SOLVABILITY CONDITIONS FOR SOME DIFFERENCE OPERATORS

N. C. APREUTESEI AND V. A. VOLPERT

Received 24 June 2004

Infinite-dimensional difference operators are studied. Under the assumption that the coefficients of the operator have limits at infinity, limiting operators and associated polynomials are introduced. Under some specific conditions on the polynomials, the operator is Fredholm and has the zero index. Solvability conditions are obtained and the exponential behavior of solutions of the homogeneous equation at infinity is proved.

1. Introduction

Infinite-dimensional difference operators may not satisfy the Fredholm property, and the Fredholm-type solvability conditions are not necessarily applicable to them. In other words, we do not know how to solve linear algebraic systems with infinite matrices. Various properties of linear and nonlinear infinite discrete systems are studied in [1, 2, 3, 4, 5, 6, 7, 8].

The goal of this paper is to establish the normal solvability for the difference operators of the form

$$(Lu)_j = a_{-m}^j u_{j-m} + \cdots + a_0^j u_j + \cdots + a_m^j u_{j+m}, \quad j \in \mathbb{Z}, \tag{1.1}$$

and to obtain the solvability conditions for the equation $Lu = f$, where $m \geq 0$ is a given integer and $f = \{f_j\}_{j=-\infty}^{\infty}$ is an element of the Banach space

$$E = \Big\{ u = \{u_j\}_{j=-\infty}^{\infty},\ u_j \in \mathbb{R},\ \sup_{j \in \mathbb{Z}} |u_j| < \infty \Big\}. \tag{1.2}$$

The right-hand side in (1.1) does not necessarily contain an odd number of summands. We use this form of the operator to simplify the presentation. We will use here the approaches developed for elliptic problems in unbounded domains [9, 10] and adapt them for infinite-dimensional difference operators.

Advances in Difference Equations 2005:1 (2005) 1–13
DOI: 10.1155/ADE.2005.1

The operator $L: E \to E$ defined in (1.1) can be regarded as $(Lu)_j = A_j U_j$, where

$$A_j = (a^j_{-m}, \dots, a^j_0, \dots, a^j_m), \qquad U_j = (u_{j-m}, \dots, u_j, \dots, u_{j+m}) \tag{1.3}$$

are $2m+1$-vectors, A_j is known, and U_j is variable.

We suppose that there exist the limits of the coefficients of the operator L as $j \to \pm\infty$

$$a^{\pm}_l = \lim_{j \to \pm\infty} a^j_l, \quad l \in \mathbb{Z},\ -m \le l \le m, \tag{1.4}$$

and $a^{\pm}_{\pm m} \neq 0$.

Denote by $L^{\pm}: E \to E$ the limiting operators

$$(L^{\pm}u)_j = a^{\pm}_{-m} u_{j-m} + \cdots + a^{\pm}_0 u_j + \cdots + a^{\pm}_m u_{j+m}, \quad j \in \mathbb{Z}. \tag{1.5}$$

Recall that a linear operator $L: E \to E$ is *normally solvable* if its image $\operatorname{Im} L$ is closed. If L is normally solvable with a finite-dimensional kernel and the codimension of its image is also finite, then L is called *Fredholm operator*. Denoting by $\alpha(L)$ and $\beta(L)$ the dimension of $\ker L$ and the codimension of $\operatorname{Im} L$, respectively, we can define *the index* $\kappa(L)$ of the operator L as $\kappa(L) = \alpha(L) - \beta(L)$. It is known that the index does not change under deformation in the class of Fredholm operators.

In Section 2 of this paper we introduce polynomials $P^+(\sigma)$ and $P^-(\sigma)$ associated with the limiting operators L^+ and L^-. We show that, if P^+ and P^- do not have roots on the unit circle, then the limiting operators are invertible and the operator L is normally solvable with a finite-dimensional kernel. If moreover the polynomials have the same number of roots inside the unit circle, then L is a Fredholm operator and its index is zero.

In Section 3 we prove that under some conditions on the polynomials P^+ and P^- corresponding to operator L in (1.1), the bounded solutions of the equation $Lu = 0$ are exponentially decreasing at $+\infty$ and $-\infty$. The idea is to approximate the equation $Lu = 0$ at $+\infty$ with the problem on half-axis:

$$\begin{aligned} &(L^+u)_j = 0, \quad j \ge 1, \\ &u_1 = a_1, \dots, u_k = a_k, \end{aligned} \tag{1.6}$$

and similarly at $-\infty$. We first prove that (1.6) has a unique solution and that this solution is exponentially decaying. Then we deduce that the solution of the equation $Lu = 0$ is also exponentially decaying.

Section 4 deals with the solvability conditions for the equation $Lu = f$, for L in (1.1) and $f = \{f_j\}_{j=-\infty}^{\infty} \in E$ being given.

Let L^* be the formally adjoint of L, $\alpha(L^*) = \dim(\ker L^*)$, and $v^l = \{v^l_j\}_{j=-\infty}^{\infty}$, $l = 1, \dots, \alpha(L^*)$ some linearly independent solutions of the equation $L^*v = 0$. One states a result which is analogous to the continuous case: equation $Lu = f$ is solvable if and only if f is orthogonal on all solutions v^l, $l = 1, \dots, \alpha(L^*)$.

