BEHAVIOR OF THE POSITIVE SOLUTIONS OF FUZZY MAX-DIFFERENCE EQUATIONS

G. STEFANIDOU AND G. PAPASCHINOPOULOS

Received 15 September 2004

We extend some results obtained in 1998 and 1999 by studying the periodicity of the solutions of the fuzzy difference equations $x_{n+1} = \max\{A/x_n, A/x_{n-1}, \dots, A/x_{n-k}\}$, $x_{n+1} = \max\{A_0/x_n, A_1/x_{n-1}\}$, where k is a positive integer, A, A_i, $i = 0,1$, are positive fuzzy numbers, and the initial values x_i, $i = -k, -k+1, \dots, 0$ (resp., $i = -1, 0$) of the first (resp., second) equation are positive fuzzy numbers.

1. Introduction

Difference equations are often used in the study of linear and nonlinear physical, physiological, and economical problems (for partial review see [3, 6]). This fact leads to the fast promotion of the theory of difference equations which someone can find, for instance, in [1, 7, 9]. More precisely, max-difference equations have increasing interest since max operators have applications in automatic control (see [2, 11, 17, 18] and the references cited therein).

Nowadays, a modern and promising approach for engineering, social, and environmental problems with imprecise, uncertain input-output data arises, the fuzzy approach. This is an expectable effect, since fuzzy logic can handle various types of vagueness but particularly vagueness related to human linguistic and thinking (for partial review see [8, 12]).

The increasing interest in applications of these two scientific fields contributed to the appearance of fuzzy difference equations (see [4, 5, 10, 13, 14, 15, 16]).

In [17], Szalkai studied the periodicity of the solutions of the ordinary difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{A}{x_{n-1}}, \dots, \frac{A}{x_{n-k}}\right\}, \tag{1.1}$$

where k is a positive integer, A is a real constant, x_i, $i = -k, -k+1, \dots, 0$ are real numbers. More precisely, if A is a positive real constant and x_i, $i = -k, -k+1, \dots, 0$ are positive real numbers, he proved that every positive solution of (1.1) is eventually periodic of period $k+2$.

Advances in Difference Equations 2005:2 (2005) 153–172
DOI: 10.1155/ADE.2005.153

In [2], Amleh et al. studied the periodicity of the solutions of the ordinary difference equation

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}\right\}, \tag{1.2}$$

where A_0, A_1 are positive real constants and x_{-1}, x_0 are real numbers. More precisely, if A_0, A_1 are positive constants, x_{-1}, x_0 are positive real numbers, $A_0 > A_1$ (resp., $A_0 = A_1$) (resp., $A_0 < A_1$), then every positive solution of (1.2) is eventually periodic of period two (resp., three) (resp., four).

In this paper, our goal is to extend the above mentioned results for the corresponding fuzzy difference equations (1.1) and (1.2) where A, A_0, A_1 are positive fuzzy numbers and x_i, $i = -k, -k+1, \dots, 0$, x_{-1}, x_0 are positive fuzzy numbers. Moreover, we find conditions so that the corresponding fuzzy equations (1.1) and (1.2) have unbounded solutions, something that does not happen in case of the ordinary difference equations (1.1) and (1.2).

We note that, in order to study the behavior of a parametric fuzzy difference equation we use the following technique: we investigate the behavior of the solutions of a related family of systems of two parametric ordinary difference equations and then, using these results and the fuzzy analog of some concepts known by the theory of ordinary difference equations, we prove our main effects concerning the fuzzy difference equation.

2. Preliminaries

We need the following definitions.

For a set B we denote by $\bar{B}$ the closure of B. We say that a function A from $\mathbb{R}^+ = (0,\infty)$ into the interval $[0,1]$ is a fuzzy number if A is normal, convex fuzzy set (see [13]), upper semicontinuous and the support $\operatorname{supp} A = \overline{\bigcup_{a\in(0,1]} [A]_a} = \overline{\{x : A(x) > 0\}}$ is compact. Then from [12, Theorems 3.1.5 and 3.1.8] the a-cuts of the fuzzy number A, $[A]_a = \{x \in \mathbb{R}^+ : A(x) \geq a\}$ are closed intervals.

We say that a fuzzy number A is positive if $\operatorname{supp} A \subset (0,\infty)$.

It is obvious that if A is a positive real number, then A is a positive fuzzy number and $[A]_a = [A,A]$, $a \in (0,1]$. In this case, we say that A is a trivial fuzzy number.

Let B_i, $i = 0,1,\dots,k$, k is a positive integer, be fuzzy numbers such that

$$[B_i]_a = [B_{i,l,a}, B_{i,r,a}], \quad i = 0,1,\dots,k,\ a \in (0,1], \tag{2.1}$$

and for any $a \in (0,1]$

$$C_{l,a} = \max\{B_{i,l,a},\ i = 0,1,\dots,k\}, \qquad C_{r,a} = \max\{B_{i,r,a},\ i = 0,1,\dots,k\}. \tag{2.2}$$

Then by [19, Theorem 2.1], $(C_{l,a}, C_{r,a})$ determines a fuzzy number C such that

$$[C]_a = [C_{l,a}, C_{r,a}], \quad a \in (0,1]. \tag{2.3}$$

According to [8] and [14, Lemma 2.3] we can define

$$C = \max\{B_i,\ i = 0,1,\dots,k\}. \tag{2.4}$$

We say that x_n is a positive solution of (1.1) (resp., (1.2)) if x_n is a sequence of positive fuzzy numbers which satisfies (1.1) (resp., (1.2)).

We say that a sequence of positive fuzzy numbers x_n persists (resp., is bounded) if there exists a positive number M (resp., N) such that

$$\operatorname{supp} x_n \subset [M,\infty), \quad (\text{resp., } \operatorname{supp} x_n \subset (0,N]), \quad n = 1,2,\dots. \tag{2.5}$$

In addition, we say that x_n is bounded and persists if there exist numbers $M,N \in (0,\infty)$ such that

$$\operatorname{supp} x_n \subset [M,N], \quad n = 1,2,\dots. \tag{2.6}$$

A solution x_n of (1.1) (resp., (1.2)) is said to be eventually periodic of period r, r is a positive integer, if there exists a positive integer m such that

$$x_{n+r} = x_n, \quad n = m, m+1,\dots. \tag{2.7}$$

3. Existence and uniqueness of the positive solutions of fuzzy difference equations (1.1) and (1.2)

In this section, we study the existence and the uniqueness of the positive solutions of the fuzzy difference equations (1.1) and (1.2).

