sup_{t \in J} \int_0^t \int_0^s \beta_i(t, s, \tau)d\tau\, ds := \beta_i < \infty$ $(i = 3, 4)$ and $D_2(\theta) = \{w(\theta) : w \in D_2\}$,

(H3) there exists $\overline{M}$, with $0 < \overline{M} < 1$ such that

$$C(1 + CB(q, \gamma))T_{p,q,\gamma}M_{p,q}(G^* + C_1^*)\|\mu\|_{L^p(J, \mathbf{R}^+)}\liminf_{\tau \to \infty}\frac{\mathcal{W}(\tau)}{\tau} < \overline{M}, \tag{3.3}$$

where $M_{p,q} := ((p-1)/(pq-1))^{(p-1)/p}$, $C_1^* = \sup_{0 \leq \eta \leq T}C_1(\eta)$ and $T_{p,q,\gamma} = \max\{T_{p,q}, T_{p,q+\gamma}\}$.

Theorem 3.1. *Suppose that* (H1)–(H3) *are satisfied, and if* $4[G^*(\beta_1 + 2\beta_3) + (\beta_2 + 2\beta_4)] < 1$, *then for* (1.1) *there exists a mild solution on* $(-\infty, T]$.

Proof. Consider the operator $\Phi : \Omega \to \Omega$ defined by

$$(\Phi u)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \displaystyle\int_0^t \psi(t - \eta, \eta)f(\eta, Gu(\eta), u_\eta)d\eta \\ \quad + \displaystyle\int_0^t\int_0^\eta \psi(t - \eta, \eta)\varphi(\eta, s)f(s, Gu(s), u_s)ds\, d\eta, & t \in J. \end{cases} \tag{3.4}$$

It is easy to see that Φ is well-defined.

Let $\overline{x}(\cdot) : (-\infty, T] \to X$ be the function defined by

$$\overline{x}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in J. \end{cases} \tag{3.5}$$

Let $u(t) = \overline{x}(t) + y(t), t \in (-\infty, T]$.

It is easy to see that y satisfies $y_0 = 0$ and

$$
\begin{aligned}
y(t) &= \int_0^t \psi(t - \eta, \eta) f(\eta, G(\overline{x}(\eta) + y(\eta)), \overline{x}_\eta + y_\eta) d\eta \\
&\quad + \int_0^t \int_0^\eta \psi(t - \eta, \eta) \varphi(\eta, s) f(s, G(\overline{x}(s) + y(s)), \overline{x}_s + y_s) ds \, d\eta, \quad t \in J
\end{aligned}
\tag{3.6}
$$

if and only if u satisfies

$$
\begin{aligned}
u(t) &= \int_0^t \psi(t - \eta, \eta) f(\eta, Gu(\eta), u_\eta) d\eta \\
&\quad + \int_0^t \int_0^\eta \psi(t - \eta, \eta) \varphi(\eta, s) f(s, Gu(s), u_s) ds \, d\eta, \quad t \in J
\end{aligned}
\tag{3.7}
$$

and $u(t) = \phi(t), t \in (-\infty, 0]$.

Let $Y_0 = \{y \in \Omega : y_0 = 0\}$. For any $y \in Y_0$,

$$
\|y\|_{Y_0} = \sup_{t \in J} \|y(t)\| + \|y_0\|_p = \sup_{t \in J} \|y(t)\|.
\tag{3.8}
$$

Thus $(Y_0, \|\cdot\|_{Y_0})$ is a Banach space.

We define the operator $\widetilde{\Phi} : Y_0 \to Y_0$ by $(\widetilde{\Phi}y)(t) = 0, t \in (-\infty, 0]$ and

$$
\begin{aligned}
\left(\widetilde{\Phi}y\right)(t) &= \int_0^t \psi(t - \eta, \eta) f(\eta, G(\overline{x}(\eta) + y(\eta)), \overline{x}_\eta + y_\eta) d\eta \\
&\quad + \int_0^t \int_0^\eta \psi(t - \eta, \eta) \varphi(\eta, s) f(s, G(\overline{x}(s) + y(s)), \overline{x}_s + y_s) ds \, d\eta, \quad t \in J.
\end{aligned}
\tag{3.9}
$$

Obviously, the operator Φ has a fixed point if and only if $\widetilde{\Phi}$ has a fixed point. So it turns out to prove that $\widetilde{\Phi}$ has a fixed point.

