POWER SERIES TECHNIQUES FOR A SPECIAL SCHRÖDINGER OPERATOR AND RELATED DIFFERENCE EQUATIONS

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We address finding solutions $y \in \mathscr{C}^2(\mathbb{R}^+)$ of the special (linear) ordinary differential equation $xy''(x) + (ax^2 + b)y'(x) + (cx + d)y(x) = 0$ for all $x \in \mathbb{R}^+$, where $a, b, c, d \in \mathbb{R}$ are constant parameters. This will be achieved in three special cases via separation and a power series method which is specified using difference equation techniques. Moreover, we will prove that our solutions are square integrable in a weighted sense—the weight function being similar to the Gaussian bell e^{-x^2} in the scenario of Hermite polynomials. Finally, we will discuss the physical relevance of our results, as the differential equation is also related to basic problems in quantum mechanics.

1. Motivation via quantum mechanics

In quantum mechanics, when considering the two-dimensional hydrogen atom in a strong magnetic field, one obtains the following radial Schrödinger equation for the "radial wave function" ψ of the electron:

$$\psi''(x) + \frac{1}{x}\psi'(x) + \left(E + m\lambda + \frac{2Z}{x} - \frac{m^2}{x^2} - \frac{\lambda^2}{4}x^2\right)\psi(x) = 0 \quad \forall x \in \mathbb{R}^+. \tag{1.1}$$

This equation is obtained by standard separation methods—decomposing the wave function into a radial and an angular part—as they are taught in every first course on quantum mechanics, see for instance [5]. For a more analytic approach to the subject, see for instance [4]. We briefly describe the motivation physicists gave us (cf. [2, 3]) to consider further analytic properties of the differential equation (1.1).

In quantum physics, the problem of hydrogen atoms in strong magnetic fields has a particular meaning: one motivation comes from experimental physics, that is, from atomic spectroscopy—one would like to understand the spectra of highly excited hydrogen atoms in the so-called superstrong magnetic fields. But also in astrophysics, spectra of hydrogen in strong magnetic fields play a role, for example, in the strongly magnetic white dwarf stars. In all of these physically interesting situations, the mathematical models behind are related to the differential equation (1.1). We mention that the occurring objects in (1.1) have the following interpretation in quantum mechanics.

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(i) x is the dimensionless radial distance (in the plane).
(ii) ψ gives information on the probability density for measuring a particle signal.
(iii) E is the dimensionless total energy.
(iv) m is the canonical angular momentum quantum number (integer!).
(v) Z is the integer multiple of the elementary charge e_0 of the nucleus.
(vi) λ is the dimensionless strength of the magnetic field.

Paying attention to the fact that the wave function ψ must yield a probability density $|\psi|^2$, we obtain the so-called Schrödinger boundary condition

$$\int_0^\infty |\psi(x)|^2 x\,dx < \infty. \tag{1.2}$$

In [3], Robnik and Romanovski make use of the separation step

$$\psi(x) \equiv x^{|m|} e^{-\lambda x^2/4} y(x). \tag{1.3}$$

The above linear differential equation then reduces to

$$y''(x) + \left(\frac{2|m|+1}{x} - \lambda x\right) y'(x) + \left[E + \lambda(m - |m| - 1) + \frac{2Z}{x}\right] y(x) = 0. \tag{1.4}$$

Multiplying this equation by x results in a differential equation of the type

$$xy''(x) + (ax^2 + b)y'(x) + (cx + d)y(x) = 0 \quad \forall x \in \mathbb{R}^+, \tag{1.5}$$

where we used the abbreviations

$$a = -\lambda, \qquad b = 2|m| + 1, \qquad c = E + \lambda(m - |m| - 1), \qquad d = 2Z. \tag{1.6}$$

Suppressing the arguments, (1.5) simply reads

$$xy'' + (ax^2 + b)y' + (cx + d)y = 0. \tag{1.7}$$

Motivated by (1.2), we furthermore want to see in which (weighted) sense the solutions of (1.7) are square integrable.

