

# HYERS-ULAM STABILITY OF THE LINEAR RECURRENCE WITH CONSTANT COEFFICIENTS

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Let  $X$  be a Banach space over the field  $\mathbb{R}$  or  $\mathbb{C}$ ,  $a_1, \dots, a_p \in \mathbb{C}$ , and  $(b_n)_{n \geq 0}$  a sequence in  $X$ . We investigate the Hyers-Ulam stability of the linear recurrence  $x_{n+p} = a_1 x_{n+p-1} + \dots + a_{p-1} x_{n+1} + a_p x_n + b_n$ ,  $n \geq 0$ , where  $x_0, x_1, \dots, x_{p-1} \in X$ .

## 1. Introduction

In 1940, S. M. Ulam proposed the following problem.

**PROBLEM 1.1.** *Given a metric group  $(G, \cdot, d)$ , a positive number  $\varepsilon$ , and a mapping  $f : G \rightarrow G$  which satisfies the inequality  $d(f(xy), f(x)f(y)) \leq \varepsilon$  for all  $x, y \in G$ , do there exist an automorphism  $a$  of  $G$  and a constant  $\delta$  depending only on  $G$  such that  $d(a(x), f(x)) \leq \delta$  for all  $x \in G$ ?*

If the answer to this question is affirmative, we say that the equation  $a(xy) = a(x)a(y)$  is stable. A first answer to this question was given by Hyers [5] in 1941 who proved that the Cauchy equation is stable in Banach spaces. This result represents the starting point theory of Hyers-Ulam stability of functional equations. Generally, we say that a functional equation is stable in Hyers-Ulam sense if for every solution of the perturbed equation, there exists a solution of the equation that differs from the solution of the perturbed equation with a small error. In the last 30 years, the stability theory of functional equations was strongly developed. Recall that very important contributions to this subject were brought by Forti [2], Găvruta [3], Ger [4], Páles [6, 7], Székelyhidi [9], Rassias [8], and Trif [10]. As it is mentioned in [1], there are much less results on stability for functional equations in a single variable than in more variables, and no surveys on this subject. In our paper, we will investigate the discrete case for equations in single variable, namely, the Hyers-Ulam stability of linear recurrence with constant coefficients.

Let  $X$  be a Banach space over a field  $K$  and

$$x_{n+p} = f(x_{n+p-1}, \dots, x_n), \quad n \geq 0, \quad (1.1)$$

a recurrence in  $X$ , when  $p$  is a positive integer,  $f : X^p \rightarrow X$  is a mapping, and  $x_0, x_1, \dots, x_{p-1} \in X$ . We say that the recurrence (1.1) is stable in Hyers-Ulam sense if for every positive  $\varepsilon$

and every sequence  $(x_n)_{n \geq 0}$  that satisfies the inequality

$$\|x_{n+p} - f(x_{n+p-1}, \dots, x_n)\| < \varepsilon, \quad n \geq 0, \quad (1.2)$$

there exist a sequence  $(y_n)_{n \geq 0}$  given by the recurrence (1.1) and a positive  $\delta$  depending only on  $f$  such that

$$\|x_n - y_n\| < \delta, \quad n \geq 0. \quad (1.3)$$

In [7], the author investigates the Hyers-Ulam-Rassias stability of the first-order linear recurrence in a Banach space. Using some ideas from [7] in this paper, one obtains a result concerning the stability of the  $n$ -order linear recurrence with constant coefficients in a Banach space, namely,

$$x_{n+p} = a_1 x_{n+p-1} + \dots + a_{p-1} x_{n+1} + a_p x_n + b_n, \quad n \geq 0, \quad (1.4)$$

where  $a_1, a_2, \dots, a_p \in K$ ,  $(b_n)_{n \geq 0}$  is a given sequence in  $X$ , and  $x_0, x_1, \dots, x_{p-1} \in X$ . Many new and interesting results concerning difference equations can be found in [1].

## 2. Main results

In what follows, we denote by  $K$  the field  $\mathbb{C}$  of complex numbers or the field  $\mathbb{R}$  of real numbers. Our stability result is based on the following lemma.