Let $\{y^k\}_{k \in \mathbf{N}}$ be a sequence such that $y^k \to y$ in Y_0 as $k \to \infty$. Since f satisfies (H1), for almost every $t \in J$, we get

$$
f\left(t, G\left(\overline{x}(t) + y^k(t)\right), \overline{x}_t + y_t^k\right) \longrightarrow f(t, G(\overline{x}(t) + y(t)), \overline{x}_t + y_t), \quad \text{as } k \longrightarrow \infty.
\tag{3.10}
$$

For $t \in (-\infty, T]$, we can prove that $\widetilde{\Phi}$ is continuous. In fact,

$$
\begin{aligned}
\left\| \left(\widetilde{\Phi} y^k \right)(t) - \left(\widetilde{\Phi} y \right)(t) \right\| \\
\leq \int_0^t \Big\| \psi(t-\eta,\eta) \Big[f\Big(\eta, G\big(\overline{x}(\eta) + y^k(\eta)\big), \overline{x}_\eta + y_\eta^k\Big) \\
- f\big(\eta, G(\overline{x}(\eta) + y(\eta)), \overline{x}_\eta + y_\eta\big) \Big] \Big\| d\eta \\
+ \int_0^t \int_0^\eta \Big\| \psi(t-\eta,\eta)\varphi(\eta,s) \Big[f\Big(s, G\big(\overline{x}(s) + y^k(s)\big), \overline{x}_s + y_s^k\Big) \\
- f\big(s, G(\overline{x}(s) + y(s)), \overline{x}_s + y_s\big) \Big] \Big\| ds\, d\eta.
\end{aligned}
\tag{3.11}
$$

Let $C_2^* = \sup_{0 \leq \eta \leq T} C_2(\eta)$, and noting (2.4), (2.5), we have

$$
\begin{aligned}
\left\| \overline{x}_t + y_t \right\|_\rho &\leq \left\| \overline{x}_t \right\|_\rho + \left\| y_t \right\|_\rho \\
&\leq C_1(t) \sup_{0\leq\tau\leq t} \|\overline{x}(\tau)\| + C_2(t)\|\overline{x}_0\|_\rho + C_1(t)\sup_{0\leq\tau\leq t}\|y(\tau)\| + C_2(t)\|y_0\|_\rho \\
&= C_2(t)\|\phi\|_\rho + C_1(t)\sup_{0\leq\tau\leq t}\|y(\tau)\| \\
&\leq C_2^*\|\phi\|_\rho + C_1^*\sup_{0\leq\tau\leq t}\|y(\tau)\|.
\end{aligned}
\tag{3.12}
$$

Moreover,

$$
\begin{aligned}
\|G(\overline{x}(t) + y(t))\| &\leq \int_0^t K(t,\tau)\|\overline{x}(\tau) + y(\tau)\| d\tau \\
&= \int_0^t K(t,\tau) \cdot \|y(\tau)\| d\tau.
\end{aligned}
\tag{3.13}
$$

Noting that $y^k \to y$ in Y_0, we can see that there exists $\varepsilon > 0$ such that $\|y^k - y\| \leq \varepsilon$, for k sufficiently large. Therefore, we have

$$
\begin{aligned}
\Big\| f\Big(t, G\big(\overline{x}(t) + y^k(t)\big), \overline{x}_t + y_t^k\Big) - f\big(t, G(\overline{x}(t) + y(t)), \overline{x}_t + y_t\big) \Big\| \\
\leq \mu(t)\Big[\mathcal{W}\Big(\big\|G\big(\overline{x}(t) + y^k(t)\big)\big\| + \big\|\overline{x}_t + y_t^k\big\|_\rho\Big) + \mathcal{W}\Big(\|G(\overline{x}(t) + y(t))\| + \|\overline{x}_t + y_t\|_\rho\Big)\Big] \\
\leq \mu(t)\Big[\omega_k^1(t) + \omega_k^2(t)\Big],
\end{aligned}
\tag{3.14}
$$

where

$$\omega_k^1(t) = \mathcal{W}\left(G^*\varepsilon + G^*\|y\|_{Y_0} + C_2^*\|\phi\|_\rho + C_1^*\varepsilon + C_1^*\|y\|_{Y_0}\right),$$
$$\omega_k^2(t) = \mathcal{W}\left(G^*\|y\|_{Y_0} + C_2^*\|\phi\|_\rho + C_1^*\|y\|_{Y_0}\right). \tag{3.15}$$