Section 5 is devoted to a particular case of the operator L related to discretization of a second-order differential equation on the real axis:

$$(Lu)_j = u_{j+1} - 2u_j + u_{j-1} + c_j(u_{j+1} - u_j) + b_j u_j, \quad j \in \mathbb{Z}, \tag{1.7}$$

where $\{b_j\}_{j=-\infty}^{\infty}$ and $\{c_j\}_{j=-\infty}^{\infty}$ are given sequences of real numbers. If there exist the limits $b^{\pm} = \lim_{j\to\pm\infty} b_j < 0$, and $c^{\pm} = \lim_{j\to\pm\infty} c_j \geq -2$, then there are no roots of the polynomials on the unit circle, and there is exactly one root inside it. Therefore the bounded solution of the equation $Lv = 0$ is exponential decaying at $\pm\infty$. If moreover $c^{\pm} \in [-2,2]$, then the solvability conditions for the equation $Lu = f$ are applicable.

2. Limiting operators and normal solvability

Let E be the Banach space of all bounded real sequences $E = \{u = \{u_j\}_{j=-\infty}^{\infty},\ u_j \in \mathbb{R}, \sup_{j\in\mathbb{Z}} |u_j| < \infty\}$ with the norm

$$\|u\| = \sup_{j\in\mathbb{Z}} |u_j|, \tag{2.1}$$

and let $L: E \to E$ be the general linear difference operator $(Lu)_j = a_{-m}^j u_{j-m} + \cdots + a_0^j u_j + \cdots + a_m^j u_{j+m}$, $j \in \mathbb{Z}$, where $m \geq 0$ is an integer and $a_{-m}^j, \dots, a_0^j, \dots, a_m^j \in \mathbb{C}$ are given coefficients. Denote by $L^+ : E \to E$ the limiting operator

$$\left(L^+ u\right)_j = a_{-m}^+ u_{j-m} + \cdots + a_0^+ u_j + \cdots + a_m^+ u_{j+m}, \quad j \in \mathbb{Z}, \tag{2.2}$$

where

$$a_l^+ = \lim_{j\to\infty} a_l^j, \quad l \in \mathbb{Z},\ -m \leq l \leq m. \tag{2.3}$$

We are going to define the associated polynomial for the operator L^+. To do this, we are looking for the solution of the equation $L^+u = 0$ under the form $u_j = \exp(\mu j)$, $j \in \mathbb{Z}$, and obtain

$$a_{-m}^+ e^{-\mu m} + \cdots + a_{-1}^+ e^{-\mu} + a_0^+ + a_1^+ e^{\mu} + \cdots + a_m^+ e^{\mu m} = 0. \tag{2.4}$$

One takes $\sigma = e^{\mu}$ and finds the polynomial associated to L^+:

$$P^+(\sigma) = a_m^+ \sigma^{2m} + \cdots + a_0^+ \sigma^m + \cdots + a_{-m}^+. \tag{2.5}$$

Recall the following auxiliary result from [1].

Lemma 2.1. *The equation $L^+u = 0$ has nonzero bounded solutions if and only if the corresponding algebraic polynomial P^+ has a root σ with $|\sigma| = 1$.*

We will find conditions in terms of P^+ for the limiting operator L^+ to be invertible. One begins with an auxiliary result concerning continuous deformations of the polynomial P^+. Without loss of generality, we may assume that the coefficient $a_m^+ = 1$. Consider the polynomial with complex coefficients

$$P(\sigma) = \sigma^n + a_1\sigma^{n-1} + \cdots + a_{n-1}\sigma + a_n. \tag{2.6}$$

Lemma 2.2. *Suppose that a polynomial $P(\sigma)$ does not have roots with $|\sigma| = 1$ and it has k roots with $|\sigma| < 1$, $0 \leq k \leq n$. Then there exists a continuous deformation $P_\tau(\sigma)$, $0 \leq \tau \leq 1$,* *such that*

$$P_0(\sigma) = P(\sigma), \quad P_1(\sigma) = (\sigma^k - a)(\sigma^{n-k} - \lambda), \tag{2.7}$$

and the polynomial $P_\tau(\sigma)$ *does not have roots with* $|\sigma| = 1$ *for any* $0 \le \tau \le 1$. *Here* $\lambda > 1$ *and* $a < 1$ *are real numbers.*

Proof. We represent the polynomial $P(\sigma)$ in the form

$$P(\sigma) = (\sigma - \sigma_1) \cdots (\sigma - \sigma_n), \tag{2.8}$$

where the roots $\sigma_1, \ldots, \sigma_k$ are inside the unit circle, and the other roots are outside it. Consider the polynomial

$$P_\tau(\sigma) = (\sigma - \sigma_1(\tau)) \cdots (\sigma - \sigma_n(\tau)) \tag{2.9}$$

that depends on the parameter τ through its roots. This means that we change the roots and find the coefficients of the polynomial through them. We change the roots in such a way that for $\tau = 0$ they coincide with the roots of the original polynomial; for $\tau = 1$ it has the roots $\sigma_1, \ldots, \sigma_k$ with $(\sigma_i)^k = a$, $i = 1, \ldots, k$ (inside the unit circle) and $n-k$ roots $\sigma_{k+1}, \ldots, \sigma_n$ such that $(\sigma_i)^{n-k} = \lambda$, $i = k+1, \ldots, n$ (outside of the unit circle). This deformation can be done in such a way that there are no roots with $|\sigma| = 1$. The lemma is proved. □

Using the associated polynomials P^+ and P^- of L^+ and L^-, we can study the normal solvability of the operator L.

