PERIODIC SOLUTIONS FOR A COUPLED PAIR OF DELAY DIFFERENCE EQUATIONS

GUANG ZHANG, SHUGUI KANG, AND SUI SUN CHENG

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Based on the fixed-point index theory for a Banach space, positive periodic solutions are found for a system of delay difference equations. By using such results, the existence of nontrivial periodic solutions for delay difference equations with positive and negative terms is also considered.

1. Introduction

The existence of positive periodic solutions for delay difference equations of the form

$$x_{n+1} = a_n x_n + h_n f\big(n, x_{n-\tau(n)}\big), \quad n \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}, \tag{1.1}$$

has been studied by many authors, see, for example, [1, 3, 5, 7, 8, 9] and the references contained therein. The above equation may be regarded as a mathematical model for a number of dynamical processes. In particular, x_n may represent the size of a population in the time period n. Since it is possible that the population may be influenced by another factor of the form $-\widehat{h}_n f_2(n, x_{n-\tau(n)})$, we are therefore interested in a more general equation of the form

$$x_{n+1} = a_n x_n + h_n f_1\big(n, x_{n-\tau(n)}\big) - \widehat{h}_n f_2\big(n, x_{n-\tau(n)}\big), \tag{1.2}$$

which includes the so-called difference equations with positive and negative terms (see, e.g., [6]).

In this paper, we will approach this equation (see Section 4) by treating it as a special case of a system of difference equations of the form

$$\begin{aligned} u_n &= \sum_{s=n}^{n+\omega-1} G(n,s) h_s f_1\big(s, u_{s-\tau(s)} - v_{s-\tau(s)}\big), \\ v_n &= \sum_{s=n}^{n+\omega-1} \widehat{G}(n,s) \widehat{h}_s f_2\big(s, u_{s-\tau(s)} - v_{s-\tau(s)}\big), \end{aligned} \tag{1.3}$$

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where $n \in \mathbb{Z}$. We will assume that ω is a positive integer, G and $\widehat{G}$ are double sequences satisfying $G(n,s) = G(n+\omega, s+\omega)$ and $\widehat{G}(n,s) = \widehat{G}(n+\omega, s+\omega)$ for $n,s \in \mathbb{Z}$, $h = \{h_n\}_{n\in\mathbb{Z}}$ and $\widehat{h} = \{\widehat{h}_n\}_{n\in\mathbb{Z}}$ are positive ω-periodic sequences, $\{\tau(n)\}_{n\in\mathbb{Z}}$ is an integer-valued ω-periodic sequence, $f_1, f_2 : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ are continuous functions, and $f_1(n+\omega,u) = f_1(n,u)$ as well as $f_2(n+\omega,u) = f_2(n,u)$ for any $u \in \mathbb{R}$ and $n \in \mathbb{Z}$.

By a solution of (1.3), we mean a pair (u,v) of sequences $u = \{u_n\}_{n\in\mathbb{Z}}$ and $v = \{v_n\}_{n\in\mathbb{Z}}$ which renders (1.3) into an identity for each $n \in \mathbb{Z}$ after substitution. A solution (u,v) is said to be ω-periodic if $u_{n+\omega} = u_n$ and $v_{n+\omega} = v_n$ for $n \in \mathbb{Z}$.

Let X be the set of all real ω-periodic sequences of the form $u = \{u_n\}_{n\in\mathbb{Z}}$ and endowed with the usual linear structure and ordering (i.e., $u \le v$ if $u_n \le v_n$ for $n \in \mathbb{Z}$). When equipped with the norm

$$\|u\| = \max_{0\le n\le\omega-1} |u_n|, \quad u \in X, \tag{1.4}$$

X is an ordered Banach space with cone $\Omega_0 = \{u = \{u_n\}_{n\in\mathbb{Z}} \in X \mid u_n \ge 0,\ n \in \mathbb{Z}\}$. $X\times X$ will denote the product (Banach) space equipped with the norm

$$\|(u,v)\| = \max\{\|u\|, \|v\|\}, \quad u,v \in X, \tag{1.5}$$

and ordering defined by $(u,v) \le (x,y)$ if $u \le x$ and $v \le y$ for any $u,v,x,y \in X$.