In view of (2.17) and the Lebesgue Dominated Convergence Theorem ensure that

$$\int_0^t \left\|\psi(t-\eta,\eta)\left[f\left(\eta,G(\overline{x}(\eta)+y^k(\eta)),\overline{x}_\eta+y_\eta^k\right)-f(\eta,G(\overline{x}(\eta)+y(\eta)),\overline{x}_\eta+y_\eta)\right]\right\|d\eta$$

$$\leq C\int_0^t (t-\eta)^{q-1}\left\|f\left(\eta,G(\overline{x}(\eta)+y^k(\eta)),\overline{x}_\eta+y_\eta^k\right)\right.$$

$$\left. -f(\eta,G(\overline{x}(\eta)+y(\eta)),\overline{x}_\eta+y_\eta)\right\|d\eta$$

$$\longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{3.16}$$

Similarly, by (2.17) and (2.18), we have

$$\int_0^t\int_0^\eta \left\|\psi(t-\eta,\eta)\varphi(\eta,s)\left[f\left(s,G(\overline{x}(s)+y^k(s)),\overline{x}_s+y_s^k\right)\right.\right.$$

$$\left.\left.-f(s,G(\overline{x}(s)+y(s)),\overline{x}_s+y_s)\right]\right\|ds\,d\eta$$

$$\leq C^2\int_0^t\int_0^\eta (t-\eta)^{q-1}(\eta-s)^{\Upsilon-1}\left\|f\left(s,G(\overline{x}(s)+y^k(s)),\overline{x}_s+y_s^k\right)\right. \tag{3.17}$$

$$\left. -f(s,G(\overline{x}(s)+y(s)),\overline{x}_s+y_s)\right\|ds\,d\eta$$

$$\longrightarrow 0, \quad \text{as } k \longrightarrow \infty.$$

Therefore, we deduce that

$$\lim_{k\to\infty}\left\|\tilde{\Phi}y^k-\tilde{\Phi}y\right\|_{Y_0}=0. \tag{3.18}$$

This means that $\tilde{\Phi}$ is continuous.

We show that $\tilde{\Phi}$ maps bounded sets of Y_0 into bounded sets in Y_0. For any $r > 0$, we set $B_r = \{y \in Y_0 : \|y\|_{Y_0} \leq r\}$. Now, for $y \in B_r$, by (3.12), (3.13), and (H1), we can see

$$\left\|f(t,G(\overline{x}(t)+y(t)),\overline{x}_t+y_t)\right\| \leq \mu(t)\mathcal{W}(M_1), \tag{3.19}$$

where $M_1 := G^*r + C_2^*\|\phi\|_\rho + C_1^*r$.

Then for any $y \in B_r$, by (2.17), (2.18), (3.19), and Remark 2.2, we have

$$
\begin{aligned}
\left\|\left(\widetilde{\Phi}y\right)(t)\right\| &\leq \int_0^t \left\|\psi(t-\eta,\eta)f\left(\eta, G(\overline{x}(\eta)+y(\eta)), \overline{x}_\eta + y_\eta\right)\right\| d\eta \\
&\quad + \int_0^t \int_0^\eta \left\|\psi(t-\eta,\eta)\varphi(\eta,s)f\left(s, G(\overline{x}(s)+y(s)), \overline{x}_s + y_s\right)\right\| ds\, d\eta \\
&\leq C \int_0^t (t-\eta)^{q-1}\mu(\eta)\mathcal{W}(M_1)d\eta + C^2 \int_0^t \int_0^\eta (t-\eta)^{q-1}(\eta-s)^{\gamma-1}\mu(s)\mathcal{W}(M_1)ds\, d\eta \\
&= M_2 \left[C \int_0^t (t-\eta)^{q-1}\mu(\eta)d\eta + C^2 B(q,\gamma) \int_0^t (t-\eta)^{q+\gamma-1}\mu(\eta)d\eta \right],
\end{aligned}
\tag{3.20}
$$

where $M_2 = \mathcal{W}(M_1)$.

Noting that the Hölder inequality, we have

$$
\begin{aligned}
\int_0^t (t-\eta)^{q-1}\mu(\eta)d\eta &= t^{(pq-1)/p} M_{p,q}\|\mu\|_{L^p(J,\mathbf{R}^+)} \leq T_{p,q} M_{p,q}\|\mu\|_{L^p(J,\mathbf{R}^+)}, \\
\int_0^t (t-\eta)^{\gamma+q-1}\mu(\eta)d\eta &\leq T_{p,q+\gamma} M_{p,q+\gamma}\|\mu\|_{L^p(J,\mathbf{R}^+)}.
\end{aligned}
\tag{3.21}
$$

Thus

$$
\left\|\left(\widetilde{\Phi}y\right)(t)\right\| \leq M_2 M_{p,q} T_{p,q,\gamma}\left[C + C^2 B(q,\gamma)\right]\|\mu\|_{L^p(J,\mathbf{R}^+)} := \widetilde{r}.
\tag{3.22}
$$

This means $\widetilde{\Phi}(B_r) \subset B_{\widetilde{r}}$.