LEMMA 2.1. *Let  $X$  be a Banach space over  $K$ ,  $\varepsilon$  a positive number,  $a \in K \setminus \{-1, 0, 1\}$ , and  $(a_n)_{n \geq 0}$  a sequence in  $X$ . Suppose that  $(x_n)_{n \geq 0}$  is a sequence in  $X$  with the following property:*

$$\|x_{n+1} - ax_n - a_n\| \leq \varepsilon, \quad n \geq 0. \quad (2.1)$$

Then there exists a sequence  $(y_n)_{n \geq 0}$  in  $X$  satisfying the relations

$$y_{n+1} = ay_n + a_n, \quad n \geq 0, \quad (2.2)$$

$$\|x_n - y_n\| \leq \frac{\varepsilon}{|a| - 1}, \quad n \geq 0. \quad (2.3)$$

*Proof.* Denote  $x_{n+1} - ax_n - a_n := b_n$ ,  $n \geq 0$ . By induction, one obtains

$$x_n = a^n x_0 + \sum_{k=0}^{n-1} a^{n-k-1} (a_k + b_k), \quad n \geq 1. \quad (2.4)$$

(1) Suppose that  $|a| < 1$ . Define the sequence  $(y_n)_{n \geq 0}$  by the relation (2.2) with  $y_0 = x_0$ . Then it follows by induction that

$$y_n = a^n x_0 + \sum_{k=0}^{n-1} a^{n-k-1} b_k, \quad n \geq 1. \quad (2.5)$$

By the relation (2.4) and (2.5), one gets

$$\begin{aligned} \|x_n - y_n\| &\leq \left\| \sum_{k=0}^{n-1} b_k a^{n-k-1} \right\| \leq \sum_{k=0}^{n-1} \|b_k\| |a|^{n-k-1} \\ &\leq \varepsilon \frac{1 - |a|^n}{1 - |a|} < \frac{\varepsilon}{1 - |a|}, \quad n \geq 1. \end{aligned} \tag{2.6}$$

(2) If  $|a| > 1$ , by using the comparison test, it follows that the series  $\sum_{n=1}^{\infty} (b_{n-1}/a^n)$  is absolutely convergent, since

$$\begin{aligned} \left\| \frac{b_{n-1}}{a^n} \right\| &\leq \frac{\varepsilon}{|a|^n}, \quad n \geq 1, \\ \sum_{n=1}^{\infty} \frac{\varepsilon}{|a|^n} &= \frac{\varepsilon}{|a| - 1}. \end{aligned} \tag{2.7}$$

Denoting

$$s := \sum_{n=1}^{\infty} \frac{b_{n-1}}{a^n}, \tag{2.8}$$

we define the sequence  $(y_n)_{n \geq 0}$  by the relation (2.2) with  $y_0 = x_0 + s$ .

Then one obtains

$$\begin{aligned} \|x_n - y_n\| &\left\| -a^n s + \sum_{k=0}^{n-1} b_k a^{n-k-1} \right\| = |a|^n \left\| -s + \sum_{k=0}^{n-1} \frac{b_k}{a^{k+1}} \right\| \\ &= |a|^n \left\| \sum_{k=n}^{\infty} \frac{b_k}{a^{k+1}} \right\| \\ &\leq \varepsilon \sum_{n=1}^{\infty} \frac{1}{|a|^n} = \frac{\varepsilon}{|a| - 1}, \quad n \geq 0. \end{aligned} \tag{2.9}$$

The lemma is proved. □

*Remark 2.2.* (1) If  $|a| > 1$ , then the sequence  $(y_n)_{n \geq 0}$  from Lemma 2.1 is uniquely determined.

(2) If  $|a| < 1$ , then there exists an infinite number of sequences  $(y_n)_{n \geq 0}$  in Lemma 2.1 that satisfy (2.2) and (2.3).

*Proof.* (1) Suppose that there exists another sequence  $(y_n)_{n \geq 0}$  defined by (2.2),  $y_0 \neq x_0 + s$ , that satisfies (2.3). Hence,

$$\|x_n - y_n\| \left\| a^n(x_0 - y_0) + \sum_{k=0}^{n-1} b_k a^{n-k-1} \right\| = |a|^n \left\| x_0 - y_0 + \sum_{k=0}^{n-1} \frac{b_k}{a^{k+1}} \right\|, \quad n \geq 1. \tag{2.10}$$

Since

$$\lim_{n \rightarrow \infty} \left\| x_0 - y_0 + \sum_{k=0}^{n-1} \frac{b_k}{a^{k+1}} \right\| = \|x_0 + s - y_0\| \neq 0, \tag{2.11}$$

it follows that

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = \infty. \quad (2.12)$$

(2) If  $|a| < 1$ , one can choose  $y_0 = x_0 + u$ ,  $\|u\| \leq \varepsilon$ . Then

$$\begin{aligned} \|x_n - y_n\| &= \left\| -a^n u + \sum_{k=0}^{n-1} b_k a^{n-k-1} \right\| \leq \varepsilon \sum_{k=0}^n |a|^k \\ &= \varepsilon \frac{1 - |a|^{n+1}}{1 - |a|} \leq \frac{\varepsilon}{1 - |a|}, \quad n \geq 1. \end{aligned} \quad (2.13)$$

□

The stability result for the  $p$ -order linear recurrence with constant coefficients is contained in the next theorem.