Theorem 2.3. *The operator* L *is normally solvable with a finite-dimensional kernel if and only if the corresponding algebraic polynomials* P^+ *and* P^- *do not have roots* σ *with* $|\sigma| = 1$.

Proof

The necessity. Suppose that the polynomials P^+, P^- do not have roots σ with $|\sigma| = 1$. We first show that the image of L is closed. To do this, let $\{f^n\}$ be a sequence in $\operatorname{Im} L$ such that $f^n \to f$ and let $\{u^n\}$ be a sequence with the property $Lu^n = f^n$.

Suppose in the beginning that $\{u^n\}$ is bounded in E. We construct a convergent subsequence. Since $\|u^n\| = \sup_{j\in\mathbb{Z}} |u_j^n| \le c$, then for every positive integer N, there exists a subsequence $\{u^{n_k}\}$ of $\{u^n\}$ and an element $u = \{u_j\}_{j=-N}^{N} \in E$ such that

$$\sup_{-N \le j \le N} \left| u_j^{n_k} - u_j \right| \longrightarrow 0, \tag{2.10}$$

that is, $u^{n_k} \to u$ as $k \to \infty$ uniformly on each bounded interval of j. Using a diagonalization process, we extend u_j to all $j \in \mathbb{Z}$.

It is clear that $\sup_{j\in\mathbb{Z}} |u_j| \le c$; that means $u \in E$. Passing to the limit as $k \to \infty$ in the linear equation $Lu^{n_k} = f^{n_k}$, we get $Lu = f$, so $f \in \operatorname{Im} L$.

We show that the convergence in (2.10) is uniform with respect to all $j \in \mathbb{Z}$. Supposing by contradiction that there exists $j_k \to \infty$ such that $|u_{j_k}^{n_k} - u_{j_k}| \geq \varepsilon > 0$, observe that the sequence $y^k = \{y_j^k\}_{j=-\infty}^{\infty}$, $y_j^k = u_{j+j_k}^{n_k} - u_{j+j_k}$ verifies the inequality $|y_0^k| = |u_{j_k}^{n_k} - u_{j_k}| \geq \varepsilon$ and the equation

$$a_{-m}^{j+j_k} y_{j-m}^k + \cdots + a_0^{j+j_k} y_j^k + \cdots + a_m^{j+j_k} y_{j+m}^k = f_{j+j_k}^{n_k} - f_{j+j_k}, \quad j \in \mathbb{Z}. \tag{2.11}$$

Since the sequence $\{y^k\}$ is bounded in E, there exists a subsequence $\{y^{k_l}\}$ which converges (say $y^{k_l} \to y^0$) uniformly with respect to j on bounded intervals. We may pass to the limit as $k_l \to \infty$ in (2.11) and obtain via (2.3),

$$a_{-m}^{+} y_{j-m}^0 + \cdots + a_0^{+} y_j^0 + \cdots + a_m^{+} y_{j+m}^0 = 0, \quad j \in \mathbb{Z}. \tag{2.12}$$

Thus, the limiting equation $L^+u = 0$ has a nonzero bounded solution $y^0 = \{y_j^0\}_{j=-\infty}^{\infty}$. Lemma 2.1 leads to a contradiction. Therefore the convergence $u_j^{n_k} - u_j \to 0$ is uniform with respect to all $j \in \mathbb{Z}$ and, since $Lu = f$, it follows that $\operatorname{Im} L$ is closed.

We analyze now the case when $\{u^n\}$ is unbounded in E. Then we write $u^n = x^n + y^n$ with $\{x^n\} \in \ker L$ and $\{y^n\}$ is in the supplement of $\ker L$. Then $Ly^n = f^n$.

If $\{y^n\}$ is bounded in E, it follows as above that $\operatorname{Im} L$ is closed. If not, then we repeat the above reasoning for $z^n = y^n/\|y^n\|$ and $g^n = f^n/\|y^n\|$. Passing to the limit on a subsequence n_k (such that $z^{n_k} \to z^0$) in the equality $Lz^{n_k} = g^{n_k}$ and using the convergence $g^{n_k} \to 0$, one obtains the contradiction that $z^0 \in \ker L$. Therefore $\operatorname{Im} L$ is closed.

In order to prove that $\ker L$ has a finite dimension, it suffices to show that every sequence u^n from $B \cap \ker L$ (where B is the unit ball) has a convergent subsequence. The reasoning is similar to that of the first part, taking $f^n = 0$.

The sufficiency. Assume that $\operatorname{Im} L$ is closed and $\dim(\ker L)$ is finite. By contradiction, one supposes that either P^+ or P^- (say P^+) has a root on the unit circle. Then the corresponding solution of $L^+u = 0$ has the form $u = \{u_j\}_{j=-\infty}^{\infty}$, where $u_j = e^{i\xi j}$, $\xi \in \mathbb{R}$, $j \in \mathbb{Z}$.