Next, we show that there exists $k \in \mathbf{N}$ such that $\widetilde{\Phi}(B_k) \subset B_k$. Suppose contrary that for every $k \in \mathbf{N}$ there exist $y^k \in B_k$ and $t_k \in J$ such that $\|(\widetilde{\Phi}y^k)(t_k)\| > k$. However, on the other hand, similar to the deduction of (3.20) and noting

$$
\left\|f\left(t, G\left(\overline{x}(t)+y^k(t)\right), \overline{x}_t + y_t^k\right)\right\| \leq \mu(t)\mathcal{W}\left(G^*k + C_2^*\|\phi\|_p + C_1^*k\right),
\tag{3.23}
$$

we have

$$
\begin{aligned}
k < \left\|\left(\widetilde{\Phi}y^k\right)(t_k)\right\| &\leq \widetilde{M_2}\left[C \int_0^{t_k} (t_k-\eta)^{q-1}\mu(\eta)d\eta + C^2 B(q,\gamma)\int_0^{t_k} (t_k-\eta)^{q+\gamma-1}\mu(\eta)d\eta\right] \\
&\leq \widetilde{M_2} M_{p,q} T_{p,q,\gamma}\left[C + C^2 B(q,\gamma)\right]\|\mu\|_{L^p(J,\mathbf{R}^+)},
\end{aligned}
\tag{3.24}
$$

where $\widetilde{M_2} = \mathcal{W}(G^*k + C_2^*\|\phi\|_p + C_1^*k)$.

Dividing both sides of (3.24) by k, and taking $k \to \infty$, we have

$$C(1 + CB(q, \gamma)) T_{p,q,\gamma} M_{p,q}(G^* + C_1^*) \|\mu\|_{L^p(J, \mathbf{R}^+)} \liminf_{\tau \to \infty} \frac{\mathcal{W}(\tau)}{\tau} \geq 1. \tag{3.25}$$

This contradicts (3.3). Hence for some positive number k, $\widetilde{\Phi}(B_k) \subset B_k$.

Let $0 < t_2 < t_1 < T$ and $y \in B_k$, then

$$\left\| \left(\widetilde{\Phi} y \right)(t_1) - \left(\widetilde{\Phi} y \right)(t_2) \right\| \leq I_1 + I_2 + I_3 + I_4, \tag{3.26}$$

where

$$I_1 = \int_0^{t_2} \left\| [\psi(t_1 - \eta, \eta) - \psi(t_2 - \eta, \eta)] f(\eta, G(\overline{x}(\eta) + y(\eta)), \overline{x}_\eta + y_\eta) \right\| d\eta,$$

$$I_2 = \int_{t_2}^{t_1} \left\| \psi(t_1 - \eta, \eta) f(\eta, G(\overline{x}(\eta) + y(\eta)), \overline{x}_\eta + y_\eta) \right\| d\eta,$$

$$I_3 = \int_0^{t_2} \int_0^\eta \left\| [\psi(t_1 - \eta, \eta) - \psi(t_2 - \eta, \eta)] \varphi(\eta, s) f(s, G(\overline{x}(s) + y(s)), \overline{x}_s + y_s) \right\| ds \, d\eta,$$

$$I_4 = \int_{t_2}^{t_1} \int_0^\eta \left\| \psi(t_1 - \eta, \eta) \varphi(\eta, s) f(s, G(\overline{x}(s) + y(s)), \overline{x}_s + y_s) \right\| ds \, d\eta. \tag{3.27}$$

It follows from Lemma 2.12, (H1) and (3.23) that $I_1, I_3 \to 0$, as $t_2 \to t_1$.