**THEOREM 2.3.** *Let  $X$  be a Banach space over the field  $K$ ,  $\varepsilon > 0$ , and  $a_1, a_2, \dots, a_p \in K$  such that the equation*

$$r^p - a_1 r^{p-1} - \dots - a_{p-1} r - a_p = 0 \quad (2.14)$$

*admits the roots  $r_1, r_2, \dots, r_p$ ,  $|r_k| \neq 1$ ,  $1 \leq k \leq p$ , and  $(b_n)_{n \geq 0}$  is a sequence in  $X$ . Suppose that  $(x_n)_{n \geq 0}$  is a sequence in  $X$  with the property*

$$\|x_{n+p} - a_1 x_{n+p-1} - \dots - a_{p-1} x_{n+1} - a_p x_n - b_n\| \leq \varepsilon, \quad n \geq 0. \quad (2.15)$$

*Then there exists a sequence  $(y_n)_{n \geq 0}$  in  $X$  given by the recurrence*

$$y_{n+p} = a_1 y_{n+p-1} + \dots + a_{p-1} y_{n+1} + a_p y_n + b_n, \quad n \geq 0, \quad (2.16)$$

*such that*

$$\|x_n - y_n\| \leq \frac{\varepsilon}{|(r_1| - 1) \cdots (|r_p| - 1)}, \quad n \geq 0. \quad (2.17)$$

*Proof.* We prove Theorem 2.3 by induction on  $p$ .

For  $p = 1$ , the conclusion of Theorem 2.3 is true in virtue of Lemma 2.1. Suppose now that Theorem 2.3 holds for a fixed  $p \geq 1$ . We have to prove the following assertion.

**ASSERTION 2.4.** *Let  $\varepsilon$  be a positive number and  $a_1, a_2, \dots, a_{p+1} \in K$  such that the equation*

$$r^{p+1} - a_1 r^p - \dots - a_p r - a_{p+1} = 0 \quad (2.18)$$

*admits the roots  $r_1, r_2, \dots, r_{p+1}$ ,  $|r_k| \neq 1$ ,  $1 \leq k \leq p+1$ , and  $(b_n)_{n \geq 0}$  is a sequence in  $X$ . If  $(x_n)_{n \geq 0}$  is a sequence in  $X$  satisfying the relation*

$$\|x_{n+p+1} - a_1 x_{n+p} - \dots - a_p x_{n+1} - a_{p+1} x_n - b_n\| \leq \varepsilon, \quad n \geq 0, \quad (2.19)$$

*then there exists a sequence  $(y_n)_{n \geq 0}$  in  $X$ , given by the recurrence*

$$y_{n+p+1} = a_1 y_{n+p} + \dots + a_p y_{n+1} + a_{p+1} y_n + b_n, \quad n \geq 0, \quad (2.20)$$

such that

$$\|x_n - y_n\| \leq \frac{\varepsilon}{|(|r_1| - 1) \cdots (|r_{p+1}| - 1)|}, \quad n \geq 0. \quad (2.21)$$

The relation (2.19) can be written in the form

$$\|x_{n+p+1} - (r_1 + \cdots + r_{p+1})x_{n+p} - \cdots + (-1)^{p+1}r_1 \cdots r_{p+1}x_n - b_n\| \leq \varepsilon, \quad n \geq 0. \quad (2.22)$$

Denoting  $x_{n+1} - r_{p+1}x_n = u_n, n \geq 0$ , one gets by (2.22)

$$\|u_{n+p} - (r_1 + \cdots + r_p)u_{n+p-1} + \cdots + (-1)^p r_1 r_2 \cdots r_p u_n - b_n\| \leq \varepsilon, \quad n \geq 0. \quad (2.23)$$

By using the induction hypothesis, it follows that there exists a sequence  $(z_n)_{n \geq 0}$  in  $X$ , satisfying the relations

$$z_{n+p} = a_1 z_{n+p-1} + \cdots + a_p z_n + b_n, \quad n \geq 0, \quad (2.24)$$

$$\|u_n - z_n\| \leq \frac{\varepsilon}{|(|r_1| - 1) \cdots (|r_p| - 1)|}, \quad n \geq 0. \quad (2.25)$$

Hence

$$\|x_{n+1} - r_{p+1}x_n - z_n\| \leq \frac{\varepsilon}{|(|r_1| - 1) \cdots (|r_p| - 1)|}, \quad n \geq 0, \quad (2.26)$$

and taking account of Lemma 2.1, it follows from (2.26) that there exists a sequence  $(y_n)_{n \geq 0}$  in  $X$ , given by the recurrence

$$y_{n+1} = r_{p+1}y_n + z_n, \quad n \geq 0, \quad (2.27)$$

that satisfies the relation

$$\|x_n - y_n\| \leq \frac{\varepsilon}{|(|r_1| - 1) \cdots (|r_{p+1}| - 1)|}, \quad n \geq 0. \quad (2.28)$$