Proposition 3.1. *Suppose that A, A_0, A_1 are positive fuzzy numbers. Then for all positive fuzzy numbers $x_{-k}, x_{-k+1},\dots,x_0$ (resp., x_{-1}, x_0) there exists a unique positive solution x_n of (1.1) (resp., (1.2)) with initial values $x_{-k}, x_{-k+1},\dots,x_0$ (resp., x_{-1}, x_0).*

Proof. Suppose that

$$[A]_a = [A_{l,a}, A_{r,a}], \quad a \in (0,1]. \tag{3.1}$$

Let x_i, $i = -k,-k+1,\dots,0$ be positive fuzzy numbers such that

$$[x_i]_a = [L_{i,a}, R_{i,a}], \quad i = -k,-k+1,\dots,0,\ a \in (0,1] \tag{3.2}$$

and let $(L_{n,a}, R_{n,a})$, $n = 0,1,\dots, a \in (0,1]$, be the unique positive solution of the system of difference equations

$$\begin{aligned} L_{n+1,a} &= \max\left\{\frac{A_{l,a}}{R_{n,a}}, \frac{A_{l,a}}{R_{n-1,a}},\dots,\frac{A_{l,a}}{R_{n-k,a}}\right\}, \\ R_{n+1,a} &= \max\left\{\frac{A_{r,a}}{L_{n,a}}, \frac{A_{r,a}}{L_{n-1,a}},\dots,\frac{A_{r,a}}{L_{n-k,a}}\right\} \end{aligned} \tag{3.3}$$

with initial values $(L_{i,a}, R_{i,a})$, $i = -k,-k+1,\dots,0$. Using [19, Theorem 2.1] and relation (3.3) and working as in [13, Proposition 2.1] and [15, Proposition 1] we can easily prove that $(L_{n,a}, R_{n,a})$, $n = 1,2,\dots$, $a \in (0,1]$ determines a sequence of positive fuzzy numbers x_n such that

$$[x_n]_a = [L_{n,a}, R_{n,a}], \quad n = 1,2,\dots,\ a \in (0,1]. \tag{3.4}$$

Now, we prove that x_n satisfies (1.1) with initial values x_i, $i = -k, -k+1, \dots, 0$. From (3.1), (3.2), (3.3), (3.4), [15, Lemma 1], and by a slight generalization of [14, Lemma 2.3] we have

$$\begin{aligned}
&\left[\max\left\{\frac{A}{x_n}, \frac{A}{x_{n-1}}, \dots, \frac{A}{x_{n-k}}\right\}\right]_a \\
&\quad = \left[\max\left\{\frac{A_{l,a}}{R_{n,a}}, \frac{A_{l,a}}{R_{n-1,a}}, \dots, \frac{A_{l,a}}{R_{n-k,a}}\right\}, \max\left\{\frac{A_{r,a}}{L_{n,a}}, \frac{A_{r,a}}{L_{n-1,a}}, \dots, \frac{A_{r,a}}{L_{n-k,a}}\right\}\right] \\
&\quad = \left[L_{n+1,a}, R_{n+1,a}\right] = \left[x_{n+1}\right]_a, \quad a \in (0,1].
\end{aligned} \tag{3.5}$$

From (3.5) and arguing as in [13, Proposition 2.1] and [15, Proposition 1] we have that x_n is the unique positive solution of (1.1) with initial values x_i, $i = -k, -k+1, \dots, 0$.

Now, suppose that

$$\left[A_i\right]_a = \left[A_{i,l,a}, A_{i,r,a}\right], \quad i = 0, 1,\ a \in (0,1]. \tag{3.6}$$

Arguing as above and using (3.6) we can easily prove that if x_i, $i = -1, 0$ are positive fuzzy numbers which satisfy (3.2) for $k = 1$, then there exists a unique positive solution x_n of (1.2) with initial values x_i, $i = -1, 0$ such that (3.4) holds and $(L_{n,a}, R_{n,a})$ satisfies the system of difference equations

$$L_{n+1,a} = \max\left\{\frac{A_{0,l,a}}{R_{n,a}}, \frac{A_{1,l,a}}{R_{n-1,a}}\right\}, \qquad R_{n+1,a} = \max\left\{\frac{A_{0,r,a}}{L_{n,a}}, \frac{A_{1,r,a}}{L_{n-1,a}}\right\}. \tag{3.7}$$

This completes the proof of the proposition. □

4. Behavior of the positive solutions of fuzzy equation (1.1)

In this section, we study the behavior of the positive solutions of (1.1). Firstly, we study the periodicity of the positive solutions of (1.1). We need the following lemmas.

Lemma 4.1. *Let A, a, b be positive numbers such that $ab \neq A$. If*

$$ab < A \qquad (\textit{resp.},\ ab > A), \tag{4.1}$$

then there exist positive numbers $\bar{y}, \bar{z}$ such that

$$\bar{y}\bar{z} = A, \tag{4.2}$$

$$a < \bar{y}, \qquad b < \bar{z} \qquad (\textit{resp.},\ a > \bar{y},\ b > \bar{z}). \tag{4.3}$$

Proof. Suppose that (4.1) is satisfied. Then if ϵ is a positive number such that

$$\begin{aligned}
&\epsilon < \frac{A - ab}{b} \qquad \left(\text{resp., } \epsilon < \frac{ab - A}{b}\right), \\
&\bar{y} = a + \epsilon, \qquad \bar{z} = \frac{A}{a+\epsilon} \qquad \left(\text{resp., } \bar{y} = a - \epsilon,\ \bar{z} = \frac{A}{a-\epsilon}\right),
\end{aligned} \tag{4.4}$$

it is obvious that (4.2) and (4.3) hold. This completes the proof of the lemma. □

LEMMA 4.2. *Consider the system of difference equations*

$$y_{n+1} = \max\left\{\frac{A}{z_n}, \frac{A}{z_{n-1}}, \dots, \frac{A}{z_{n-k}}\right\}, \qquad z_{n+1} = \max\left\{\frac{A}{y_n}, \frac{A}{y_{n-1}}, \dots, \frac{A}{y_{n-k}}\right\}, \tag{4.5}$$

where A is a positive real constant, k is a positive integer, and y_i, z_i, $i = -k, -k+1, \dots, 0$ are positive real numbers. Then every positive solution (y_n, z_n) of (4.5) is eventually periodic of period $k+2$.

Proof. Let (y_n, z_n) be an arbitrary positive solution of (4.5). Firstly, suppose that there exists a $\lambda \in \{1, 2, \dots, k+2\}$ such that

$$y_\lambda z_\lambda < A. \tag{4.6}$$

Then from (4.6) and Lemma 4.1 there exist positive constants $\bar{y}$, $\bar{z}$ such that (4.2) holds and

$$y_\lambda < \bar{y}, \qquad z_\lambda < \bar{z}. \tag{4.7}$$

From (4.2), (4.5), and (4.7) we have, for $i = \lambda+1, \lambda+2, \dots, k+\lambda+1$,

$$y_i = \max\left\{\frac{A}{z_{i-1}}, \frac{A}{z_{i-2}}, \dots, \frac{A}{z_{i-k-1}}\right\} \geq \frac{A}{z_\lambda} > \frac{A}{\bar{z}} = \bar{y}, \quad z_i > \bar{z}. \tag{4.8}$$

Then relations (4.2), (4.5), and (4.8) imply that

$$y_{k+\lambda+2} = \max\left\{\frac{A}{z_{k+\lambda+1}}, \frac{A}{z_{k+\lambda}}, \dots, \frac{A}{z_{\lambda+1}}\right\} < \frac{A}{\bar{z}} = \bar{y}, \quad z_{k+\lambda+2} < \bar{z}. \tag{4.9}$$

Therefore, from (4.2), (4.5), (4.8), and (4.9) we take, for $j = k+\lambda+3, k+\lambda+4, \dots, 2k+\lambda+3$,

$$y_j = \max\left\{\frac{A}{z_{j-1}}, \frac{A}{z_{j-2}}, \dots, \frac{A}{z_{j-k-1}}\right\} = \frac{A}{z_{k+\lambda+2}}, \quad z_j = \frac{A}{y_{k+\lambda+2}}. \tag{4.10}$$

So, from (4.5), (4.9), (4.10) and working inductively for $i = 0, 1, \dots$ and $j = 3, 4, \dots, k+3$ we can easily prove that

$$\begin{aligned} y_{k+\lambda+2+i(k+2)} &= y_{k+\lambda+2}, \qquad y_{k+\lambda+j+i(k+2)} = \frac{A}{z_{k+\lambda+2}}, \\ z_{k+\lambda+2+i(k+2)} &= z_{k+\lambda+2}, \qquad z_{k+\lambda+j+i(k+2)} = \frac{A}{y_{k+\lambda+2}} \end{aligned} \tag{4.11}$$

and so it is obvious that (y_n, z_n) is eventually periodic of period $k+2$.