For I_2, from (2.17), (3.23), and (H1), we have

$$I_2 = \int_{t_2}^{t_1} \left\| \psi(t_1 - \eta, \eta) f(\eta, G(\overline{x}(\eta) + y(\eta)), \overline{x}_\eta + y_\eta) \right\| d\eta$$

$$\leq C \widetilde{M_2} \int_{t_2}^{t_1} (t_1 - \eta)^{q-1} \mu(\eta) d\eta \longrightarrow 0, \quad \text{as } t_2 \longrightarrow t_1. \tag{3.28}$$

Similarly, by (2.17), (2.18), (H1), and Remark 2.2, we have

$$I_4 = \int_{t_2}^{t_1} \int_0^\eta \left\| \psi(t_1 - \eta, \eta) \varphi(\eta, s) f(s, G(\overline{x}(s) + y(s)), \overline{x}_s + y_s) \right\| ds \, d\eta$$

$$\leq C^2 \widetilde{M_2} \int_{t_2}^{t_1} (t_1 - \eta)^{q-1} \int_0^\eta (\eta - s)^{\gamma - 1} \mu(s) ds \, d\eta \longrightarrow 0, \quad \text{as } t_2 \longrightarrow t_1. \tag{3.29}$$

So, the set $\{ (\widetilde{\Phi} y)(\cdot) : y \in B_k \}$ is equicontinuous.

For every bounded set $H \subset B_k$ and any $\varepsilon > 0$, we can take a sequence $\{h_n\}_{n=1}^{\infty} \subset H$ such that $\alpha(H) \le 2\alpha(\{h_n\}) + \varepsilon$ (see [32]), thus from Lemmas 2.6–2.8, and 2.12 and (H2), we have

$$
\begin{aligned}
\alpha\left(\widetilde{\Phi}H\right) &\le 2\alpha\left(\left\{\widetilde{\Phi}h_n\right\}\right) + \varepsilon = 2\sup_{t\in J}\alpha\left(\left\{\widetilde{\Phi}h_n(t)\right\}\right) + \varepsilon \\
&= 2\sup_{t\in J}\alpha\left(\left\{\int_0^t \psi(t-\eta,\eta)f(\eta,G(\overline{x}(\eta)+h_n(\eta)),\overline{x}_\eta + h_{n\eta})d\eta\right\}\right. \\
&\qquad\qquad \left. + \left\{\int_0^t\int_0^\eta \psi(t-\eta,\eta)\varphi(\eta,s)f(s,G(\overline{x}(s)+h_n(s)),\overline{x}_s + h_{ns})ds\,d\eta\right\}\right) + \varepsilon \\
&\le 4\sup_{t\in J}\left(\int_0^t \alpha(\{\psi(t-\eta,\eta)f(\eta,G(\overline{x}(\eta)+h_n(\eta)),\overline{x}_\eta + h_{n\eta})\})d\eta\right) \\
&\quad + 8\sup_{t\in J}\left(\int_0^t\int_0^\eta \alpha(\{\psi(t-\eta,\eta)\varphi(\eta,s)f(s,G(\overline{x}(s)+h_n(s)),\overline{x}_s + h_{ns})\})ds\,d\eta\right) + \varepsilon \\
&\le 4\sup_{t\in J}\left(\int_0^t\left[\beta_1(t,\eta)G^*\alpha(\{h_n\}) + \beta_2(t,\eta)\sup_{-\infty<\theta\le 0}\alpha(\{h_n(\theta+\eta)\})\right]d\eta\right) \\
&\quad + 8\sup_{t\in J}\left(\int_0^t\int_0^\eta\left[\beta_3(t,\eta,s)G^*\alpha(\{h_n\}) + \beta_4(t,\eta,s)\sup_{-\infty<\theta\le 0}\alpha(\{h_n(\theta+s)\})\right]ds\,d\eta\right) + \varepsilon \\
&\le 4\sup_{t\in J}\left(\int_0^t\left[\beta_1(t,\eta)G^*\alpha(\{h_n\}) + \beta_2(t,\eta)\sup_{0\le\tau\le\eta}\alpha(\{h_n(\tau)\})\right]d\eta\right) \\
&\quad + 8\sup_{t\in J}\left(\int_0^t\int_0^\eta\left[\beta_3(t,\eta,s)G^*\alpha(\{h_n\}) + \beta_4(t,\eta,s)\sup_{0\le\tau\le s}\alpha(\{h_n(\tau)\})\right]ds\,d\eta\right) + \varepsilon \\
&\le 4\left[G^*(\beta_1 + 2\beta_3) + (\beta_2 + 2\beta_4)\right]\alpha(\{h_n\}) + \varepsilon \le 4\left[G^*(\beta_1 + 2\beta_3) + (\beta_2 + 2\beta_4)\right]\alpha(H) + \varepsilon,
\end{aligned}
\tag{3.30}
$$

since ε is arbitrary, we can obtain

$$
\alpha\left(\widetilde{\Phi}H\right) \le 4\left[G^*(\beta_1 + 2\beta_3) + (\beta_2 + 2\beta_4)\right]\alpha(H) < \alpha(H).
\tag{3.31}
$$