Let $\alpha = \{\alpha_j\}_{j=-\infty}^{\infty}$, $\beta^N = \{\beta_j^N\}_{j=-\infty}^{\infty}$, $\gamma^N = \{\gamma_j^N\}_{j=-\infty}^{\infty}$ be a partition of unity ($\alpha_j + \beta_j^N + \gamma_j^N = 1$) given by

$$\begin{gathered}
\alpha_j = \begin{cases} 1, & j \leq 0, \\ 0, & j \geq 1, \end{cases} \\
\beta_j^N = \begin{cases} 1, & 1 \leq j \leq N, \\ 0, & j \leq 0,\ j \geq N+1, \end{cases} \\
\gamma_j^N = \begin{cases} 1, & j \geq N+1, \\ 0, & j \leq N. \end{cases}
\end{gathered} \tag{2.13}$$

For a fixed $\varepsilon_n \to 0$ (as $n \to \infty$), let $u^n = \{u_j^n\}_{j=-\infty}^{\infty}$, $v^n = \{v_j^n\}_{j=-\infty}^{\infty}$, $f^n = \{f_j^n\}_{j=-\infty}^{\infty}$ be the sequences defined by $u_j^n = e^{i(\xi+\varepsilon_n)j}$, $v_j^n = (1 - \alpha_j)(u_j^n - u_j)$, and $f_j^n = Lv_j^n$, $j \in \mathbb{Z}$. It is clear that $u_j^n \to u_j$ $(n \to \infty)$ uniformly on every bounded interval of integers j.

It is sufficient to prove that $f^n \to 0$. Indeed, in this case, by hypothesis it follows that $v^n \to 0$. But this is in contradiction with

$$||v^n|| = \sup_{j>0} \left| e^{i(\xi+\varepsilon_n)j} - e^{i\xi j} \right| \geq m > 0 \tag{2.14}$$

for some m.

In order to show that $f^n \to 0$ as $n \to \infty$, observe that f_j^n can be written under the form

$$\begin{aligned} f_j^n &= (\alpha_j + \beta_j^N + \gamma_j^N)(L[(\beta^N + \gamma^N)(u^n - u)])_j \\ &= \alpha_j(L[(\beta^N + \gamma^N)(u^n - u)])_j + \beta_j^N(L[(\beta^N + \gamma^N)(u^n - u)])_j \\ &\quad + \gamma_j^N(L[\beta^N(u^n - u)])_j + \gamma_j^N((L - L^+)[\gamma^N(u^n - u)])_j \\ &\quad + \gamma_j^N(L^+[\gamma^N(u^n - u)])_j. \end{aligned} \tag{2.15}$$

A simple computation implies that the first three terms tend to zero as $n \to \infty$, uniformly with respect to all integers j.

Next, condition (2.3) and the boundedness $\|u^n\| = \|u\| = 1$ lead to the convergence

$$\left| \gamma_j^N((L - L^+)[\gamma^N(u^n - u)])_j \right| \leq \left| \gamma^N(L - L^+) \right|_0 \cdot ||\gamma^N(u^n - u)|| \longrightarrow 0, \tag{2.16}$$

as $N \to \infty$, where $|\cdot|_0$ is the norm of the operator. For a given N, one estimates the last term of (2.15). Since $u_j = e^{i\xi j}$, $j \in \mathbb{Z}$ is a solution of the equation $L^+u = 0$, then

$$\begin{aligned} (L^+(u^n - u))_j &= (L^+u^n)_j = (L^+u^n)_j - e^{i\varepsilon_n j}(L^+u)_j \\ &= e^{i(\xi+\varepsilon_n)j}\left[a_{-m}^+ e^{-i\xi m}(e^{-i\varepsilon_n m} - 1) + \cdots + a_{-1}^+ e^{-i\xi}(e^{-i\varepsilon_n} - 1)\right. \\ &\qquad \left. + a_1^+ e^{i\xi}(e^{i\varepsilon_n} - 1) + \cdots + a_m^+ e^{i\xi m}(e^{i\varepsilon_n m} - 1)\right], \end{aligned} \tag{2.17}$$

so

$$\begin{aligned} (L^+(u^n - u))_j = i\varepsilon_n e^{i(\xi+\varepsilon_n)j}&\left[a_{-m}^+(-m)e^{-i\xi m}e^{ib_{-m}} + \cdots - a_{-1}^+ e^{-i\xi}e^{ib_{-1}}\right. \\ &\left. + a_1^+ e^{i\xi}e^{ib_1} + \cdots + a_m^+ m e^{i\xi m}e^{ib_m}\right], \quad j \in \mathbb{Z}, \end{aligned} \tag{2.18}$$

where $b_{-m}, \ldots, b_{-1}, b_1, \ldots, b_m$ are intermediate points. Thus the last term in (2.15) goes to zero as $n \to \infty$ and therefore $f^n \to 0$. This completes the proof. □

Now we are ready to establish the invertibility of L^+.

Theorem 2.4. *If the operator L^+ is such that the corresponding polynomial does not have roots with $|\sigma| = 1$, then it is invertible.*

Proof. Lemma 2.2 for P^+ implies the existence of a continuous deformation $P_\tau(\sigma)$, $0 \leq \tau \leq 1$, from the polynomial $P_0 = P^+$ to $P_1(\sigma) = (\sigma^k - a)(\sigma^{2m-k} - \lambda)$ such that $P_\tau(\sigma)$ does not admit solutions with $|\sigma| = 1$. Here $\lambda > 1$, $a < 1$ are given. The operator which corresponds to P_1 is L_1^+ defined by

$$\left(L_1^+ u\right)_j = u_{j+k} - a u_j - \lambda u_{j+2k-2m} + a\lambda u_{j+k-2m}. \tag{2.19}$$

Indeed, looking for the solution of L_1^+ in the form $u_j = e^{\mu j}$, we arrive at

$$e^{\mu k} - a - \lambda e^{\mu(2k-2m)} + a\lambda e^{\mu(k-2m)} = 0. \tag{2.20}$$

We put $\sigma = e^{\mu}$ and get

$$\left(\sigma^k - a\right)\left(\sigma^{2m-k} - \lambda\right) = 0, \tag{2.21}$$

so P_1 is the above polynomial.