2. Analytically solvable special cases

When trying to solve (1.7), we might first notice that we should rather appreciate a second-order differential equation of the "Bessel-like" type

$$x^2 y'' + xy' + p(x)y = 0, \tag{2.1}$$

where $p(x)$ is a polynomial in x. But that is exactly what we obtain when we multiply (1.1) by x^2. Thus, our first step towards solving (1.7) is indeed to *undo* the separation step made in [3] via

$$y(x) \equiv x^{(1-b)/2} e^{-ax^2/4} \psi(x) = x^{-|m|} e^{\lambda x^2/4} \psi(x), \tag{2.2}$$

recovering the following Bessel-like equation for the wave function ψ:

$$x^2\psi'' + x\psi' + \left[-\frac{a^2}{4}x^4 + \left(c - \frac{a(b+1)}{2}\right)x^2 + dx - \frac{(b-1)^2}{4}\right]\psi = 0. \tag{2.3}$$

(In detail, (2.3) is either obtained by inserting the "undoing" step into (1.7) directly, or by substituting a, b, c, d into (1.1)—multiplied by x^2.) The polynomial $p(x)$ in (2.1) is now given by

$$p(x) = -\frac{a^2}{4}x^4 + \left(c - \frac{a(b+1)}{2}\right)x^2 + dx - \frac{(b-1)^2}{4}. \tag{2.4}$$

Remark 2.1. At this point, one might suspect that the whole separation procedure was in vain. Anyway, this is not totally true. The separation approach from [3] has led us to (1.7), which is of analytical interest. Working back the whole argumentation, we have found a way to tackle this equation—by transforming it into (2.3).

In general, the polynomial $p(x)$ is not of the form $C_1x^N + C_2$, where $N \in \mathbb{N}$ and $C_1, C_2 \in \mathbb{R}$. However, this is the case in Bessel's differential equation, which one can solve analytically via power series techniques. We therefore consider the special cases where we have

$$p(x) = C_1x^N + C_2, \quad N \in \mathbb{N},\ C_1, C_2 \in \mathbb{R}. \tag{2.5}$$

Having a close look at the polynomial $p(x)$ in (2.3) results in three such cases:

(i) $a = c = 0 \Rightarrow p(x) \equiv dx - B$;
(ii) $a = d = 0 \Rightarrow p(x) \equiv cx^2 - B$;
(iii) $c = a(b+1)/2,\ d = 0 \Rightarrow p(x) = -Ax^4 - B$.

Here we have made use of the abbreviations

$$A \equiv \frac{a^2}{4} \geq 0, \qquad B \equiv \frac{(b-1)^2}{4} \geq 0 \tag{2.6}$$

for the sake of simplicity. In the next section, we are going to solve (2.3) in these three special cases—the same as in Bessel's case via power series. However, the analytically most challenging and "new" case is (iii), since it conserves most of the structure of the general differential equation. But we have to keep in mind that all the special cases no more depend on four parameters, but only on two of them. Physically speaking, we have "annihilated" two degrees of freedom.

3. Power series and related difference equations

There is a quite general power series method for solving second-order differential equations of "Bessel-type" as discussed above; we refer for instance to [1]. We give an outline of this technique.

Lemma 3.1. *Let $p_1(x) = \sum_{k=0}^{\infty} \alpha_k x^k$ and let $p_0(x) = \sum_{k=0}^{\infty} \beta_k x^k$ be convergent power series, moreover let the differential equation*

$$x^2\psi'' + xp_1(x)\psi' + p_0(x)\psi = 0 \quad \forall x \in \mathcal{D} \tag{3.1}$$

be given on some domain $\mathcal{D} \subset \mathbb{R}$. Denote $F(z) = z(z-1) + \alpha_0 z + \beta_0$. If a solution of (3.1) is of the form

$$\psi(x) = x^r \sum_{n=0}^{\infty} \eta_n x^n \quad \forall x \in \mathcal{D}, \tag{3.2}$$

then there must exist $F(r) = 0$. If, in addition, $F(r+n) \neq 0$ for all $n \in \mathbb{N}$, then the coefficients η_n can be computed explicitly by the difference equation

$$\eta_n F(r+n) = -\sum_{k=1}^{n} \eta_{n-k}\big(\beta_k + \alpha_k(r+n-k)\big) \quad \forall n \in \mathbb{N}, \tag{3.3}$$

where $\eta_0 \in \mathbb{R}$ is arbitrary. If the thus constructed series in (3.2) converges, it indeed constitutes a solution of (3.1).

