CONSTRUCTION OF UPPER AND LOWER SOLUTIONS FOR SINGULAR DISCRETE INITIAL AND BOUNDARY VALUE PROBLEMS VIA INEQUALITY THEORY

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We present new existence results for singular discrete initial and boundary value problems. In particular our nonlinearity may be singular in its dependent variable and is allowed to change sign.

1. Introduction

An upper- and lower-solution theory is presented for the singular discrete boundary value problem

$$\begin{gathered} -\Delta\left(\varphi_p\left(\Delta u(k-1)\right)\right) = q(k)f\left(k,u(k)\right), \quad k \in N = \{1,\dots,T\}, \\ u(0) = u(T+1) = 0, \end{gathered} \tag{1.1}$$

and the singular discrete initial value problem

$$\begin{gathered} \Delta u(k-1) = q(k)f\left(k,u(k)\right), \quad k \in N = \{1,\dots,T\}, \\ u(0) = 0, \end{gathered} \tag{1.2}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\Delta u(k-1) = u(k) - u(k-1)$, $T \in \{1,2,\dots\}$, $N^+ = \{0,1,\dots,T\}$, and $u : N^+ \to \mathbb{R}$. Throughout this paper, we will assume $f : N \times (0,\infty) \to \mathbb{R}$ is continuous. As a result, our nonlinearity $f(k,u)$ may be singular at $u = 0$ and may change sign.

Remark 1.1. Recall a map $f : N \times (0,\infty) \to \mathbb{R}$ is continuous if it is continuous as a map of the topological space $N \times (0,\infty)$ into the topological space $\mathbb{R}$. Throughout this paper, the topolopy on N will be the discrete topology.

We will let $C(N^+,\mathbb{R})$ denote the class of map u continuous on N^+ (discrete topology) with norm $\|u\| = \max_{k\in N^+} \|u(k)\|$. By a solution to (1.1) (resp., (1.2)) we mean a $u \in C(N^+,\mathbb{R})$ such that u satisfies (1.1) (resp., (1.2)) for $i \in N$ and u satisfies the boundary (resp., initial) condition.

It is interesting to note here that the existence of solutions to singular initial and boundary value problems in the continuous case have been studied in great detail in

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the literature (see [2, 4, 5, 6, 7, 9, 10, 11] and the references therein). However, only a few papers have discussed the discrete singular case (see [1, 3, 8] and the references therein).

In [7], the following result has been proved.

Theorem 1.2. *Let* $n_0 \in \{1,2,\dots\}$ *be fixed and suppose the following conditions are satisfied:*

$$f : N \times (0,\infty) \longrightarrow \mathbb{R} \text{ is continuous,} \tag{1.3}$$

$$q \in C(N,(0,\infty)), \tag{1.4}$$

$$\begin{gathered} \text{there exists a function } \alpha \in C(N^+,\mathbb{R}) \text{ with} \\ \alpha(0) = \alpha(T+1) = 0, \quad \alpha > 0 \text{ on } N \text{ such that} \\ q(k)f(k,\alpha(k)) \geq -\Delta(\varphi_p(\alpha(k-1))) \quad \text{for } k \in N, \end{gathered} \tag{1.5}$$

$$\begin{gathered} \text{there exists a function } \beta \in C(N^+,\mathbb{R}) \text{ with} \\ \beta(k) \geq \alpha(k), \quad \beta(k) \geq \frac{1}{n_0} \quad \text{for } k \in N^+ \text{ with} \\ q(k)f(k,\beta(k)) \leq -\Delta(\varphi_p(\beta(k-1))) \quad \text{for } k \in N. \end{gathered} \tag{1.6}$$

Then (1.1) has a solution $u \in C(N^+,\mathbb{R})$ *with* $u(k) \geq \alpha(k)$ *for* $k \in N^+$.

In [1], the following result has been proved.

Theorem 1.3. *Let* $n_0 \in \{1,2,\dots\}$ *be fixed and suppose the following conditions are satisfied:*

$$f : N \times (0,\infty) \longrightarrow \mathbb{R} \text{ is continuous,} \tag{1.7}$$

$$q \in C(N,(0,\infty)), \tag{1.8}$$

$$\begin{gathered} \text{there exists a function } \alpha \in C(N^+,\mathbb{R}) \text{ with} \\ \alpha(0) = 0, \quad \alpha > 0 \text{ on } N \text{ such that} \\ q(k)f(k,\alpha(k)) \geq \Delta\alpha(k-1) \quad \text{for } k \in N, \end{gathered} \tag{1.9}$$

$$\begin{gathered} \text{there exists a function } \beta \in C(N^+,\mathbb{R}) \text{ with} \\ \beta(k) \geq \alpha(k), \quad \beta(k) > \frac{1}{n_0} \quad \text{for } k \in N^+ \text{ with} \\ q(k)f(k,\beta(k)) \leq \Delta\beta(k-1) \quad \text{for } k \in N. \end{gathered} \tag{1.10}$$

Then (1.2) has a solution $u \in C(N^+,\mathbb{R})$ *with* $u(k) \geq \alpha(k)$ *for* $k \in N^+$.