Taking $a = 1/\lambda$, we obtain

$$\left(L_1^+ u\right)_j = (Mu)_j - \frac{1}{\lambda} u_j, \tag{2.22}$$

where

$$(Mu)_j = u_{j+k} - \lambda u_{j+2k-m} + u_{j+k-2m} \tag{2.23}$$

is invertible for large $\lambda \geq 0$ (see, e.g., [1, Lemma 4.9]). Since L_1^+ is close to M (for $\lambda \geq 0$ large enough), one deduces that L_1^+ is also invertible. Hence the index of L_1^+ is zero.

Since the continuous deformation P_τ does not have solutions σ with $|\sigma| = 1$, we find that the corresponding continuous deformation of the operator L_τ^+ does not admit nonzero bounded solutions (see Lemma 2.1). By Theorem 2.3 one obtains that L_τ^+ is normally solvable with a finite-dimensional kernel. From the general theory of Fredholm operators, we know that the index of such homotopies does not change. Since the index of L_1^+ is $\kappa(L_1^+) = 0$, we deduce that $\kappa(L^+) = 0$. This, together with the fact that $\ker L^+ = \Phi$, implies $\operatorname{Im} L^+ = E$, therefore L^+ is invertible. The theorem is proved. □

Remark 2.5. An analogous result can be stated for L^-.

As a consequence, we may study the Fredholm property of L with the aid of the polynomials P^+ and P^-.

Corollary 2.6. *If the limiting operators L^+ and L^- for an operator L are such that the corresponding polynomials $P^+(\sigma)$ and $P^-(\sigma)$ do not have roots with $|\sigma| = 1$ and have the same number of roots inside the unit circle, then L is a Fredholm operator with the zero index.*

Proof. We construct a homotopy of L in such a way that L^+ and L^- are reduced independently to the operator in Theorem 2.4. Then, this homotopy is in the class of the normally solvable operators with finite-dimensional kernels.

Since at $+\infty$ and $-\infty$ the operators L^+ and L^- coincide, we finally reduce L to an operator with constant coefficients. According to Theorem 2.4, it is invertible. Therefore, L is a Fredholm operator and has the zero index, as claimed. □

3. Exponential decay

We consider now the problem

$$(L^+u)_j = 0, \quad j \geq 1, \tag{3.1}$$

assuming that the corresponding polynomial $P^+(\sigma)$ does not have roots with $|\sigma| = 1$ and has k roots with $|\sigma| < 1$. One associates to (3.1) the boundary conditions

$$u_1 = a_1, \dots, u_k = a_k, \tag{3.2}$$

with $a_1, \dots, a_k \in \mathbb{R}$ given.

Since there are k roots inside the unit circle, then there are k linearly independent solutions of the equation $L^+u = 0$ decaying as $j \to \infty$. Denote them by $u^1, \dots, u^k$. Consider their values for $j = 1, \dots, 2m$:

$$\begin{gathered} u_1^1, u_2^1, \dots, u_{2m}^1, \\ \vdots \\ u_1^k, u_2^k, \dots, u_{2m}^k. \end{gathered} \tag{3.3}$$

Each of these solutions is completely determined by the above values. Therefore the corresponding k vectors are linearly independent. Indeed, otherwise the solutions would have been linearly dependent. Therefore there exist k linearly independent columns. Without loss of generality we can assume that these are the first k columns. Hence the corresponding $k \times k$ matrix is invertible.

Any bounded solution of (3.1) can be represented in the form

$$u = c_1 u^1 + \cdots + c_k u^k. \tag{3.4}$$

Substituting it in (3.2), we uniquely determine the coefficients $c_1, \dots, c_k$. Therefore we have proved the following result.

Proposition 3.1. *If the corresponding polynomial $P^+(\sigma)$ for L^+ does not admit roots on the unit circle $|\sigma| = 1$ and has k roots with $|\sigma| < 1$, then for each $(a_1, \dots, a_k) \in \mathbb{R}^k$, problem (3.1)-(3.2) has a unique bounded solution. In addition, this solution is exponentially decreasing.*

This result holds also for L^-. Thus, for the solution of $Lu = 0$, we may conclude the following.

Theorem 3.2. *Suppose that the polynomials $P^+(\sigma)$ and $P^-(\sigma)$ corresponding to L^+ and L^-, respectively, do not have roots with $|\sigma| = 1$ and have the same number of roots with $|\sigma| < 1$. Then the bounded solutions of the equation $Lu = 0$ are exponentially decreasing at $\pm\infty$.*

Proof. Let $\tilde{u} = \{\tilde{u}_j\}_{j=-\infty}^{\infty}$ be a bounded solution of the equation $Lu = 0$. Consider the problem

$$(Lu)_j = 0, \quad u_{N+1} = a_1, \dots, u_{N+k} = a_k. \tag{3.5}$$

For N sufficiently large this problem is uniquely solvable for any $a_1,\dots,a_k$ since problem (3.1)-(3.2) is uniquely solvable, and the operator L is close to the operator L^+. If we put $a_i = \tilde{u}_{N+i}$, $i = 1,\dots,k$, then the solution of problem (3.5) coincides with $\tilde{u}$ for $j \geq N$. Therefore it is sufficient to prove that the solution of problem (3.5) is exponentially decreasing for any a_i, $i = 1,\dots,k$.