Now we see more clearly why we wanted a differential equation like (2.1), just being a special case of (3.1). The crucial difficulty lies, of course, in solving the difference equation (3.3). However, we are able to do this in our special cases (i)–(iii). What are the coefficients α_k and β_k in the cases (i)–(iii)?

(i) $\alpha_0 = 1$, $\alpha_k = 0$ for all $k \in \mathbb{N}$, and $\beta_0 = -B$, $\beta_1 = d$, $\beta_k = 0$ for all $k \in \mathbb{N} \setminus \{1\}$.

(ii) $\alpha_0 = 1$, $\alpha_k = 0$ for all $k \in \mathbb{N}$, and $\beta_0 = -B$, $\beta_2 = c$, $\beta_k = 0$ for all $k \in \mathbb{N} \setminus \{2\}$.

(iii) $\alpha_0 = 1$, $\alpha_k = 0$ for all $k \in \mathbb{N}$, and $\beta_0 = -B$, $\beta_4 = -A$, $\beta_k = 0$ for all $k \in \mathbb{N} \setminus \{4\}$.

Now, making the choice (3.2) from Lemma 3.1, all the cases yield the same relation for the exponent r to be specified:

$$F(r) = r(r-1) + r - B = r^2 - B \stackrel{!}{=} 0. \tag{3.4}$$

This equation leaves us with two choices for r, namely,

$$r_\pm = \pm\sqrt{B} \equiv \pm\varrho. \tag{3.5}$$

The result for $F(r+n)$ where $n \in \mathbb{N}$ is also the same, since the function F only depends on the parameters α_0 and β_0:

$$F(r+n) = (r+n)(r+n-1) + r + n - r^2 = 2rn + n^2 = n(n+2r). \tag{3.6}$$

The condition on r in order to guarantee $F(r+n) \neq 0$ for all $n \in \mathbb{N}$ thus reads $-2r \notin \mathbb{N}$. Since we have by (3.5) the representation

$$r_\pm = \pm\varrho \stackrel{!}{=} \pm\frac{|b-1|}{2} \tag{3.7}$$

for the two possible values of r, we have to consider the cases $b \geq 1$ and $b < 1$ separately.

Case 1 ($b \geq 1$). We denote $r_\pm \equiv r_{1,2}$. Here we have $\varrho/2 = b-1$, thus $r_1/2 = b-1$ and $r_2/2 = 1-b$. Hence $r_1 \geq 0$ satisfies the above condition, but r_2 satisfies it only if

$$b - 1 \notin \mathbb{N} \Longleftrightarrow b \notin \mathbb{N} \setminus \{1\}. \tag{3.8}$$

Case 2 ($b < 1$). We denote $r_\pm \equiv \tilde{r}_{1,2}$. Now we have $\varrho/2 = 1 - b$, therefore $\tilde{r}_1/2 = 1 - b$ and $\tilde{r}_2/2 = b - 1$. Thus $\tilde{r}_1 > 0$ satisfies the condition, whereas for $\tilde{r}_2$, we must guarantee that

$$1 - b \notin \mathbb{N} \Longleftrightarrow b \notin \mathbb{Z} \setminus \mathbb{N}. \tag{3.9}$$

Dropping the above separation, one easily notices that the corresponding "power series solutions" $\psi_{1,2}$ will coincide in the case of convergence in the sense of the identities $r_1 = \tilde{r}_2$ and $r_2 = \tilde{r}_1$. Therefore, the separation into the two different cases from above can be avoided, combining the conditions (3.8) and (3.9) to the stronger condition

$$b \notin \mathbb{Z} \setminus \{1\}. \tag{3.10}$$

If (3.10) is satisfied, the difference equation (3.3) will make sense for both r-values, whereas it will only provide a sequence of coefficients η_n for one of the r-values if (3.10) is not fulfilled. (We will come back to the physical consequences later.)