Also some results from the literature, which will be needed in Section 2 are presented.

Lemma 1.4 [8]. *Let* $u \in C(N^+,\mathbb{R})$ *satisfy* $u(k) \geq 0$ *for* $k \in N^+$. *If* $u \in C(N^+,\mathbb{R})$ *satisfies*

$$\begin{gathered} -\Delta^2 u(k-1) = u(k), \quad k \in N = \{1,2,\dots,T\}, \\ u(0) = u(T+1) = 0, \end{gathered} \tag{1.11}$$

then

$$u(k) \geq \mu(k)\|u\| \quad \text{for } k \in N^+; \tag{1.12}$$

here

$$\mu(k) = \min\left\{\frac{T+1-k}{T+1}, \frac{k}{T}\right\}. \tag{1.13}$$

Lemma 1.5 [8]. *Let* $[a,b] = \{a, a+1, \dots, b\} \subset N$. *If* $u \in C(N^+, \mathbb{R})$ *satisfies*

$$\begin{gathered} \Delta(\varphi_p(\Delta u(k-1))) \leq 0, \quad k \in [a,b], \\ u(a-1) \geq 0, \quad u(b+1) \geq 0, \end{gathered} \tag{1.14}$$

then $u(k) \geq 0$ *for* $k \in [a-1, b+1] = \{a-1, a, \dots, b+1\} \subset N^+$.

In Theorems 1.2 and 1.3 the construction of a lower solution α and an upper solution β is critical. We present an easily verifiable condition in Section 2.

2. Main results

We begin with a result for boundary value problems.

Theorem 2.1. *Let* $n_0 \in \{1, 2, \dots\}$ *be fixed and suppose (1.3), (1.4) hold. Also assume the following conditions are satisfied:*

there exists a constant $c_0 > 0$ *such that*

$$q(k)f(k,u) \geq c_0 \quad \text{for } k \in N,\ 0 < u \leq \frac{1}{n_0}, \tag{2.1}$$

there exist $h > 0$ *continuous and nondecreasing on* $[0, \infty)$ *such that*

$$|f(k,u)| \leq h(u) \quad \text{for } (k,u) \in N \times \left[\frac{1}{n_0}, \infty\right), \tag{2.2}$$

there exist $M > \frac{1}{n_0}$ *such that*

$$M - \frac{1}{n_0} > \varphi_p^{-1}(h(M))\, b_0; \tag{2.3}$$

here

$$b_0 = \max_{k \in N}\left\{\sum_{i=1}^{k} \varphi_p^{-1}\left(\sum_{j=i}^{k} q(j)\right), \sum_{i=1}^{k} \varphi_p^{-1}\left(\sum_{j=i}^{k} q(j)\right)\right\}. \tag{2.4}$$

Then (1.1) has a solution $u \in C(N^+, \mathbb{R})$ *with* $u(k) > 0$ *for* $k \in N$.

Proof. First we construct the lower solution α in (1.5). Let $\alpha(k) = cv(k)$, $k \in N^+$, where $v \in C(N^+, [0,\infty))$ is the solution of

$$\begin{gathered} -\Delta(\varphi_p(\Delta v(k-1))) = 1, \quad k \in N, \\ v(0) = v(T+1) = 0, \end{gathered} \tag{2.5}$$

$$0 < c < \min\left\{c_0^{1/(p-1)}, \frac{1}{n_0\|v\|}\right\}. \tag{2.6}$$

Since $-\Delta(\varphi_p(\Delta v(k-1))) > 0$ implies $\Delta^2 v(k-1) < 0$ for $k \in N$, it follows from Lemma 1.4 that $v(k) \geq \mu(k)\|v\|$ for $k \in N^+$. Thus,

$$0 < \alpha(k) \leq \frac{1}{n_0} \quad \text{for } k \in N, \tag{2.7}$$

$$\begin{gathered} -\Delta(\varphi_p(\Delta\alpha(k-1))) = c^{p-1} \leq c_0 \quad \text{for } k \in N, \\ \alpha(0) = \alpha(T+1) = 0. \end{gathered} \tag{2.8}$$