By (2.24) and (2.27), one gets

$$y_{n+p+1} = a_1 y_{n+p} + \cdots + a_{p+1} y_n + b_n, \quad n \geq 0. \quad (2.29)$$

The theorem is proved. □

*Remark 2.5.* If  $|r_k| > 1, 1 \leq k \leq p$ , in Theorem 2.3, then the sequence  $(y_n)_{n \geq 0}$  is uniquely determined.

*Proof.* The proof follows from Remark 2.2. □

*Remark 2.6.* If there exists an integer  $s, 1 \leq s \leq p$ , such that  $|r_s| = 1$ , then the conclusion of Theorem 2.3 is not generally true.

*Proof.* Let  $\varepsilon > 0$ , and consider the sequence  $(x_n)_{n \geq 0}$ , given by the recurrence

$$x_{n+2} + x_{n+1} - 2x_n = \varepsilon, \quad n \geq 0, x_0, x_1 \in K. \quad (2.30)$$

A particular solution of this recurrence is

$$x_n = \frac{\varepsilon}{3}n, \quad n \geq 0, \quad (2.31)$$

hence the general solution of the recurrence is

$$x_n = \alpha + \beta(-2)^n + \frac{\varepsilon}{3}n, \quad n \geq 0, \alpha, \beta \in K. \quad (2.32)$$

Let  $(y_n)_{n \geq 0}$  be a sequence satisfying the recurrence

$$y_{n+2} + y_{n+1} - 2y_n = 0, \quad n \geq 0, y_0, y_1 \in K. \quad (2.33)$$

Then  $y_n = \gamma + \delta(-2)^n$ ,  $n \geq 0$ ,  $\gamma, \delta \in K$ , and

$$\sup_{n \in \mathbb{N}} |x_n - y_n| = \infty. \quad (2.34)$$

□

*Example 2.7.* Let  $X$  be a Banach space and  $\varepsilon$  a positive number. Suppose that  $(x_n)_{n \geq 0}$  is a sequence in  $X$  satisfying the inequality

$$\|x_{n+2} - x_{n+1} - x_n\| \leq \varepsilon, \quad n \geq 0. \quad (2.35)$$

Then there exists a sequence  $(f_n)_{n \geq 0}$  in  $X$  given by the recurrence

$$f_{n+2} - f_{n+1} - f_n = 0, \quad n \geq 0, \quad (2.36)$$

such that

$$\|x_n - f_n\| \leq (2 + \sqrt{5})\varepsilon, \quad n \geq 0. \quad (2.37)$$

*Proof.* The equation  $r^2 - r - 1 = 0$  has the roots  $r_1 = (1 + \sqrt{5})/2$ ,  $r_2 = (1 - \sqrt{5})/2$ . By the Theorem 2.3, it follows that there exists a sequence  $(f_n)_{n \geq 0}$  in  $X$  such that

$$\|x_n - f_n\| \leq \frac{\varepsilon}{|(|r_1| - 1)(|r_2| - 1)|} = (2 + \sqrt{5})\varepsilon, \quad n \geq 0. \quad (2.38)$$

□

## References

- [1] R. P. Agarwal, *Difference Equations and Inequalities. Theory, Methods, and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 228, Marcel Dekker, New York, 2000.
- [2] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, *Aequationes Math.* **50** (1995), no. 1-2, 143–190.
- [3] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, *J. Math. Anal. Appl.* **184** (1994), no. 3, 431–436.

- [4] R. Ger, *A survey of recent results on stability of functional equations*, Proc. of the 4th International Conference on Functional Equations and Inequalities (Cracow), Pedagogical University of Cracow, Poland, 1994, pp. 5–36.
- [5] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.
- [6] Z. Páles, *Generalized stability of the Cauchy functional equation*, Aequationes Math. **56** (1998), no. 3, 222–232.
- [7] ———, *Hyers-Ulam stability of the Cauchy functional equation on square-symmetric groupoids*, Publ. Math. Debrecen **58** (2001), no. 4, 651–666.
- [8] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300.
- [9] L. Székelyhidi, *Note on Hyers's theorem*, C. R. Math. Rep. Acad. Sci. Canada **8** (1986), no. 2, 127–129.
- [10] T. Trif, *On the stability of a general gamma-type functional equation*, Publ. Math. Debrecen **60** (2002), no. 1-2, 47–61.

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