Now the question arises whether the corresponding "solutions" $\psi(x) = x^r \sum_{n=0}^{\infty} \eta_n x^n$ of (2.3) are really *solutions*, that is, converge for every $x \in \mathbb{R}^+$. In the next theorem, we are going to compute the coefficients explicitly, which allows us to prove this convergence easily by the *ratio test*—not going into the details however. We rather state the result.

Theorem 3.2. *If $b \notin \mathbb{Z} \setminus \{\pm 1\}$, the differential equation (2.3) in each of the three cases mentioned above has a basis $\{\psi_1, \psi_2\}$ of solutions of type (3.2), explicitly given by the following representations.*

(i) *Special case 1 ($a = c = 0$, $d \neq 0$):*

$$\psi_1(x) = x^{(b-1)/2} \sum_{n=0}^{\infty} \frac{(-dx)^n}{\prod_{j=1}^{n} j(j+b-1)}, \qquad \psi_2(x) = x^{(1-b)/2} \sum_{n=0}^{\infty} \frac{(-dx)^n}{\prod_{j=1}^{n} j(j+1-b)}. \tag{3.11}$$

(ii) *Special case 2 ($a = d = 0$, $c \neq 0$):*

$$\psi_1(x) = x^{(b-1)/2} \sum_{n=0}^{\infty} \frac{(-cx^2/2)^n}{\prod_{j=1}^{n} j(2j+b-1)}, \qquad \psi_2(x) = x^{(1-b)/2} \sum_{n=0}^{\infty} \frac{(-cx^2/2)^n}{\prod_{j=1}^{n} j(2j+1-b)}. \tag{3.12}$$

(iii) *Special case 3 ($a \neq 0$, $c = a(b+1)/2$, $d = 0$):*

$$\psi_1(x) = x^{(b-1)/2} \sum_{n=0}^{\infty} \frac{(Ax^4/4)^n}{\prod_{j=1}^{n} j(4j+b-1)}, \qquad \psi_2(x) = x^{(1-b)/2} \sum_{n=0}^{\infty} \frac{(Ax^4/4)^n}{\prod_{j=1}^{n} j(4j+1-b)}. \tag{3.13}$$

If otherwise $b \in \mathbb{Z} \setminus \{\pm 1\}$, only one of those solutions of type (3.2) will exist.

Proof. We will only prove this for the "most interesting" case (iii). Consider $2r_1 = b - 1$ first. (Remember that we dropped the separation into $b \geq 1$ and $b < 1$ before!) By Lemma 3.1, we can choose η_0 arbitrarily, say $\eta_0 \equiv 1$. The difference equation (3.3) then yields

$\eta_1 = \eta_2 = \eta_3 = 0$ and

$$\eta_n = \frac{A}{n(n+b-1)}\eta_{n-4} \quad \forall n \geq 4. \tag{3.14}$$

Exploiting this recursion, we end up exactly with ψ_1 from the theorem. In the same way, we obtain ψ_2 when considering the other r-value, that is, $2r_2 = 1 - b$. (Convergence of those series is—as was already mentioned—guaranteed by the ratio test.) □

Corollary 3.3. *In the case $b \notin \mathbb{Z} \setminus \{1\}$, a basis of solutions of (1.7) is, in our three special cases, given by*

$$y_{1,2}(x) \equiv x^{(1-b)/2} e^{-ax^2/4} \psi_{1,2}(x) \quad \forall x \in \mathbb{R}^+, \tag{3.15}$$

where $\psi_{1,2}$ denote the solutions of (2.3) as written down in Theorem 3.2.