As a result (1.5) holds, since

$$q(k)f(k,\alpha(k)) \geq c_0 \geq -\Delta(\varphi_p(\Delta\alpha(k-1))) \quad \text{for } k \in N. \tag{2.9}$$

Next we discuss the boundary value problem

$$\begin{gathered} -\Delta(\varphi_p(\Delta u(k-1))) = q(k)h(M), \quad k \in N, \\ u(0) = u(T+1) = \frac{1}{n_0}. \end{gathered} \tag{2.10}$$

It follows from [8] that (2.10) has a solution $u \in C(N^+, \mathbb{R})$. Let $v(k) = u(k) - 1/n_0$ for $k \in N^+$. Then $\Delta(\varphi_p(\Delta u(k-1))) = -\Delta(\varphi_p(\Delta v(k-1))) \leq 0$ for $k \in N$, and $v(0) = v(T+1) = 0$. Lemma 1.5 guarantees that $v(k) \geq 0$ and so $u(k) \geq 1/n_0$ for $k \in N^+$. Next we prove $u(k) \leq M$ for $k \in N^+$. Now since $\Delta(\varphi_p(\Delta u(k-1))) \leq 0$ on N implies $\Delta^2 u(k-1) \leq 0$ on N, then there exists $k_0 \in N$ with $\Delta u(k) \geq 0$ on $[0,k_0) = \{0,1,\dots,k_0-1\}$ and $\Delta u(k) \leq 0$ on $[k_0, T+1) = \{k_0, k_0+1, \dots, T\}$, and $u(k_0) = \|u\|$. Suppose $u(k_0) > M$.

Also notice that for $k \in N$, we have

$$-\Delta(\varphi_p(\Delta u(k-1))) = q(k)h(M). \tag{2.11}$$

We sum (2.11) from $j+1$ $(0 \leq j < k_0)$ to k_0 to obtain

$$\varphi_p(\Delta u(j)) = \varphi_p(\Delta u(k_0)) + h(M)\sum_{k=j+1}^{k_0} q(k). \tag{2.12}$$

Now since $\Delta u(k_0) \leq 0$, we have

$$\varphi_p(\Delta u(j)) \leq h(M)\sum_{k=j+1}^{k_0} q(k) \quad \text{for } 0 \leq j < k_0, \tag{2.13}$$

that is,

$$\Delta u(j) \leq \varphi_p^{-1}(h(M))\varphi_p^{-1}\left(\sum_{k=j+1}^{k_0} q(k)\right) \quad \text{for } 0 \leq j < k_0. \tag{2.14}$$

Then we sum the above from 0 to $k_0 - 1$ to obtain

$$\begin{aligned} u(k_0) - u(0) &\leq \varphi_p^{-1}(h(M)) \sum_{j=0}^{k_0-1} \varphi_p^{-1}\left(\sum_{k=j+1}^{k_0} q(k)\right) \\ &\leq \varphi_p^{-1}(h(M)) \sum_{j=1}^{k_0} \varphi_p^{-1}\left(\sum_{k=j}^{k_0} q(k)\right). \end{aligned} \tag{2.15}$$

Similarly, we sum (2.11) from k_0 to j ($k_0 \leq j \leq T+1$) to obtain

$$-\varphi_p(\Delta u(j)) = -\varphi_p(\Delta u(k_0 - 1)) + h(M) \sum_{k=k_0}^{j} q(k) \quad \text{for } j \geq k_0. \tag{2.16}$$

Now since $\Delta u(k_0 - 1) \geq 0$, we have

$$-\Delta u(j) = \varphi_p^{-1}(h(M))\varphi_p^{-1}\left(\sum_{k=k_0}^{j} q(k)\right) \quad \text{for } j \geq k_0. \tag{2.17}$$

We sum the above from k_0 to T to obtain

$$u(k_0) - u(T+1) \leq \varphi_p^{-1}(h(M)) \sum_{j=k_0}^{T} \varphi_p^{-1}\left(\sum_{k=k_0}^{j} q(k)\right). \tag{2.18}$$