In view of the Sadovskii's fixed point theorem, we conclude that $\widetilde{\Phi}$ has at least one fixed point $\widetilde{y}$ in B_k. Let $u(t) = \overline{x}(t) + \widetilde{y}(t)$, $t \in (-\infty, T]$, then $u(t)$ is a fixed point of the operator Φ which is a mild solution of (1.1). $\qquad\square$

Now we assume that

(H1′) there exists a positive function $l(\cdot) \in L^1(J, \mathbf{R}^+)$, such that

$$
\begin{aligned}
&\| f(t, v_1, w_1) - f(t, v_2, w_2) \| \\
&\qquad \le l(t)(\|v_1 - v_2\| + \|w_1 - w_2\|_\rho), \quad (v_1, v_2) \in X^2, \ (w_1, w_2) \in \mathcal{P}^2,
\end{aligned}
\tag{3.32}
$$

(H2′) there exists a constant ϖ, with $0 < \varpi < 1$, such that the function $\Lambda : J \to \mathbf{R}^+$ defined by

$$
\Lambda(t) = C(G^* + C_1^*)\Gamma(q)\left[I^q l(t) + C\Gamma(\gamma)I^{\gamma+q}l(t)\right] \le \varpi, \quad t \in J.
\tag{3.33}
$$

Theorem 3.2. *Assume that (H1′) and (H2′) are satisfied, then (1.1) has a unique mild solution.*

Proof. Let $\widetilde{\Phi}$ be defined as in Theorem 3.1. For any $y, y^* \in Y_0$, we have

$$
\begin{aligned}
&\left\| f\left(t, G(\overline{x}(t) + y(t)), \overline{x}_t + y_t\right) - f\left(t, G(\overline{x}(t) + y^*(t)), \overline{x}_t + y_t^*\right) \right\| \\
&\qquad \le l(t)\left(\left\| G(y(t) - y^*(t)) \right\| + \left\| y_t - y_t^* \right\|_\rho\right) \\
&\qquad \le l(t)\left(G^* \|y - y^*\|_{Y_0} + C_1(t)\sup_{0 \le \tau \le t}\|y(\tau) - y^*(\tau)\| \right) \\
&\qquad \le (G^* + C_1^*)l(t)\|y - y^*\|_{Y_0}.
\end{aligned}
\tag{3.34}
$$

Thus, from (2.17), (2.18), Definition 2.1 and Remark 2.2, we have

$$
\begin{aligned}
&\left\| \left(\widetilde{\Phi}y\right)(t) - \left(\widetilde{\Phi}y^*\right)(t) \right\| \\
&\quad \le \int_0^t \|\psi(t - \eta, \eta)\| \, \| f\left(\eta, G(\overline{x}(\eta) + y(\eta)), \overline{x}_\eta + y_\eta\right) \\
&\qquad\qquad\qquad - f\left(\eta, G(\overline{x}(\eta) + y^*(\eta)), \overline{x}_\eta + y_\eta^*\right) \| \, d\eta \\
&\qquad + \int_0^t \int_0^\eta \|\psi(t - \eta, \eta)\varphi(\eta, s)\| \, \| f\left(s, G(\overline{x}(s) + y(s)), \overline{x}_s + y_s\right) \\
&\qquad\qquad\qquad\qquad - f\left(s, G(\overline{x}(s) + y^*(s)), \overline{x}_s + y_s^*\right) \| \, ds \, d\eta \\
&\quad \le C(G^* + C_1^*)\|y - y^*\|_{Y_0} \cdot \left[\int_0^t (t - \eta)^{q-1}l(\eta)d\eta + C\int_0^t\int_0^\eta (t - \eta)^{q-1}(\eta - s)^{\gamma-1}l(s)ds\,d\eta \right] \\
&\quad = C(G^* + C_1^*) \cdot \left[\Gamma(q)I^q l(t) + C\Gamma(q)\Gamma(\gamma)I^{\gamma+q}l(t) \right] \cdot \|y - y^*\|_{Y_0} \\
&\quad = \Lambda(t)\|y - y^*\|_{Y_0}.
\end{aligned}
\tag{3.35}
$$