Therefore, if relation

$$y_{k+2} z_{k+2} < A \tag{4.12}$$

holds, then (y_n, z_n) is eventually periodic of period $k+2$.

Now, suppose that relation

$$y_{k+2}z_{k+2} > A \tag{4.13}$$

is satisfied. Then from (4.13) and Lemma 4.1 there exist positive constants $\bar{y}$, $\bar{z}$ such that (4.2) holds and

$$y_{k+2} > \bar{y}, \qquad z_{k+2} > \bar{z}. \tag{4.14}$$

Moreover, from (4.5) and (4.14) there exist $\lambda, \mu \in \{1, 2, \dots, k+1\}$ such that

$$y_{k+2} = \max\left\{\frac{A}{z_{k+1}}, \frac{A}{z_k}, \dots, \frac{A}{z_1}\right\} = \frac{A}{z_\lambda} > \bar{y}, \qquad z_{k+2} = \frac{A}{y_\mu} > \bar{z}. \tag{4.15}$$

Hence, from (4.2) and (4.15) it follows that

$$z_\lambda < \bar{z}, \qquad y_\mu < \bar{y}. \tag{4.16}$$

We prove that $\lambda = \mu$. Suppose on the contrary that $\lambda \neq \mu$. Without loss of generality we may suppose that $1 \leq \mu \leq \lambda - 1$. Then from (4.2), (4.5), and (4.16) we get

$$z_\lambda = \max\left\{\frac{A}{y_{\lambda-1}}, \frac{A}{y_{\lambda-2}}, \dots, \frac{A}{y_{\lambda-k-1}}\right\} \geq \frac{A}{y_\mu} > \bar{z} \tag{4.17}$$

which contradicts to (4.16). Hence, $\lambda = \mu$ and from (4.2) and (4.16) we have

$$y_\lambda z_\lambda < A \tag{4.18}$$

and so (y_n, z_n) is eventually periodic of period $k+2$ if (4.13) holds.

Finally, suppose that

$$y_{k+2}z_{k+2} = A. \tag{4.19}$$

From (4.5) it is obvious that

$$y_{k+2} \geq \frac{A}{z_i}, \quad z_{k+2} \geq \frac{A}{y_i}, \quad i = 1, 2, \dots, k+1. \tag{4.20}$$

Therefore, relations (4.5), (4.19), and (4.20) imply that

$$y_{k+3} = \max\left\{y_{k+2}, \frac{A}{z_{k+1}}, \dots, \frac{A}{z_2}\right\} = y_{k+2}, \qquad z_{k+3} = z_{k+2}. \tag{4.21}$$

Hence, using (4.19), (4.20), (4.21) and working inductively we can easily prove that

$$y_{k+i} = y_{k+2}, \quad z_{k+i} = z_{k+2}, \quad i = 3, 4, \dots \tag{4.22}$$

and so it is obvious that (y_n, z_n) is eventually periodic of period $k+2$ if (4.19) holds. This completes the proof of the lemma. □

Proposition 4.3. *Consider (1.1) where A is a positive real constant and* $x_{-k}, x_{-k+1}, \dots, x_0$ *are positive fuzzy numbers. Then every positive solution of (1.1) is eventually periodic of period* $k+2$.

Proof. Let x_n be a positive solution of (1.1) with initial values $x_{-k}, x_{-k+1}, \dots, x_0$ such that (3.2) and (3.4) hold. From Proposition 3.1, $(L_{n,a}, R_{n,a})$, $n = 1, 2, \dots$, $a \in (0,1]$ satisfies system (3.3). Using Lemma 4.2 we have that

$$L_{n+k+2,a} = L_{n,a}, \quad R_{n+k+2,a} = R_{n,a}, \quad n = 2k+4, 2k+5, \dots, \ a \in (0,1]. \tag{4.23}$$

Therefore, from (3.4) and (4.23) we have that x_n is eventually periodic of period $k+2$. This completes the proof of the proposition. □

Now, we find conditions so that every positive solution of (1.1) neither is bounded nor persists. We need the following lemma.

Lemma 4.4. *Consider the system of difference equations*

$$y_{n+1} = \max\left\{\frac{B}{z_n}, \frac{B}{z_{n-1}}, \dots, \frac{B}{z_{n-k}}\right\}, \qquad z_{n+1} = \max\left\{\frac{C}{y_n}, \frac{C}{y_{n-1}}, \dots, \frac{C}{y_{n-k}}\right\}, \tag{4.24}$$

where k is a positive integer, y_i, z_i, $i = -k, -k+1, \dots, 0$ *are positive real numbers, and B, C are positive real constants such that*

$$B < C. \tag{4.25}$$

Then for every positive solution (y_n, z_n) *of (4.24) the following relations hold:*

$$\lim_{n\to\infty} z_n = \infty, \qquad \lim_{n\to\infty} y_n = 0. \tag{4.26}$$

Proof. Since for any $n \geq 1$ we have

$$\frac{C}{y_n} = \frac{C}{\max\{B/z_{n-1}, B/z_{n-2}, \dots, B/z_{n-k-1}\}} = \lambda \min\{z_{n-1}, z_{n-2}, \dots, z_{n-k-1}\}, \tag{4.27}$$

where $\lambda = C/B$, from (4.24) we get

$$z_{n+1} = \max\left\{\lambda \min\{z_{n-1}, z_{n-2}, \dots, z_{n-k-1}\}, \frac{C}{y_{n-1}}, \dots, \frac{C}{y_{n-k}}\right\} \tag{4.28}$$

and clearly

$$z_{n+1} \geq \lambda \min\{z_{n-1}, z_{n-2}, \dots, z_{n-k-1}\}, \quad n = 1, 2, \dots. \tag{4.29}$$

Using (4.29) we can easily prove that

$$z_n \geq \lambda \min\{z_1, z_0, \dots, z_{-k}\}, \quad n = 2, 3, \dots, k+3, \tag{4.30}$$

and so

$$z_n \geq \lambda^2 \min\{z_1, z_0, \dots, z_{-k}\}, \quad n = k+4, k+5, \dots, 2k+5. \tag{4.31}$$

From (4.31) and working inductively we get, for $r = 3, 4, \dots$,

$$z_n \geq \lambda^r \min\{z_1, z_0, \dots, z_{-k}\}, \quad n = (r-1)k + 2r, (r-1)k + 2r + 1, \dots, r(k+2) + 1. \tag{4.32}$$