Now (2.15) and (2.18) imply

$$M - \frac{1}{n_0} \leq b_0 \varphi_p^{-1}(h(M)). \tag{2.19}$$

This contradicts (2.3). Thus

$$\frac{1}{n_0} \leq u(k) \leq M \quad \text{for } k \in N^+. \tag{2.20}$$

Let $\beta(k) \equiv u(k)$ for $k \in N^+$. Now (2.7) and (2.20) guarantee

$$\alpha(k) \leq \beta(k) \quad \text{for } k \in N^+. \tag{2.21}$$

Now (2.2) and (2.20) imply $f(k,\beta(k)) \leq h(\beta(k)) \leq h(M)$ so

$$\begin{gathered} \beta \in C(N^+, \mathbb{R}) \text{ with} \\ \beta(k) \geq \alpha(k), \quad \beta(k) \geq \frac{1}{n_0} \quad \text{for } k \in N^+ \text{ with} \\ q(k) f(k,\beta(k)) \leq -\Delta(\varphi_p(\beta(k-1))) \quad \text{for } k \in N. \end{gathered} \tag{2.22}$$

Now Theorem 1.2 guarantees that (1.1) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) \geq \alpha(k) > 0$ for $k \in N$. □

Example 2.2. Consider the boundary value problem

$$\Delta^2 u(k-1) = \frac{k}{[u(k)]^\alpha} + [u(k)]^\beta - A, \quad k \in N, \qquad u(0) = u(T+1) = 0 \tag{2.23}$$

with $p = 2$, $\alpha > 0$, $0 \le \beta < 1$, and $A > 0$. Then (2.23) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

To see this, we will apply Theorem 2.1 with

$$q(k) = 1, \qquad f(k,u) = \frac{k}{u^\alpha} + u^\beta - A. \tag{2.24}$$

Let $n_0 > (2A)^{1/\alpha}$ and $c_0 = A$. Then for $k \in N$ and $0 < u \le 1/n_0$,

$$q(k)f(k,u) = \frac{k}{u^\alpha} + u^\beta - A \ge \frac{k}{u^\alpha} - A \ge \frac{1}{u^\alpha} - A \ge 2A - A = A = c_0, \tag{2.25}$$

so (2.1) is satisfied. Let $h(u) = u^\beta + n_0^\alpha T + A$. Then (2.2) is immediate. Also since $0 \le \beta < 1$, we see that

$$\text{there exist } M > \frac{1}{n_0} \text{ such that } M - \frac{1}{n_0} > b_0\left(M^\beta + n_0^\alpha T + A\right); \tag{2.26}$$

here

$$b_0 = \max_{k\in N}\left(\sum_{j=1}^{k}(k-j+1), \sum_{j=k}^{T}(j-k+1)\right). \tag{2.27}$$

Thus (2.3) holds. Theorem 2.1 guarantees that (2.23) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

Next we present a result for initial value problems.

Theorem 2.3. *Let $n_0 \in \{1, 2, \ldots\}$ be fixed and suppose (1.2), (1.3) hold. Also assume the following conditions are satisfied:*

there exists a constant $c_0 > 0$ such that

$$q(k)f(k,u) \ge c_0 \quad \text{for } k \in N,\ 0 < u \le \frac{1}{n_0}, \tag{2.28}$$

there exist $h > 0$ continuous and nondecreasing on $[0, \infty)$ such that

$$|f(k,u)| \le h(u) \quad \text{for } (k,u) \in N \times \left[\frac{1}{n_0}, \infty\right), \tag{2.29}$$

there exist $M > \dfrac{1}{n_0}$ such that

$$M - \frac{1}{n_0} > h(M)\sum_{k=1}^{T} q(k). \tag{2.30}$$

Then (1.2) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

Proof. First we construct the lower solution α in (1.9). Let

$$\alpha(k) = \begin{cases} c\sum_{i=1}^{k} q(i), & k \in N, \\ 0, & k = 0, \end{cases} \tag{2.31}$$

where

$$0 < c < \frac{1}{n_0 \sum_{i=1}^{T} q(i)}, \quad c \max_{k\in N} q(k) \le c_0. \tag{2.32}$$

Then (2.7) holds, and $\alpha(0) = 0$, $\Delta\alpha(k-1) = \alpha(k) - \alpha(k-1) = cq(k) \le c_0$ for $k \in N$ with (1.9) holding, since

$$q(k) f(k, \alpha(k)) \ge c_0 \ge \Delta\alpha(k-1) \quad \text{for } k \in N. \tag{2.33}$$

Next we discuss the initial value problem

$$\begin{gathered} \Delta u(k-1) = q(k) f^*(k, u(k)), \quad k \in N, \\ u(0) = \frac{1}{n_0}; \end{gathered} \tag{2.34}$$

here

$$f^*(k,u) = \begin{cases} f\left(k, \dfrac{1}{n_0}\right), & u \le \dfrac{1}{n_0}, \\ f(k,u), & \dfrac{1}{n_0} \le u \le M, \\ f(k,M), & u \ge M. \end{cases} \tag{2.35}$$