We have a closer look at the functions $y_{1,2}$ in Corollary 3.3 at least in the case (iii). They possess the explicit representation

$$\begin{aligned} y_1(x) &= e^{-ax^2/4} \sum_{n=0}^{\infty} \frac{(ax^2/4)^{2n}}{\prod_{j=1}^{\infty} j(4j+b-1)}, \\ y_2(x) &= x^{1-b} e^{-ax^2/4} \sum_{n=0}^{\infty} \frac{(ax^2/4)^{2n}}{\prod_{j=1}^{\infty} j(4j+1-b)}. \end{aligned} \tag{3.16}$$

It is a bit mysterious that the term $ax^2/4$ appears both in the exponential function and in the series. However, we have not found a suitable interpretation for this behavior. From now on, we will be concerned with the weighted square integrability of our solutions. Indeed, we will provide weight functions which allow such a scenario.

4. Square integrability with respect to Gaussian weights

We first have to specify what we understand by *square integrability with respect to Gaussian weights*. This means for some solution y_* of (1.7) that there is a constant $\gamma \equiv \gamma(a,b,c,d) > 0$ such that

$$\int_0^{\infty} |y_*(x)|^2 e^{-\gamma x^2} dx < \infty. \tag{4.1}$$

How this is related to (1.2) will be discussed in the last section. We provide an auxiliary statement known from calculus.

Lemma 4.1. *For arbitrary $k > 0$ and $l \geq 0$, the following "Gaussian" integrability condition holds:*

$$\int_0^{\infty} x^l e^{-kx^2} dx < \infty. \tag{4.2}$$

It is, for example, exploited in the context of Hermite polynomials.

Now we can state a quite general result concerning the existence of solutions y_* which satisfy (4.1) for some $\gamma > 0$.

Theorem 4.2. *Provided that the condition (3.10) is satisfied, one obtains in the special cases the following integrability situation:*

(i) y_1 *is square integrable with respect to a Gaussian weight if* $b \geq 1/2$;

(ii) y_1 *is square integrable with respect to a Gaussian weight if* $b > 0$;

(iii) y_1 *is square integrable with respect to a Gaussian weight if* $b > -1$.

In all the cases, y_2 *fulfills (4.1) for some* $\gamma > 0$ *if* $b < -1$.

Proof. We are going to prove the parts (i)–(iii) separately.

(i) Assume that $b \geq 1/2$. We write down y_1 a bit differently:

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-4dx)^n}{\prod_{j=1}^{n} 4j(j+b-1)}. \tag{4.3}$$

Now we have for all $j \in \mathbb{N}$,

$$4j(j+b-1) \geq 4j\left(j - \frac{1}{2}\right) = 2j(2j-1), \tag{4.4}$$

and therefore,

$$|y_1(x)| \leq \sum_{n=0}^{\infty} \frac{(4|d|x)^n}{\prod_{j=1}^{n} 4j(j+b-1)} \leq \sum_{n=0}^{\infty} \frac{(4|d|x)^n}{(2n)!} \quad \forall x > 0. \tag{4.5}$$

Now fix $x_0 = (4|d|)^{-1}$ in order to obtain

$$(4|d|x)^n \leq (4|d|x)^{4n} = (16d^2x^2)^{2n} \equiv (C_1x^2)^{2n} \quad \forall x \geq x_0, \tag{4.6}$$

consequently,

$$|y_1(x)| \leq \cosh(C_1x^2) \quad \forall x \geq x_0 \tag{4.7}$$

by the Taylor expansion of cosh. Moreover, we thus obtain the estimation

$$|y_1(x)|^2 \leq \cosh^2(C_1x^2) = \cdots = \frac{1}{4}\left(e^{2C_1x^2} + 2 + e^{-2C_1x^2}\right) \quad \forall x \geq x_0. \tag{4.8}$$

Finally, let $\gamma \equiv 2C_1 + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Then we get with $\delta \equiv \gamma + 2C_1$

$$\int_{x_0}^{\infty} |y_1(x)|^2 e^{-\gamma x^2} dx \leq \frac{1}{4} \int_{x_0}^{\infty} \left(e^{-\varepsilon x^2} + e^{-\gamma x^2} + e^{-\delta x^2}\right) dx < \infty \tag{4.9}$$

by Lemma 4.1. This establishes the statement.