Obviously, from (4.25) and (4.32) we have that

$$\lim_{n\to\infty} z_n = \infty. \tag{4.33}$$

Hence, relations (4.24) and (4.33) imply that

$$\lim_{n\to\infty} y_n = 0 \tag{4.34}$$

and so from (4.33) and (4.34) we have that relations (4.26) are true. This completes the proof of the lemma. □

Proposition 4.5. *Consider (1.1) where k is a positive integer, A is a nontrivial positive fuzzy number, and $x_{-k}, x_{-k+1}, \dots, x_0$ are positive fuzzy numbers. Then every positive solution of (1.1) is unbounded and does not persist.*

Proof. Let x_n be a positive solution of (1.1) with initial values $x_{-k}, x_{-k+1}, \dots, x_0$ such that (3.2) and (3.4) hold. Since A is a nontrivial positive fuzzy number there exists an $\bar{a} \in (0,1]$ such that

$$A_{l,\bar{a}} < A_{r,\bar{a}}. \tag{4.35}$$

Moreover, since (4.35) holds and $(L_{n,a}, R_{n,a})$, $a \in (0,1]$ satisfies system (3.3), then from Lemma 4.4 we have that

$$\lim_{n\to\infty} R_{n,\bar{a}} = \infty, \qquad \lim_{n\to\infty} L_{n,\bar{a}} = 0. \tag{4.36}$$

Therefore, from (4.36) there are no positive numbers M, N such that $\bigcup_{a\in(0,1]}[L_{n,a}, R_{n,a}] \subset [M, N]$. This completes the proof of the proposition. □

From Propositions 4.3 and 4.5 the following corollary results.

Corollary 4.6. *Consider the fuzzy difference equation (1.1) where A is a positive fuzzy number. Then the following statements are true.*

(i) *Every positive solution of (1.1) is eventually periodic of period $k+2$ if and only if A is a trivial fuzzy number.*

(ii) *Every positive solution of (1.1) neither is bounded nor persists if and only if A is a nontrivial fuzzy number.*

5. Behavior of the positive solutions of fuzzy equation (1.2)

Firstly, we study the periodicity of the positive solutions of (1.2). We need the following lemma.

Lemma 5.1. *Consider the system of difference equations*

$$y_{n+1} = \max\left\{\frac{B}{z_n}, \frac{D}{z_{n-1}}\right\}, \qquad z_{n+1} = \max\left\{\frac{C}{y_n}, \frac{E}{y_{n-1}}\right\}, \tag{5.1}$$

where B, D, C, E are positive real constants and the initial values y_{-1}, y_0, z_{-1}, z_0 are positive real numbers. Then the following statements are true.

(i) *If*

$$B = C, \quad B \geq E \geq D, \quad B, D, C, E \text{ are not all equal}, \tag{5.2}$$

then every positive solution of system (5.1) is eventually periodic of period two.

(ii) *If*

$$D = E, \quad D \geq C \geq B, \quad B, D, C, E \text{ are not all equal}, \tag{5.3}$$

then every positive solution of system (5.1) is eventually periodic of period four.

Proof. We give a sketch of the proof (for more details see the appendix). Let (y_n, z_n) be a positive solution of (5.1).

(i) Firstly, we prove that if there exists an $m \in \{1,2,\ldots\}$ such that

$$E \leq y_m z_m \leq \frac{B^2}{E}, \tag{5.4}$$

then (y_n, z_n) is eventually periodic of period two.

Moreover, we prove that if for an $m \in \{1,2\}$ relation (5.4) does not hold, then there exists a $w \in \{1,2,3\}$ such that

$$u_w = y_w z_w < E. \tag{5.5}$$

In addition, we prove that if

$$D \leq u_w < E, \tag{5.6}$$

then u_m for $m = w+2$ satisfies relation (5.4) which implies that (y_n, z_n) is eventually periodic of period two.

Finally, if

$$u_w < D, \tag{5.7}$$

then we prove that there exists an $r \in \{0,1,\ldots\}$ such that

$$\left(\frac{DE}{B^2}\right)^{r+1} \leq \frac{u_w}{D} \leq \left(\frac{DE}{B^2}\right)^{r} \tag{5.8}$$

and u_m for $m = w + 3r + 3$ satisfies relation (5.4) or (5.6) and so (y_n, z_n) is eventually periodic of period two.

(ii) Firstly, we prove that if there exists an $m \in \{1, 2, \ldots\}$ such that

$$\frac{C^2}{D} \le y_m z_m \le D, \tag{5.9}$$

then (y_n, z_n) is eventually periodic of period four.

In addition, we prove that if relation (5.9) does not hold for $m \in \{1, 2, 3\}$ then there exists a $p \in \{1, 2, 3, 4\}$ such that

$$u_p = y_p z_p < \frac{C^2}{D}. \tag{5.10}$$

Furthermore, if

$$\frac{B^2}{D} \le u_p < \frac{C^2}{D}, \tag{5.11}$$

we prove that (5.9) holds for $m = p + 4$ or $m = p + 5$. Therefore, the solution (y_n, z_n) is eventually periodic of period four.

Finally, if

$$u_p < \frac{B^2}{D}, \tag{5.12}$$

then we prove that there exists a $q \in \{0, 1, \ldots\}$ such that

$$\left(\frac{BC}{D^2}\right)^{q+1} \le \frac{u_p D}{B^2} \le \left(\frac{BC}{D^2}\right)^{q} \tag{5.13}$$

and either (5.9) or (5.11) holds for $m = p + 3q + 3$ and so (y_n, z_n) is eventually periodic of period four. □

Proposition 5.2. *Consider the fuzzy difference equation (1.2) where A_i, $i = 0, 1$ are nonequal positive fuzzy numbers such that (3.6) holds and the initial values x_i, $i = -1, 0$ are positive fuzzy numbers. Then the following statements are true.*

(i) *If A_0 is a positive trivial fuzzy number such that*

$$A_{0,l,a} = A_{0,r,a} = A_0, \quad a \in (0, 1], \ \max\{A_0 - \epsilon, A_1\} = A_0 - \epsilon, \tag{5.14}$$

where ϵ is a real constant, $0 < \epsilon < A_0$, then every positive solution of (1.2) is eventually periodic of period two.

(ii) *If A_1 is a positive trivial fuzzy number such that*

$$A_{1,l,a} = A_{1,r,a} = A_1, \quad a \in (0, 1], \ \max\{A_0, A_1 - \epsilon\} = A_1 - \epsilon, \tag{5.15}$$

where ϵ is a real constant, $0 < \epsilon < A_1$, then every positive solution of (1.2) is eventually periodic of period four.

Proof. Let x_n be a positive solution of (1.2) with initial values x_i, $i = -1,0$ such that relations (3.2) for $k = 1$ and (3.4) hold, then $(L_{n,a}, R_{n,a})$, $n = 1,2,\dots$, $a \in (0,1]$ satisfies system (3.7).