Consider the operator S of multiplication by $\exp(\mu\sqrt{1+j^2})$, that is,

$$(Su)_j = e^{\mu\sqrt{1+j^2}}u_j, \quad j = 0, \pm 1, \dots. \tag{3.6}$$

Here $\mu > 0$. Let $L_\mu = SLS^{-1}$. Then

$$(L_\mu u)_j = a^j_{-m}u_{j-m}e^{\mu(\sqrt{1+j^2}-\sqrt{1+(j-m)^2})} + \cdots + a^j_0 u_j + \cdots + a^j_m u_{j+m}e^{\mu(\sqrt{1+j^2}-\sqrt{1+(j+m)^2})}. \tag{3.7}$$

For μ sufficiently small, the operator L_μ is close to the operator L. Therefore the problem

$$(L_\mu v)_j = 0, \quad v_{N+1} = b_1,\dots,v_{N+k} = b_k \tag{3.8}$$

is uniquely solvable for any $b_1,\dots,b_k$.

If we put $b_i = \exp(\mu\sqrt{1+(N+i)^2})a_i$, $i = 1,\dots,k$, then the solution u of problem (3.5) can be expressed through the solution v of problem (3.8): $u = S^{-1}v$. Since v is bounded, then u is exponentially decreasing.

Thus we have proved that $\tilde{u}$ is exponentially decreasing as $j \to \infty$. Similarly it can be proved for $j \to -\infty$. The theorem is proved. □

4. Solvability conditions

In this section, we establish solvability conditions for the equation

$$Lu = f. \tag{4.1}$$

Here L is the operator in (1.1) and $f = \{f_j\}_{j=-\infty}^{\infty}$ is fixed in E.

For the operator L, denote $\alpha(L) = \dim(\ker L)$ and $\beta(L) = \operatorname{codim}(\operatorname{Im} L)$. If (u,v) is the inner product of two sequences $u = \{u_j\}_{j=-\infty}^{\infty}$, $v = \{v_j\}_{j=-\infty}^{\infty}$ in the sense l^2, that is,

$$(u,v) = \sum_{j=-\infty}^{\infty} u_j v_j, \tag{4.2}$$

then we may define the formally adjoint L^* of the operator L by the equality

$$(Lu,v) = (u,L^*v). \tag{4.3}$$

Let L^+, L^- and L_*^+, L_*^- be the limiting operators associated with L and L^*, respectively. We work under the following hypothesis:

(H) the polynomials P^+, P^- corresponding to L^+ and L^- do not have roots with $|\sigma| = 1$ and have the same number of roots with $|\sigma| < 1$. Similarly for the polynomials P_*^+ and P_*^- corresponding to L_*^+ and L_*^-.

Corollary 2.6 implies that L and L^* are Fredholm operators with the zero index.

LEMMA 4.1. *Under hypothesis (H), it holds that* $\beta(L) \geq \alpha(L^*)$.

Proof. By the definition of the Fredholm operator it follows that (4.1) is solvable for a given $f = \{f_j\}_{j=-\infty}^{\infty} \in E$ if and only if there exist linearly independent functionals $\varphi_k \in E^*$, $k = 1,\dots,\beta(L)$ such that

$$\varphi_k(f) = 0, \quad k = 1,\dots,\beta(L). \tag{4.4}$$

On the other hand consider the functionals ψ_l given by

$$\psi_l(f) = \sum_{j=-\infty}^{\infty} f_j v_j^l, \quad l = 1,\dots,\alpha(L^*), \tag{4.5}$$

where $v^l = \{v_j^l\}_{j=-\infty}^{\infty}$, $l = 1,\dots,\alpha(L^*)$ are linearly independent solutions of the homogeneous equation $L^*v = 0$. We know from Theorem 3.2 that the values v_j^l are exponentially decreasing with respect to j. Therefore the functionals ψ_l are well defined.

Obviously, ψ_l is linear for each l. If $f^{(n)} \to f$ in E (in the norm supremum), then we may pass to the limit in (4.5) under the sum to find that $\psi_l(f^{(n)}) \to \psi_l(f)$, as $n \to \infty$, for all $l = 1,\dots,\alpha(L^*)$, that is, ψ_l are continuous. Therefore $\psi_l \in E^*$, $l = 1,\dots,\alpha(L^*)$, where E^* denotes the dual space of E.