So, we get

$$\left\|\left(\widetilde{\Phi}y\right)(t) - \left(\widetilde{\Phi}y^*\right)(t)\right\|_{Y_0} < \|y - y^*\|_{Y_0}, \tag{3.36}$$

and the result follows from the contraction mapping principle.   □

Example 3.3. We consider the following model:

$$\frac{\partial^q}{\partial t^q} v(t,\xi) = a(t,\xi)\frac{\partial^2 v}{\partial \xi^2}(t,\xi) + \frac{t^n}{n^2} \sin\left(\left|\int_0^t (t-s)v(s,\xi)ds\right|\right)$$

$$\cdot \int_0^t e^{-\left|\int_0^s (s-\tau)v(\tau,\xi)d\tau\right|} ds + \frac{t^n}{n^2} \int_{-\infty}^0 \zeta(\theta) \sin\left|t^2 v(t+\theta,\xi)\right| d\theta, \tag{3.37}$$

$$v(t,0) = v(t,1) = 0,$$

$$v(\theta,\xi) = v_0(\theta,\xi), \quad -\infty < \theta \le 0,$$

where $0 \le t \le 1$, $\xi \in [0,1]$, $n \in \mathbf{N}$, $a(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t, that is, there exist $C > 0$ and $\overline{\gamma} \in (0,1)$ such that

$$\|a(t_1,\xi) - a(t_2,\xi)\| \le C|t_1 - t_2|^{\overline{\gamma}}, \quad \xi \in [0,1], \ 0 \le t_1 \le t_2 \le 1, \tag{3.38}$$

$\zeta : (-\infty,0] \to \mathbf{R}$, $v_0 : (-\infty,0] \times [0,1] \to \mathbf{R}$ are continuous functions, and $\int_{-\infty}^0 |\zeta(\theta)|d\theta < \infty$.
     Set $X = L^2([0,1],\mathbf{R})$ and define $A(t)$ by

$$D(A(t)) = H^2(0,1) \cap H_0^1(0,1),$$

$$A(t)u = -a(t,\xi)u''. \tag{3.39}$$

Then $-A(s)$ generates an analytic semigroup $\exp(-tA(s))$ satisfying assumptions (A1)–(A3) (see [33]).
     Let the phase space $\mathcal{P}$ be $\text{BUC}(\mathbf{R}^-, X)$, the space of bounded uniformly continuous functions endowed with the following norm:

$$\|\varphi\|_{\mathcal{P}} = \sup_{-\infty < \theta \le 0} |\varphi(\theta)|, \quad \forall \varphi \in \mathcal{P}, \tag{3.40}$$

then we can see that $C_1(t) = 1$ in (2.5).

For $t \in [0,1]$, $\xi \in [0,1]$ and $\varphi \in \mathrm{BUC}(\mathbf{R}^-, X)$, we set

$$u(t)(\xi) = v(t,\xi),$$

$$\phi(\theta)(\xi) = v_0(\theta,\xi), \quad \theta \in (-\infty, 0],$$

$$f(t, Gu(t), \varphi)(\xi) = \frac{t^n}{n^2} \sin(|(Gu(t))(\xi)|) \cdot \int_0^t e^{-|(Gu(s))(\xi)|} ds + \frac{t^n}{n^2} \int_{-\infty}^0 \zeta(\theta) \sin\left|t^2 \varphi(\theta)(\xi)\right| d\theta, \tag{3.41}$$

where

$$(Gu(t))(\xi) = \int_0^t (t-s)u(s)(\xi) ds, \tag{3.42}$$

now $G^* = \sup_{t \in [0,1]} \int_0^t (t-s) ds = 1/2 < \infty$.

Then the above equation (3.37) can be written in the abstract form as (1.1).

Moreover,

$$\begin{aligned}
\left\| f(t, Gu(t), \varphi)(\xi) \right\| &\le \frac{t^{n+1}}{n^2} \|Gu(t)\| + \frac{t^{n+2}}{n^2} \|\varphi\|_p \int_{-\infty}^0 |\zeta(\theta)| d\theta \\
&\le \frac{1}{n^2} \max\left\{ t^{n+1}, t^{n+2} \int_{-\infty}^0 |\zeta(\theta)| d\theta \right\} \left(\|Gu(t)\| + \|\varphi\|_p \right) \\
&= \mu(t) \mathcal{W}\left(\|Gu(t)\| + \|\varphi\|_p \right),
\end{aligned} \tag{3.43}$$

where $\mu(t) := \max\{t^{n+1}, t^{n+2} \int_{-\infty}^0 |\zeta(\theta)| d\theta\}$, $\mathcal{W}(z) = z/n^2$ satisfy (H1).