Then (2.34) is equivalent to

$$u(k) = \begin{cases} \dfrac{1}{n_0} + \sum_{i=1}^{k} q(i) f^*(i, u(i)), & k \in N, \\ \dfrac{1}{n_0}, & k = 0. \end{cases} \tag{2.36}$$

From Brouwer's fixed point theorem, we know that (2.34) has a solution $u \in C(N^+, \mathbb{R})$. We first show

$$u(k) \ge \frac{1}{n_0} \quad \text{for } k \in N^+. \tag{2.37}$$

Suppose (2.37) is not true. Then there exists a $\tau \in N$ such that

$$u(\tau) < \frac{1}{n_0}, \quad u(\tau - 1) \ge \frac{1}{n_0} \tag{2.38}$$

since $u(0) = 1/n_0$. Thus we have, from (2.28)

$$\Delta u(\tau - 1) = q(\tau) f^*(\tau, u(\tau)) = q(\tau) f\left(\tau, \frac{1}{n_0}\right) > 0, \tag{2.39}$$

so

$$u(\tau) - \frac{1}{n_0} > u(\tau - 1) - \frac{1}{n_0} \geq 0, \tag{2.40}$$

a contradiction. Thus (2.37) is satisfied. Next we show

$$u(k) \leq M \quad \text{for } k \in N^+. \tag{2.41}$$

Suppose (2.41) is false. Then since $u(0) = 1/n_0$, there exists $\tau \in N$ such that

$$u(\tau) > M, \qquad u(k) \leq M \quad \text{for } k \in \{0, 1, \ldots, \tau - 1\}. \tag{2.42}$$

Thus, we have

$$\begin{aligned}
\Delta u(\tau - 1) &= u(\tau) - u(\tau - 1) \leq q(\tau) h(M), \\
\Delta u(\tau - 2) &= u(\tau - 1) - u(\tau - 2) \leq q(\tau - 1) h(M), \\
&\vdots \\
\Delta u(0) &= u(1) - u(0) \leq q(1) h(M).
\end{aligned} \tag{2.43}$$

Adding both sides of the above formula gives

$$u(\tau) - u(0) \leq h(M) \sum_{k=1}^{\tau} q(k) \leq h(M) \sum_{k=1}^{T} q(k), \tag{2.44}$$

that is,

$$M - \frac{1}{n_0} \leq h(M) \sum_{k=1}^{T} q(k). \tag{2.45}$$

This contradicts (2.30). Thus, we have (2.20). Let $\beta(k) \equiv u(k)$ for $k \in N^+$. By (2.7) and (2.37), we have $\alpha(k) \leq \beta(k)$ for $k \in N^+$. Then

$$\begin{gathered}
\beta \in C(N^+, \mathbb{R}) \text{ with} \\
\beta(k) \geq \alpha(k), \quad \beta(k) > \frac{1}{n_0} \quad \text{for } k \in N^+ \text{ with} \\
q(k) f(k, \beta(k)) = \Delta \beta(k - 1) \quad \text{for } k \in N.
\end{gathered} \tag{2.46}$$

Now Theorem 1.3 guarantees that (1.2) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) \geq \alpha(k) > 0$ for $k \in N$. □

Example 2.4. Consider the initial value problem

$$\Delta u(k-1) = k[u(k)]^{-\alpha} + [u(k)]^{\beta} - A, \quad k \in N, \\ u(0) = 0 \tag{2.47}$$

with $\alpha > 0$, $0 \leq \beta < 1$, and $A > 0$. Now (2.47) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

To see this we will apply Theorem 2.3 with (2.24). Let $n_0 > (2A)^{1/\alpha}$ and $c_0 = A$. Then for $k \in N$ and $0 < u \leq 1/n_0$, (2.25) holds and so (2.28) is satisfied. Let $h(u) = u^{\beta} + n_0^{\alpha}\, T + A$. Then (2.29) is immediate. Also since $0 \leq \beta < 1$, we see

$$\text{there exist } M > \frac{1}{n_0} \quad \text{such that } M - \frac{1}{n_0} > T\left(M^{\beta} + n_0^{\alpha} T + A\right), \tag{2.48}$$

so (2.30) holds. Theorem 2.3 guarantees that (2.47) has a solution $u \in C(N^+, \mathbb{R})$ with $u(k) > 0$ for $k \in N$.

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