In order to prove that $\beta(L) \geq \alpha(L^*)$, suppose that it is not true. Then among the functionals ψ_l there exists at least one functional (say ψ_1) which is linearly independent with respect to all φ_k, $k = 1,\dots,\beta(L)$. This means that there exists $f \in E$ such that (4.4) holds, but

$$\psi_1(f) = \sum_{j=-\infty}^{\infty} f_j v_j^1 \neq 0. \tag{4.6}$$

From (4.4) it follows that (4.1) is solvable. We multiply it by v^1 and find $(Lu, v^1) = (f, v^1)$. By (4.6) observe that the right-hand side is different from zero. But since v^1 is a solution of the equation $L^*v = 0$, we deduce that $(Lu, v^1) = (u, L^*v^1) = 0$. The contradiction we arrive at, proves the lemma. □

Remark 4.2. Analogously we find $\beta(L^*) \geq \alpha(L)$. Therefore, if one denotes by $\kappa(L) = \alpha(L) - \beta(L)$ the index of the operator L, we get

$$\kappa(L) + \kappa(L^*) \leq 0. \tag{4.7}$$

Since in our case $\kappa(L) = \kappa(L^*) = 0$, it follows that

$$\beta(L) = \alpha(L^*), \qquad \beta(L^*) = \alpha(L). \tag{4.8}$$

Theorem 4.3. *Equation (4.1) is solvable if and only if*

$$\sum_{j=-\infty}^{\infty} f_j v_j^l = 0, \quad l = 1,\ldots,\alpha(L^*), \tag{4.9}$$

where $v^l = \{v_j^l\}_{j=-\infty}^{\infty}$, $l = 1,\ldots,\alpha(L^*)$ *are linearly independent solutions of the equation* $L^* v = 0$.

Proof. Equation (4.1) is solvable for a given $f \in E$ if and only if (4.4) holds for some functionals $\varphi_k \in E^*$, $k = 1,\ldots,\beta(L)$.

Consider the subspaces Φ and Ψ of E^* generated by the functionals φ_k, $k = 1,\ldots,\beta(L)$ and by ψ_l from (4.5), $l = 1,\ldots,\alpha(L^*)$, respectively. By (4.8) we deduce that their dimensions coincide. We show that actually $\Phi = \Psi$. We show first that $\Psi \subseteq \Phi$. Indeed, if not, there exists $\psi \in \Psi$, $\psi \notin \Phi$. Then there exists $f \in E$ such that (4.4) holds, but at least one $\psi_l(f) \neq 0$, so we get the same contradiction as in the proof of Lemma 4.1. Therefore, $\Psi \subseteq \Phi$ and since they have the same dimensions, we get that $\Psi = \Phi$. The theorem is proved. □

5. Example

We are concerned with the difference operator $L : E \to E$,

$$\begin{aligned}(Lu)_j &= u_{j+1} - 2u_j + u_{j-1} + c_j(u_{j+1} - u_j) + b_j u_j \\ &= (1 + c_j)u_{j+1} + (b_j - c_j - 2)u_j + u_{j-1}, \quad j \in \mathbb{Z},\end{aligned} \tag{5.1}$$

where b_j, c_j are real sequences. It is a discretization of the second-order differential equation on the real axis. Here we find sufficient conditions on the coefficients b_j, c_j in order to establish the Fredholm property and the solvability conditions.

Suppose that there exist the limits

$$b^{\pm} = \lim_{j\to\pm\infty} b_j < \infty, \qquad c^{\pm} = \lim_{j\to\pm\infty} c_j < \infty. \tag{5.2}$$

Let $L^* : E \to E$ be the formally adjoint operator of L,

$$\begin{aligned}(L^* u)_j &= u_{j+1} - 2u_j + u_{j-1} - c_j(u_{j+1} - u_j) + b_j u_j \\ &= (1 - c_j)u_{j+1} + (b_j + c_j - 2)u_j + u_{j-1}, \quad j \in \mathbb{Z}.\end{aligned} \tag{5.3}$$

Denote by $L^{\pm}$ and $L_*^{\pm}$ the operators with constant coefficients

$$\begin{aligned}(L^{\pm} u)_j &= (1 + c^{\pm})u_{j+1} + (b^{\pm} - c^{\pm} - 2)u_j + u_{j-1}, \quad j \in \mathbb{Z}, \\ (L_*^{\pm} u)_j &= (1 - c^{\pm})u_{j+1} + (b^{\pm} + c^{\pm} - 2)u_j + u_{j-1}, \quad j \in \mathbb{Z}.\end{aligned} \tag{5.4}$$

They are the limiting operators associated with L and L^*, respectively. One defines the polynomials P^+, P^- corresponding to L^+, L^- and the polynomials P_*^+, P_*^- corresponding to L_*^+, L_*^-. To this end, we search for the solution of the equation $L^+u = 0$ in the form $u_j = e^{\mu j}$. Denoting $\sigma = e^{\mu}$, one obtains the polynomial P^+ corresponding to L^+,

$$P^+(\sigma) = (1+c^+)\sigma^2 + (b^+ - c^+ - 2)\sigma + 1. \tag{5.5}$$

Analogously, we have

$$\begin{aligned} P^-(\sigma) &= (1+c^-)\sigma^2 + (b^- - c^- - 2)\sigma + 1, \\ P_*^+(\sigma) &= (1-c^+)\sigma^2 + (b^+ + c^+ - 2)\sigma + 1, \\ P_*^-(\sigma) &= (1-c^-)\sigma^2 + (b^- + c^- - 2)\sigma + 1. \end{aligned} \tag{5.6}$$

We first show that, under some certain conditions on the coefficients, the polynomial

$$P(\sigma) = (1+c)\sigma^2 + (b-c-2)\sigma + 1 \tag{5.7}$$

does not have solutions σ with $|\sigma| = 1$.