(ii) Assume that $b > 0$. We again write down y_1:

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-c)^n x^{2n}}{\prod_{j=1}^{n} 2j(2j+b-1)}. \tag{4.10}$$

Once more, we obtain by the condition on b

$$2j(2j+b-1) \geq 2j(2j-1) \quad \forall j \in \mathbb{N}. \tag{4.11}$$

Performing the same estimation steps as above, now letting $x_0 \equiv c^{-1/2}$ and $C_2 \equiv \sqrt{|c|}$, we gain

$$|\, y_1(x)\,| \leq \cosh\left(C_2 x^2\right) \quad \forall x \geq x_0. \tag{4.12}$$

The remainder of the argumentation is left to the reader.

(iii) Now suppose $b > -1$. Then, slightly different to the parts (i) and (ii), we obtain

$$|\, y_1(x)\,| = y_1(x) \leq e^{-ax^2/4} \cosh\left(\frac{ax^2}{4}\right) \quad \forall x > 0, \tag{4.13}$$

and therefore,

$$|\, y_1(x)\,|^2 \leq \frac{1}{4} e^{-ax^2/2}\left(e^{ax^2/4} + e^{-ax^2/4}\right)^2 = \frac{1}{4}\left(1 + 2e^{-ax^2/2} + e^{-ax^2}\right) \quad \forall x > 0. \tag{4.14}$$

For $a > 0$, the statement is hence an immediate consequence of Lemma 4.1. Rather let $a < 0$—as in the physical situation where $a \equiv -\lambda$. Choosing $\gamma \equiv \varepsilon - a$ where $\varepsilon > 0$ results in the same conclusion as in the previous cases, now letting $\delta \equiv \gamma + a/2 > 0$.

Having shown the part of the theorem concerned with y_1, we now arrive with the statement concerning y_2. In each case, we define the function $z : \mathbb{R}^+ \to \mathbb{R}$ via $z(x) = x^{b-1} y_2(x)$. Assume throughout that $b < -1$. It is clear from the first part of the proof that z is square integrable with respect to some Gaussian weight—taking over the role of y_1. Indeed, for some fixed $x_0 > 0$ and $\gamma > 0$, the integral $\int_{x_0}^{\infty} |z(x)|^2 e^{-\gamma x^2}\, dx$ can be estimated from above by a linear combination of "Gaussian" integrals $\int_0^{\infty} e^{-kx^2}\, dx$, where $k > 0$. But now we have

$$|\, y_2(x)\,|^2 = x^{2(1-b)}\,|\, z(x)\,|^2 \quad \forall x > 0, \tag{4.15}$$

yielding the claim by Lemma 4.1, since $l \equiv 2(1 - b) > 0$. □

Especially, in the important case (iii), Theorem 4.2 provides a one-dimensional subspace of solutions satisfying (4.1) for some $\gamma > 0$, whenever the condition (3.10) is guaranteed. The question is what happens if $b \in \mathbb{Z} \setminus \{1\}$.

Case 1 ($b > 1$). Then the solution y_1 in Corollary 3.3 exists, whereas y_2 does not. But during the proof of Theorem 4.2, we have formally shown that y_1 satisfies (4.1) for some $\gamma > 0$ even if $b > -1$. Therefore, y_1 is also square integrable in our weighted sense in this integer case.

Case 2 ($b < 1$). This time y_2 exists, not y_1. And we know that y_2 fulfills our integrability condition if $b \leq -1$—having a closer look at the last part of the preceding proof. Thus we do not know what happens in the case $b = 0$, but can guarantee that y_2 satisfies the condition if $b < 0$.