For any $u_1, u_2 \in X$, $\varphi, \widetilde{\varphi} \in p$,

$$\begin{aligned}
&\left\| \psi(t-s,s) f(s, Gu_1(s), \varphi)(\xi) - \psi(t-s,s) f(s, Gu_2(s), \widetilde{\varphi})(\xi) \right\| \\
&\le \frac{C}{n^2}(t-s)^{q-1} \left[s^{n+1} \|Gu_1(s) - Gu_2(s)\| + s^n \int_0^s \|Gu_1(\tau) - Gu_2(\tau)\| d\tau \right] \\
&+ \frac{C}{n^2}(t-s)^{q-1} s^{n+2} \int_{-\infty}^0 |\zeta(\theta)| \|\varphi(\theta)(\xi) - \widetilde{\varphi}(\theta)(\xi)\| d\theta.
\end{aligned} \tag{3.44}$$

Therefore, for any bounded sets $D_1 \subset X$, $D_2 \subset p$, we have

$$\begin{aligned}
\alpha\left(\psi(t-s,s) f(s, D_1, D_2) \right) &\le \frac{2C}{n^2}(t-s)^{q-1} s^{n+1} \cdot \alpha(D_1) \\
&+ \frac{C}{n^2}(t-s)^{q-1} s^{n+2} \int_{-\infty}^0 |\zeta(\theta)| \alpha(D_2(\theta)) d\theta.
\end{aligned} \tag{3.45}$$

Moreover,

$$
\frac{2C}{n^2} \sup_{t \in [0,1]} \int_0^t (t-s)^{q-1} s^{n+1} ds = \frac{2C}{n^2} \sup_{t \in [0,1]} t^{n+q+1} B(q, n+2) = \frac{2C}{n^2} B(q, n+2) := \beta_1,
$$

$$
\frac{C}{n^2} \sup_{t \in [0,1]} \int_0^t (t-s)^{q-1} s^{n+2} \int_{-\infty}^0 |\zeta(\theta)| d\theta ds = \frac{C}{n^2} B(q, n+3) \int_{-\infty}^0 |\zeta(\theta)| d\theta := \beta_2.
$$

(3.46)

Similarly, we obtain

$$
\alpha\big(\psi(t-s,s)\varphi(s,\tau)f(\tau, D_1, D_2)\big)
$$

$$
\leq (t-s)^{q-1}(s-\tau)^{\gamma-1}\tau^{n+1}\left[\frac{2C^2}{n^2}\alpha(D_1) + \frac{C^2}{n^2}\tau\int_{-\infty}^0 |\zeta(\theta)| d\theta \sup_{\theta \leq 0} \alpha(D_2(\theta))\right],
$$

$$
\frac{2C^2}{n^2} \sup_{t \in [0,1]} \int_0^t \int_0^s (t-s)^{q-1}(s-\tau)^{\gamma-1}\tau^{n+1} d\tau\, ds
$$

$$
= \frac{2C^2}{n^2} \sup_{t \in [0,1]} t^{q+\gamma+n+1} B(q, \gamma) B(q+\gamma, n+2) = \frac{2C^2}{n^2} B(q, \gamma) B(q+\gamma, n+2) := \beta_3,
$$

$$
\frac{C^2}{n^2} \sup_{t \in [0,1]} \int_0^t \int_0^s (t-s)^{q-1}(s-\tau)^{\gamma-1}\tau^{n+2} d\tau\, ds \int_{-\infty}^0 |\zeta(\theta)| d\theta
$$

$$
= \frac{C^2}{n^2} B(q, \gamma) B(q+\gamma, n+3) \int_{-\infty}^0 |\zeta(\theta)| d\theta := \beta_4.
$$

(3.47)

Suppose further that

(1) there exists $\overline{M} \in (0,1)$ such that $(3/2n^2)C(1 + CB(q,\gamma))((p-1)/(pq-1))^{(p-1)/p}\|\mu\|_{L^p([0,1],\mathbf{R}^+)} < \overline{M}$,

(2) $2(\beta_1 + 2\beta_3) + 4(\beta_2 + 2\beta_4) < 1$,

then (3.37) has a mild solution by Theorem 3.1.

Acknowledgments

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