(i) Firstly, suppose that (5.14) is satisfied. We define the set $E \subset (0,1]$ as follows: for any $a \in E$ there exists an $m_a \in \{1,2\}$ such that

$$A_{1,l,a} \le u_{m_a,a} \le \frac{A_0^2}{A_{1,r,a}}, \quad u_{n,a} = L_{n,a}R_{n,a}, \quad n = 1,2,\dots,\ a \in E. \tag{5.16}$$

Then from statement (i) of Lemma 5.1 the sequences $L_{n,a}$, $R_{n,a}$, $a \in E$ are periodic sequences of period two for $n \ge 5$. Moreover, since for any $a \in (0,1] - E$ the relation (5.16) does not hold, then from statement (i) of Lemma 5.1 for any $a \in (0,1] - E$ there exists a $w_a \in \{1,2,3\}$ and an $r_a \in \{0,1,\dots\}$ such that

$$u_{w_a,a} < A_{1,l,a}, \quad \left(\frac{A_{1,l,a}A_{1,r,a}}{A_0^2}\right)^{r_a+1} \le \frac{u_{w_a,a}}{A_{1,l,a}} \le \left(\frac{A_{1,l,a}A_{1,r,a}}{A_0^2}\right)^{r_a}. \tag{5.17}$$

Hence, from statement (i) of Lemma 5.1, $L_{n,a}$, $R_{n,a}$, $a \in (0,1] - E$ are periodic sequences of period two for $n \ge w_a + 3r_a + 3$ and so for $n \ge 3r_a + 6$.

We prove that there exists an $r \in \{1,2,\dots\}$ such that

$$r \ge r_a, \quad a \in (0,1] - E. \tag{5.18}$$

Since x_i, $i = 1,2,3$ are positive fuzzy numbers there exist positive real numbers K, L such that $[L_{i,a}, R_{i,a}] \subset [K, L]$, $i = 1,2,3$, $a \in (0,1] - E$. Then from (5.14) and (5.17) there exists an $r \in \{1,2,\dots\}$ such that, for $a \in (0,1] - E$,

$$\left(\frac{A_{1,l,a}A_{1,r,a}}{A_0^2}\right)^{r} \le \left(\frac{A_0 - \epsilon}{A_0}\right)^{2r} \le \frac{K^2}{A_0 - \epsilon} \le \frac{u_{w_a,a}}{A_{1,l,a}} \le \left(\frac{A_{1,l,a}A_{1,r,a}}{A_0^2}\right)^{r_a} \tag{5.19}$$

and so from (5.14) relation (5.18) is satisfied. Therefore, from (5.18) it follows that $L_{n,a}$, $R_{n,a}$, $a \in (0,1] - E$ are periodic sequences of period two for $n \ge 3r + 6$ and so x_n is eventually periodic of period two.

Arguing as above and using statement (ii) of Lemma 5.1 we can easily prove that every positive solution of (1.2) is eventually periodic of period four if relation (5.15) holds. This completes the proof of the proposition. □

In the last proposition of this paper we find conditions so that every positive solution of (1.2) neither is bounded nor persists. We need the following lemma.

Lemma 5.3. *Consider system (5.1) where B, D, C, E are positive real constants, z_{-1}, z_0, y_{-1}, y_0 are positive real numbers. If one of the following statements:*

(i) $B < C$, $D < E$,

(ii) $B < C$, $D < C$,

(iii) $D < E$, $B < E$,

is satisfied, then for every positive solution (y_n, z_n) of (5.1) relations (4.26) hold.

Proof. Firstly, suppose that conditions (i) of Lemma 5.3 are satisfied then we have that either

$$C > D \tag{5.20}$$

or

$$E > B \tag{5.21}$$

holds. Suppose that (5.20) holds. From (5.1) it is obvious that for $n = 1,2,\dots,$

$$\frac{C}{y_n} = \frac{C}{\max\{B/z_{n-1}, D/z_{n-2}\}} \geq \lambda \min\{z_{n-1}, z_{n-2}\}, \quad \lambda = \min\left\{\frac{C}{B}, \frac{C}{D}\right\}. \tag{5.22}$$

Hence, from (5.1), (5.22) it follows that relation (4.29) holds for $k = 1$. Then arguing as in Lemma 4.4 we can prove relations (4.26).

Now, consider that relation (5.21) holds. From (5.1) it is obvious that for $n = 2,3,\dots,$

$$\frac{E}{y_{n-1}} = \frac{E}{\max\{B/z_{n-2}, D/z_{n-3}\}} \geq \mu \min\{z_{n-2}, z_{n-3}\}, \quad \mu = \min\left\{\frac{E}{B}, \frac{E}{D}\right\}, \tag{5.23}$$

then from (5.1), (5.23) it follows that

$$z_{n+1} \geq \mu \min\{z_{n-2}, z_{n-3}\}, \quad n = 2,3,\dots. \tag{5.24}$$

In view of (5.24) and using the same argument to prove (4.32) we get for $r = 1,2,\dots,$

$$z_n \geq \mu^r \min\{z_2, z_1, z_0, z_{-1}\}, \quad n = 4r-1, 4r, 4r+1, 4r+2. \tag{5.25}$$

Thus, from (5.25) it is obvious that relations (4.26) are satisfied.

Now, suppose that relations (ii) (resp., (iii)) of Lemma 5.3 hold. Then relation (4.29) for $k = 1$ (resp., (5.25)) holds which implies that (4.26) is true. This completes the proof of the lemma. □

Proposition 5.4. *Consider the fuzzy difference equation (1.2) where A_i, $i = 0,1$ are positive fuzzy numbers such that (3.6) holds and the initial values x_i, $i = -1,0$ are positive fuzzy numbers. If there exists an $a \in (0,1]$ which satisfies one of the the following conditions:*

(i) $A_{0,l,a} < A_{0,r,a}$, $A_{1,l,a} < A_{1,r,a}$,

(ii) $A_{0,l,a} < A_{0,r,a}$, $A_{1,l,a} < A_{0,r,a}$,

(iii) $A_{0,l,a} < A_{1,r,a}$, $A_{1,l,a} < A_{1,r,a}$,

then the solution x_n of (1.2) neither is bounded nor persists.

Proof. Let x_n be a positive solution of (1.2) with initial values x_{-1}, x_0 such that relations (3.2) for $k = 1$ and (3.4) hold. Since there exists an $a \in (0,1]$ such that one of the relations (i), (ii), (iii) of Proposition 5.4 holds and $(L_{n,a}, R_{n,a})$, $a \in (0,1]$ satisfies (3.7) then from Lemma 5.3 and arguing as in Proposition 4.5 we can easily prove that the solution x_n of (1.2) neither is bounded nor persists. This completes the proof of the proposition. □

Appendix

Proof of Lemma 5.1. Let (y_n, z_n) be a positive solution of (5.1).

(i) Firstly, we prove that if there exists an $m \in \{1,2,\dots\}$ such that (5.4) holds, then (y_n, z_n) is eventually periodic of period two. Relations (5.1) and (5.2) imply that

$$z_n y_{n-1} \geq B, \quad y_n z_{n-1} \geq B, \quad n = 1,2,\dots. \tag{A.1}$$

From (5.2), (5.4), and (A.1) we get

$$\frac{D}{z_{m-1}} \leq \frac{D}{B} y_m \leq \frac{B}{z_m}, \qquad \frac{E}{y_{m-1}} \leq \frac{E}{B} z_m \leq \frac{B}{y_m}. \tag{A.2}$$

Using (5.1), (5.2), and (A.2) it follows that

$$y_{m+1} = \max\left\{\frac{B}{z_m}, \frac{D}{z_{m-1}}\right\} = \frac{B}{z_m}, \qquad z_{m+1} = \frac{B}{y_m}. \tag{A.3}$$

From (5.1), (5.2), (5.4), and (A.3) we can easily prove that

$$\begin{aligned} y_{m+2} &= \max\left\{y_m, \frac{D}{z_m}\right\} = y_m, \qquad z_{m+2} = z_m, \\ y_{m+3} &= \max\left\{\frac{B}{z_m}, \frac{D}{B} y_m\right\} = \frac{B}{z_m} = y_{m+1}, \quad z_{m+3} = z_{m+1}. \end{aligned} \tag{A.4}$$

Therefore, using (5.1), (A.4) and working inductively we can easily prove that

$$y_{n+2} = y_n, \quad z_{n+2} = z_n, \quad n = m+2, m+3, \dots \tag{A.5}$$

and so (y_n, z_n) is eventually periodic of period two.