Proposition 4.3. *In the case $2c = a(b + 1)$, $a, b \neq 0$, $d = 0$, a one-dimensional subspace of solutions y_* is obtained—being of type (3.2)—which satisfies the integrability condition (4.1) for some $\gamma > 0$. On the one hand, if $b \geq -1$, this subspace is spanned by y_1. On the other hand, if $b < -1$, it is spanned by y_2.*

Remark 4.4. Similar statements hold in the cases (i) and (ii). But we will not go into further details here, leaving them to the reader.

Last but not least, we have to come back to the physical motivation—having been a starting point for the analysis of our special differential equation.

5. Physical interpretation and perspectives

Although the cases (i) and (ii) are of mathematical interest, we rather concentrate on a discussion of the physical relevance of the solutions established in case (iii). What is the meaning of the parameters a, b, c, d in quantum mechanics?

(a) $a = -\lambda$ stands for the strength of the magnetic field. Since a is arbitrary in our consideration, we cover a lot of physical scenarios from this viewpoint.

(b) $b = 2|m| + 1$ corresponds to a quantum number. In the physical setting, we can only have $b \in \mathbb{N}$. In this integer case, we only obtain one (linearly independent) power series solution like (3.2), namely, y_1. However, this solution satisfies (4.1) for some $\gamma > 0$ by Proposition 4.3.

(c) $c = E - a(m + |m| - b)$ is related to quantum numbers (energy and angular momentum) and to the magnetic field. Here we have $2c = a(b+1)$, and therefore,

$$2E = a(2m + 2|m| - b + 1) \overset{!}{\Longleftrightarrow} E = -\lambda m. \tag{5.1}$$

The requirement (5.1) means that the energy in our case must be quantized by integer multiples of the magnetic field strength—after having chosen suitable units.

(d) $d = 2Z$ is twice the number of the protons in the nucleus. However, the restriction $d = 0$ is quite unfortunate, since it says that there are *no* protons in the nucleus. Hence we are somehow treating the case of a free electron.

The interpretations (a)–(d) show that the scenario we addressed in (iii) is not the "typical" physical scenario, nevertheless it is interesting for quantum mechanics because of the energy quantization in (c). Maybe something similar can be obtained if $d \neq 0$—probably via perturbation methods. (One has to take into consideration that, in a *strong* magnetic field, one can sort of neglect the influence of the Coulomb forces.)

We finally pay attention to our integrability condition, in comparison to (1.2). By the relation

$$y_1(x) = x^{-|m|} e^{\lambda x^2/4} \psi_1(x), \tag{5.2}$$

the condition (4.1) is equivalent to the following condition for the "wave function":

$$\int_0^\infty |\psi_1(x)|^2 x^{-2|m|} e^{(\lambda/2-\gamma)x^2} dx < \infty. \tag{5.3}$$

Comparing (5.3) and (1.2), we see that the Schrödinger boundary condition (square integrability of the wave function) is satisfied if, for some $x_0 > 0$, the estimation

$$x \leq \text{const} \cdot x^{-2|m|} e^{(\lambda/2-\gamma)x^2} \quad \forall x > x_0 \tag{5.4}$$

holds true. For $m \neq 0$, this is clearly equivalent to $2\gamma < \lambda$. But in step (iii) of the proof of Theorem 4.2, we had chosen $\gamma > \lambda$, not being compatible with the above assertion. Therefore, we cannot guarantee the Schrödinger boundary condition via our estimations from above.

The very last problem immediately leads us to a question for possible future research on the topic: can one weaken the assertion $\gamma > \lambda$ such that it is possible to have $2\gamma < \lambda$—at least for certain magnetic fields? Another weak point is that physicists are mainly interested in wave functions which can be decomposed into some *ground state* times a *polynomial*, preferably yielding a sequence of orthogonal functions when considering all quantizations. In our approach, we did not get polynomials, but just power series—being quite similar to Bessel functions. Can we ameliorate this scenario? And at last, how can we get rid of the unnatural assertion $d = 0$, forcing the nucleus to be uncharged? A lot of work has to be done in approaching these questions from a purely analytical viewpoint.

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