Lemma 5.1. *If $b < 0$, $c \geq -2$, then $P(\sigma)$ does not admit roots on the unit circle and admits one and only one root inside the unit circle.*

Proof. If $c = -1$, we get $P(\sigma) = (b-1)\sigma + 1$, so in view of hypothesis $b < 0$, one finds that the root $\sigma = 1/(1-b)$ belongs to the unit circle, that is, to the interval $(-1,1)$.

Assume now $c \neq -1$. Then we may divide the equation $P(\sigma) = 0$ by $1+c$ to obtain

$$\sigma^2 + p\sigma + q = 0 \quad \text{with } p = \frac{b-c-2}{1+c},\ q = \frac{1}{1+c}. \tag{5.8}$$

Since $\Delta = p^2 - 4q = (b-c)^2 - 4b/(1+c)^2 > 0$, (5.8) has two different real roots

$$\sigma_1 = \frac{-p+\sqrt{p^2-4q}}{2}, \qquad \sigma_2 = \frac{-p-\sqrt{p^2-4q}}{2}. \tag{5.9}$$

Case 1 ($c > -1$). From $b < 0$, $c > -1$, we derive $q > 0$ and $p + q + 1 < 0$. Consequently $p < -1$. A simple computation leads to the conclusion that $\sigma_1 > 1$ and $\sigma_2 \in (-1,1)$.
Case 2 ($-2 \leq c < -1$). In this case, $q \leq -1$ and $p + q + 1 > 0$. This implies that $p > 0$. Hence $\sigma_1 \in (-1,1)$ and $\sigma_2 < -1$.

The lemma is proved. □

Applying Theorem 2.3, Theorem 2.4, Corollary 2.6, and Theorem 3.2, we deduce with the aid of Lemma 5.1 for the polynomials P^+ and P^-, some properties of the operator L given in (5.1).

Proposition 5.2. *If $b^+ < 0$ and $c^+ \geq -2$, then the limiting operator L^+ does not have nonzero bounded solutions. In addition, it is invertible.*

An analogous result can be deduced for L^-.

Proposition 5.3. *If $b^+, b^- < 0$ and $c^+, c^- \geq -2$, then L is a Fredholm operator and its index is zero.*

Proposition 5.4. *Assume that* $b^+, b^- < 0$ *and* $c^+, c^- \geq -2$. *Then every bounded solution of the equation* $Lu = 0$ *is exponentially decreasing at* $\pm\infty$.

Similar results can be obtained for the adjoint operator L^* of L, defined in (5.3), under the hypotheses that $b^+, b^- < 0$ and $c^+, c^- \leq 2$.

Now we are ready to find the solvability conditions for the equation $Lu = f$. Denote $\alpha(L^*) = \dim(\ker L^*)$.

Proposition 5.5. *Let* $b^+, b^- < 0$ *and* $c^+, c^- \in [-2, 2]$. *Then the equation* $Lu = f$ *is solvable for a given* $f \in E$, *if and only if*

$$\sum_{j=-\infty}^{\infty} f_j v_j^l = 0, \quad l = 1, \ldots, \alpha(L^*), \tag{5.10}$$

where $v^l = \{v_j^l\}_{j=-\infty}^{\infty}$, $l = 1, \ldots, \alpha(L^*)$ *are linearly independent solutions of the equation* $L^* v = 0$.

References

[1] N. C. Apreutesei and V. A. Volpert, *Some properties of infinite dimensional discrete operators*, Topol. Methods Nonlinear Anal. **24** (2004), no. 1, 159–182.

[2] L. G. Arabajian and M. T. Hakopian, *On one infinite algebraic system with a Toeplitz matrix*, J. Integral Equations Math. Phys. **1** (1992), no. 1, 32–43.

[3] K. N. Bagisbaev and A. B. Bazarbekov, *Solvability of a class of infinite algebraic systems*, Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat. **153** (1990), no. 2, 79–81 (Russian).

[4] P. W. Bates, K. Lu, and B. Wang, *Attractors for lattice dynamical systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **11** (2001), no. 1, 143–153.

[5] S. Goldberg, *Introduction to Difference Equations*, Dover Publications, New York, 1986.

[6] A. S. Kosmodamianskiĭ and V. I. Khrestenko, *A proof of the normality of infinite algebraic systems obtained in the solution of two-dimensional problems of elasticity theory*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **1985** (1985), no. 5, 33–35 (Russian).

[7] M. Pituk, *More on Poincaré's and Perron's theorems for difference equations*, J. Differ. Equations Appl. **8** (2002), no. 3, 201–216.

[8] A. P. Shilin, *An infinite algebraic system with a nearly piecewise-Toeplitz matrix*, Vestsī Akad. Navuk Belarusī Ser. Fīz.-Mat. Navuk **1995** (1995), no. 3, 36–40 (Russian).

[9] V. A. Volpert and A. I. Volpert, *Properness and topological degree for general elliptic operators*, Abstr. Appl. Anal. **2003** (2003), no. 3, 129–181.

[10] V. A. Volpert, A. I. Volpert, and J. F. Collet, *Topological degree for elliptic operators in unbounded cylinders*, Adv. Differential Equations **4** (1999), no. 6, 777–812.

N. C. Apreutesei: Department of Mathematics, "Gh. Asachi" Technical University of Iasi, 700506 Iasi, Romania

E-mail address: napreut@math.tuiasi.ro

V. A. Volpert: Laboratoire de Mathématiques Appliquées, Université Claude Bernard Lyon 1, 69622 Villeurbanne, France

E-mail address: volpert@maply.univ-lyon1.fr