Now, we prove that there exists an $m \in \{1,2,\dots\}$ such that (5.4) holds. If there exists an $m \in \{1,2\}$ such that (5.4) is satisfied, then the proof is completed. Now, suppose that for any $m \in \{1,2\}$ relation (5.4) is not true. We claim that there exists a $w \in \{1,2,3\}$ such that (5.5) holds. If for $w = 1,2$ relation (5.5) does not hold, then from (5.2) and since (5.4) is not true for $m = 1,2$ we have

$$u_w > \frac{B^2}{E} > E, \quad w = 1,2. \tag{A.6}$$

Hence, from (5.1), (5.2), (A.1), and (A.6) we get

$$y_3 z_3 = \max\left\{\frac{B^2}{y_2 z_2}, \frac{BE}{y_1 z_2}, \frac{DB}{y_2 z_1}, \frac{DE}{y_1 z_1}\right\} < E \tag{A.7}$$

and so our claim is true.

Then since from (A.1) and (5.5), relations (A.2) for $m = w$ hold, from (5.1) and (5.2) we have that relations (A.3) for $m = w$ are true. Using (5.1), (5.2), (5.5), and (A.3) for $m = w$ we can easily prove that

$$u_{w+2} = \max\left\{u_w, \frac{BE}{y_w z_{w+1}}, \frac{DB}{y_{w+1} z_w}, \frac{DE}{u_w}\right\} = \max\left\{E, \frac{DE}{u_w}\right\}. \tag{A.8}$$

Since (5.5) holds we have that either (5.6) or (5.7) is satisfied.

Firstly, suppose that (5.6) holds then from (A.8) we get $u_{w+2} = E$ and so relation (5.4) is satisfied for $m = w+2$, which means that (y_n, z_n) is eventually periodic of period two.

Now, suppose that (5.7) holds. From (5.2) and (5.7) there exists an $r \in \{0,1,\dots\}$ such that (5.8) holds. Now, we prove that, for all $s = 0,1,\dots,r+1$,

$$y_{w+3s} = \frac{y_w B^s}{E^s}, \qquad z_{w+3s} = \frac{z_w B^s}{D^s}, \qquad y_{w+3s+1} = \frac{D^s}{z_w B^{s-1}}, \qquad z_{w+3s+1} = \frac{E^s}{y_w B^{s-1}}. \tag{A.9}$$

Relations (A.3) for $m = w$ imply that (A.9) is true for $s = 0$. Suppose that (A.9) is true for an $s = j \in \{0,1,\dots,r\}$. Then from (5.1), (5.2), (5.8), (A.9) we have

$$\begin{aligned} y_{w+3j+2} &= \max\left\{\frac{B^j y_w}{E^j}, \frac{D^{j+1}}{z_w B^j}\right\} = \frac{D^{j+1}}{z_w B^j}, \\ z_{w+3j+2} &= \max\left\{\frac{B^j z_w}{D^j}, \frac{E^{j+1}}{y_w B^j}\right\} = \frac{E^{j+1}}{y_w B^j}. \end{aligned} \tag{A.10}$$

Moreover, using (5.1), (5.2), (5.8), (A.9), and (A.10) it follows that

$$y_{w+3j+3} = \frac{B^{j+1} y_w}{E^{j+1}}, \qquad z_{w+3j+3} = \frac{B^{j+1} z_w}{D^{j+1}}, \qquad y_{w+3j+4} = \frac{D^{j+1}}{B^j z_w}, \qquad z_{w+3j+4} = \frac{E^{j+1}}{B^j y_w}. \tag{A.11}$$

From relations (5.8) and (A.9) for $j = r+1$ we take that (A.9) is true for $s = 0,1,\dots,r+1$. Finally, from relations (A.11) it follows that

$$D \le u_{w+3r+3} \le \frac{B^2}{E} \tag{A.12}$$

which means that either (5.4) or (5.6) holds for $m = w+3r+3$. Therefore, (y_n, z_n) is eventually periodic of period two. This completes the proof of statement (i).

(ii) Firstly, we prove that if there exists an $m \in \{1,2,\dots\}$ such that (5.9) holds, then (y_n, z_n) is eventually periodic of period four. Relations (5.1), (5.3) imply that

$$z_n y_{n-1} \ge C, \quad y_n z_{n-1} \ge B, \quad z_n y_{n-2} \ge D, \quad n = 1,2,\dots. \tag{A.13}$$

Then from (5.1), (5.3), (5.9), and (A.13) we can easily prove that

$$y_{m+1} = \max\left\{\frac{B}{z_m}, \frac{D}{z_{m-1}}\right\} \le \frac{D}{B} y_m, \qquad z_{m+1} \le \frac{D}{C} z_m. \tag{A.14}$$

In addition, from (5.1), (5.3), (5.9), and (A.13), we get

$$\frac{B}{z_{m+1}} \le \frac{B}{C} y_m \le \frac{BD}{C} \frac{1}{z_m} \le \frac{D}{z_m} \tag{A.15}$$

and so from (5.1) we have

$$y_{m+2} = \max\left\{\frac{B}{z_{m+1}}, \frac{D}{z_m}\right\} = \frac{D}{z_m}. \tag{A.16}$$

In what follows, we consider the following four cases:
(A1) $y_m z_m \le BD/C$,
(A2) $y_m z_{m-1} \le D^2/C$, $z_m/z_{m-1} \le D/C$,
(A3) $y_m z_{m-1} \le D^2/C$, $z_m/z_{m-1} > D/C$,
(A4) $y_m z_m > BD/C$, $y_m z_{m-1} > D^2/C$.
Suppose that (A1) or (A2) is satisfied, then from (5.1) it is obvious that

$$\frac{C}{D} y_m \le \max\left\{\frac{B}{z_m}, \frac{D}{z_{m-1}}\right\} = y_{m+1} \tag{A.17}$$

which implies that

$$z_{m+2} = \max\left\{\frac{C}{y_{m+1}}, \frac{D}{y_m}\right\} = \frac{D}{y_m}. \tag{A.18}$$

Also, since relations (5.3), (5.9), and (A.14) imply that

$$\frac{B}{D} y_m \le \frac{B}{z_m} \le \frac{BD}{C} \frac{1}{z_{m+1}} \le \frac{D}{z_{m+1}}, \tag{A.19}$$

then from (5.1), (A.18), and (A.19) we have

$$y_{m+3} = \max\left\{\frac{B}{D} y_m, \frac{D}{z_{m+1}}\right\} = \frac{D}{z_{m+1}}. \tag{A.20}$$

In addition, if $z_m/z_{m-1} \le D/C$, then from (5.1), (5.3) we can easily prove that

$$z_m y_{m+1} = \max\left\{B, D\frac{z_m}{z_{m-1}}\right\} \le \frac{D^2}{C}. \tag{A.21}$$

Moreover, if (A1) is true then from (A.13), we get that $z_m/z_{m-1} = z_m y_m / y_m z_{m-1} \le D/C$ and so if (A1) or (A2) is satisfied, then from (5.1), (5.3), (A.16), and (A.21) we take

$$z_{m+3} = \max\left\{\frac{C}{D} z_m, \frac{D}{y_{m+1}}\right\} = \frac{D}{y_{m+1}}. \tag{A.22}$$

According to relations (5.1), (5.3), (A.13), (A.14), (A.16), (A.18), (A.20), and (A.22) it is easy to prove that

$$y_{m+4} = y_m, \qquad z_{m+4} = z_m, \qquad y_{m+5} = y_{m+1}, \qquad z_{m+5} = z_{m+1}. \tag{A.23}$$

Therefore, using (5.1), (5.3), (A.23) and working inductively we can easily prove that for $n = m+2, m+3, \ldots$ the following relations hold:

$$y_{n+4} = y_n, \qquad z_{n+4} = z_n, \tag{A.24}$$

which means that (y_n, z_n) is eventually periodic of period four.

Now, suppose that condition (A3) holds then obviously, relations (A.18) and (A.20) are satisfied. From (5.1), (5.3) and arguing as in (A.21) we have that $z_m y_{m+1} > D^2/C$ and so from (5.1), (5.3), and (A.16) it follows that

$$z_{m+3} = \frac{C}{y_{m+2}} = \frac{C}{D} z_m. \tag{A.25}$$

Since from (A.13) and condition (A3) we get $(C/D) z_m y_m > z_{m-1} y_m \geq B$ then from (5.1), (5.3), (5.9), (A.13), (A.14), (A.16), (A.18), (A.20), and (A.25) we can prove that

$$\begin{aligned} &y_{m+4} = y_m, \quad z_{m+4} = z_m, \quad y_{m+5} = \frac{D^2}{C z_m}, \quad z_{m+5} = z_{m+1}, \\ &y_{m+6} = y_{m+2}, \quad z_{m+6} = \frac{D}{y_m} = z_{m+2}, \quad y_{m+7} = y_{m+3}, \quad z_{m+7} = \frac{C}{D} z_m = z_{m+3}. \end{aligned} \tag{A.26}$$

Using (5.1), (5.3), (A.26) and working inductively we can easily prove that (A.24) is true for $n = m+4, m+5, \ldots$ and so (y_n, z_n) is eventually periodic of period four.

Finally, consider that condition (A4) is satisfied. From (5.1), (5.3), (5.9), (A.14) and condition (A4) we get

$$y_{m+1} < \frac{C}{D} y_m, \quad y_{m+1} z_m < \frac{C}{D} y_m z_m \leq C \leq \frac{D^2}{C}, \quad z_{m+1} y_{m+1} < y_m z_m \leq D. \tag{A.27}$$

Then, in view of (5.1), (5.3), (A.13), (A.14), (A.16), and (A.27) we have

$$\begin{aligned} &z_{m+2} = \frac{C}{y_{m+1}}, \quad y_{m+3} = \frac{D}{z_{m+1}}, \quad z_{m+3} = \frac{D}{y_{m+1}}, \quad y_{m+4} = \frac{D y_{m+1}}{C}, \\ &z_{m+4} = z_m, \quad y_{m+5} = y_{m+1}, \quad z_{m+5} = \max\left\{\frac{C^2}{D y_{m+1}}, z_{m+1}\right\}. \end{aligned} \tag{A.28}$$

If $z_{m+1} y_{m+1} > C^2/D$, then from (5.1), (5.3), and (A.28) we get

$$z_{m+5} = z_{m+1}, \quad y_{m+6} = y_{m+2}, \quad z_{m+6} = \frac{C}{y_{m+1}} = z_{m+2}. \tag{A.29}$$

Then from relations (5.1), (5.3), (A.28), (A.29) and working inductively we take relations (A.24) for $n = m+3, m+4, \ldots$ and so y_n, z_n is eventually periodic of period four.

Finally, if $z_{m+1} y_{m+1} \leq C^2/D$ from the last relation of (A.28) we get

$$z_{m+5} = \frac{C^2}{D} \frac{1}{y_{m+1}}. \tag{A.30}$$

Then from (A.27), (A.28), and (A.30) we get

$$y_{m+5} z_{m+5} = \frac{C^2}{D}, \quad y_{m+5} z_{m+4} \leq \frac{D^2}{C}. \tag{A.31}$$

Therefore, from (A.31) it is obvious that relations (5.9) and (A2) or (A3) for $m = m+5$ are satisfied and so (y_n, z_n) is eventually periodic of period four.

Now, we prove that there exists an $m \in \{1, 2, \ldots\}$ such that (5.9) holds. If there exists an $m \in \{1, 2, 3, 4\}$ such that (5.9) is satisfied, then the proof is completed. Now, suppose that for any $m \in \{1,2,3,4\}$ relation (5.9) is not true. We claim that there exists a $p \in \{1,2,3,4\}$ such that relation (5.10) holds. If for $p = 1,2$ relation (5.10) does not hold and since (5.9) is not true for $m = 1,2$, we have

$$u_1, u_2 > D. \tag{A.32}$$

Firstly, suppose that

$$z_1 y_2 > D. \tag{A.33}$$

Then since from (5.1) and (5.3) it follows that

$$u_{n+1} = \max\left\{\frac{BC}{u_n}, \frac{BD}{z_n y_{n-1}}, \frac{CD}{y_n z_{n-1}}, \frac{D^2}{u_{n-1}}\right\}, \quad u_n = y_n z_n, \tag{A.34}$$

using relations (5.3), (A.13), (A.32), (A.33), (A.34) we have

$$u_3 < \max\left\{\frac{BC}{D}, \frac{BD}{C}, C, D\right\} = D \tag{A.35}$$

and since (5.9) does not hold for $m = 3$, we get that (5.10) is true for $p = 3$.

Now, suppose that

$$z_1 y_2 \le D, \qquad u_3 \ge D. \tag{A.36}$$

Relations (5.1), (5.3), (A.32), and (A.36) imply that

$$y_3 z_2 = \max\left\{B, \frac{D z_2 y_2}{z_1 y_2}\right\} > D. \tag{A.37}$$

Then from (5.3), (A.13), (A.32), (A.34), (A.36), and (A.37) we can prove that

$$u_4 < \max\left\{\frac{BC}{D}, \frac{BD}{C}, C, D\right\} = D \tag{A.38}$$

and so (5.10) is true for $p = 4$. Thus, our claim is true.

In view of (5.10) and (A.13) it follows that

$$\frac{D}{y_{p-1}} \le \frac{D}{C} z_p < \frac{C}{y_p} \tag{A.39}$$

and so from (5.1) and (5.3) we get that

$$z_{p+1} = \frac{C}{y_p}. \tag{A.40}$$

Since (5.10) holds we have that either (5.11) or (5.12) is satisfied.

Firstly, suppose that (5.11) holds then using (5.1), (A.13) and arguing as in (A.14) we get

$$y_{p+1} \le \frac{D}{B} y_p. \tag{A.41}$$

From (5.1), (5.3), (5.11), and (A.40) we get

$$y_{p+2} = \frac{D}{z_p}, \qquad z_{p+2} = \max\left\{\frac{C}{y_{p+1}}, \frac{D}{y_p}\right\}. \tag{A.42}$$

Firstly, suppose

$$y_{p+1} < \frac{C}{D} y_p, \tag{A.43}$$

then from (A.42) we have

$$z_{p+2} = \frac{C}{y_{p+1}}. \tag{A.44}$$

Using (5.3), (5.11), (A.43) it follows that

$$\frac{C}{D} z_p < \frac{C^3}{D^2} \frac{1}{y_p} < \frac{C^4}{D^3} \frac{1}{y_{p+1}} \le \frac{D}{y_{p+1}}, \qquad \frac{BD}{C^2} y_p < \frac{B}{z_p} \le y_{p+1} \tag{A.45}$$

and so from relations (5.1), (5.3), (5.11), (A.40), (A.41), (A.42), (A.43), and (A.44) we get

$$y_{p+3} = \frac{D y_p}{C}, \qquad z_{p+3} = \frac{D}{y_{p+1}}, \qquad y_{p+4} = \frac{D y_{p+1}}{C}, \qquad z_{p+4} = \frac{C^2}{D y_p}, \tag{A.46}$$

$$y_{p+5} = y_{p+1}, \qquad z_{p+5} = \frac{C^2}{D y_{p+1}}. \tag{A.47}$$

From (A.47) clearly,

$$y_{p+5} z_{p+5} = \frac{C^2}{D} \tag{A.48}$$

and so relation (5.9) holds for $m = p + 5$ which means that (y_n, z_n) is eventually periodic of period four.

Now, suppose that

$$y_{p+1} \ge \frac{C}{D} y_p. \tag{A.49}$$

Then (5.1), (5.3), and (A.49) imply that

$$z_{p+2} = \frac{D}{y_p}. \tag{A.50}$$

Since from (A.13) and (A.41) it results that $B/z_{p+3} \le (B/D) y_{p+1} \le y_p$ then using (5.1), (5.3), (5.11), (A.40), (A.42), and (A.50) we get

$$y_{p+3} = \frac{D y_p}{C}, \qquad z_{p+3} \ge \frac{D}{y_{p+1}}, \qquad y_{p+4} = y_p, \qquad z_{p+4} = \frac{C^2}{D y_p}. \tag{A.51}$$

From (A.51) we get

$$y_{p+4}z_{p+4} = \frac{C^2}{D} \tag{A.52}$$

and so relation (5.9) holds for $m = p+4$, which means that (y_n, z_n) is eventually periodic of period four.

Finally, suppose that relation (5.12) holds. From (5.3) and (5.12) there exists a $q \in \{0,1,\dots\}$ such that relation (5.13) holds. Using the same argument to prove (A.9), we can prove that, for all $s = 0,1,\dots,q+1$,

$$y_{p+3s} = \frac{y_p D^s}{C^s}, \qquad z_{p+3s} = \frac{z_p D^s}{B^s}, \qquad y_{p+3s+1} = \frac{B^{s+1}}{z_p D^s}, \qquad z_{p+3s+1} = \frac{C^{s+1}}{y_p D^s}. \tag{A.53}$$

From (5.3), (5.13), and (A.53) for $s = q+1$ it easily results that

$$\frac{B^2}{D} \le u_{p+3q+3} \le \frac{BD}{C} \le D \tag{A.54}$$

and so we have that either (5.9) or (5.11) is satisfied for $m = p+3q+3$, which means that (y_n, z_n) is eventually periodic of period four. Thus, the proof of the lemma is completed. □

Acknowledgment

This work is a part of the Doctoral thesis of G. Stefanidou.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities. Theory, Methods, and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 155, Marcel Dekker, New York, 1992.

[2] A. M. Amleh, J. Hoag, and G. Ladas, *A difference equation with eventually periodic solutions*, Comput. Math. Appl. **36** (1998), no. 10–12, 401–404.

[3] D. Benest and C. Froeschlé (eds.), *Analysis and Modelling of Discrete Dynamical Systems*, Advances in Discrete Mathematics and Applications, vol. 1, Gordon and Breach Science Publishers, Amsterdam, 1998.

[4] E. Y. Deeba and A. De Korvin, *Analysis by fuzzy difference equations of a model of* CO_2 *level in the blood*, Appl. Math. Lett. **12** (1999), no. 3, 33–40.

[5] E. Y. Deeba, A. De Korvin, and E. L. Koh, *A fuzzy difference equation with an application*, J. Differ. Equations Appl. **2** (1996), no. 4, 365–374.

[6] L. Edelstein-Keshet, *Mathematical Models in Biology*, The Random House/Birkhäuser Mathematics Series, Random House, New York, 1988.

[7] S. N. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, 1996.

[8] G. J. Klir and Bo Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice-Hall PTR, New Jersey, 1995.

[9] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Mathematics and Its Applications, vol. 256, Kluwer Academic Publishers, Dordrecht, 1993.

[10] V. Lakshmikantham and A. S. Vatsala, *Basic theory of fuzzy difference equations*, J. Difference Equ. Appl. **8** (2002), no. 11, 957–968.

[11] D. P. Mishev, W. T. Patula, and H. D. Voulov, *A reciprocal difference equation with maximum*, Comput. Math. Appl. **43** (2002), no. 8-9, 1021–1026.
[12] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, CRC Press, Florida, 1997.
[13] G. Papaschinopoulos and B. K. Papadopoulos, *On the fuzzy difference equation* $x_{n+1} = A + x_n/x_{n-m}$, Fuzzy Sets and Systems **129** (2002), no. 1, 73–81.
[14] ______, *On the fuzzy difference equation* $x_{n+1} = A + B/x_n$, Soft Comput. **6** (2002), no. 6, 456–461.
[15] G. Papaschinopoulos and G. Stefanidou, *Boundedness and asymptotic behavior of the solutions of a fuzzy difference equation*, Fuzzy Sets and Systems **140** (2003), no. 3, 523–539.
[16] G. Stefanidou and G. Papaschinopoulos, *Trichotomy, stability, and oscillation of a fuzzy difference equation*, Adv. Difference Equ. **2004** (2004), no. 4, 337–357.
[17] I. Szalkai, *On the periodicity of the sequence* $x_{n+1} = \max\{A_0/x_n, A_1/x_{n-1}, \dots, A_k/x_{n-k}\}$, J. Differ. Equations Appl. **5** (1999), no. 1, 25–29.
[18] H. D. Voulov, *On the periodic character of some difference equations*, J. Difference Equ. Appl. **8** (2002), no. 9, 799–810.
[19] C. Wu and B. Zhang, *Embedding problem of noncompact fuzzy number space* $E^{\sim}$*. I*, Fuzzy Sets and Systems **105** (1999), no. 1, 165–169.

G. Stefanidou: Department of Electrical and Computer Engineering, Democritus University of Thrace, 67100 Xanthi, Greece
E-mail address: tfele@yahoo.gr

G. Papaschinopoulos: Department of Electrical and Computer Engineering, Democritus University of Thrace, 67100 Xanthi, Greece
E-mail address: gpapas@ee.duth.gr