We remark that a recent paper [4] is concerned with the differential system

$$\begin{aligned} y' &= -a(t)y(t) + f(t, y(t-\tau(t))), \\ x' &= -a(t)x(t) + f(t, x(t-\tau(t))). \end{aligned} \tag{1.6}$$

There are some ideas in the proof of Theorem 2.1 which are similar to those in [4]. But the techniques in the other results are new.

2. Main result

In this section, we assume that

$$\begin{aligned} &0 < m \le G(n,s) \le M < +\infty, \quad n \le s \le n+\omega-1, \\ &0 < m' \le \widehat{G}(n,s) \le M' < +\infty, \quad n \le s \le n+\omega-1. \end{aligned} \tag{2.1}$$

Then,

$$\Omega = \Big\{\{u_n\}_{n\in\mathbb{Z}} \in X : u_n \geqslant \sigma\|u\|,\ n \in \mathbb{Z}\Big\}, \quad \text{where } \sigma = \min\left\{\frac{m}{M}, \frac{m'}{M'}\right\} \tag{2.2}$$

is a cone in X and $\Omega\times\Omega$ is a cone in $X\times X$.

THEOREM 2.1. *In addition to the assumptions imposed on the functions G, $\widehat{G}$, h, $\widehat{h}$, f_1, and f_2 in Section 1, suppose that G and $\widehat{G}$ satisfy (2.1). Suppose further that f_1, f_2 are nonnegative and satisfy $f_1(n,0)=0=f_2(n,0)$ for $n\in\mathbb{Z}$ as well as*

$$\lim_{|x|\to 0}\frac{f_1(n,x)}{|x|}=+\infty, \tag{2.3}$$

$$\lim_{|x|\to 0}\frac{f_2(n,x)}{|x|}<+\infty, \tag{2.4}$$

$$\lim_{x\to+\infty}\frac{f_1(n,x)}{x}=0, \tag{2.5}$$

$$\lim_{|x|\to+\infty}\frac{f_2(n,x)}{|x|}=0, \tag{2.6}$$

uniformly with respect to all $n\in\mathbb{Z}$. Then (1.3) has an ω -periodic solution (u,v) in $\Omega\times\Omega$ such that $\|(u,v)\|>0$. In the sequel, $(\Omega\times\Omega)_\alpha$ will denote the set $\{(u,v)\in\Omega\times\Omega \mid \|(u,v)\|=\alpha\}$.

Proof. Let $A_1,A_2:\Omega\times\Omega\to X$ and $A:\Omega\times\Omega\to X\times X$ be defined, respectively, by

$$\begin{gathered}
(A_1(u,v))_n=\sum_{s=n}^{n+\omega-1}G(n,s)h_sf_1(s,u_{s-\tau(s)}-v_{s-\tau(s)}),\quad n\in\mathbb{Z},\\
(A_2(u,v))_n=\sum_{s=n}^{n+\omega-1}\widehat{G}(n,s)\widehat{h}_sf_2(s,u_{s-\tau(s)}-v_{s-\tau(s)}),\quad n\in\mathbb{Z},\\
(A(u,v))_n=(A_1(u,v)_n,A_2(u,v)_n),\quad n\in\mathbb{Z},
\end{gathered} \tag{2.7}$$

for $u,v\in\Omega$. For any $n,\check{n}\in\mathbb{Z}$, we have

$$\begin{aligned}
(A_1(u,v))_n&=\sum_{s=n}^{n+\omega-1}G(n,s)h_sf_1(s,u_{s-\tau(s)}-v_{s-\tau(s)})\\
&\le M\sum_{s=0}^{\omega-1}h_sf_1(s,u_{s-\tau(s)}-v_{s-\tau(s)}),\\
(A_1(u,v))_{\check{n}}&=\sum_{s=\check{n}}^{\check{n}+\omega-1}G(\check{n},s)h_sf_1(s,u_{s-\tau(s)}-v_{s-\tau(s)})\\
&\geqslant m\sum_{s=0}^{\omega-1}h_sf_1(s,u_{s-\tau(s)}-v_{s-\tau(s)})\\
&\geqslant \sigma(A_1(u,v))_n.
\end{aligned} \tag{2.8}$$

Similarly, we can prove that $(A_2(u,v))_{\check{n}}\geqslant\sigma(A_2(u,v))_n$ for any $n,\check{n}\in\mathbb{Z}$. Thus, $A:\Omega\times\Omega\to\Omega\times\Omega$. Furthermore, in view of the boundedness of G and $\widehat{G}$, and the continuity of f_1 and f_2, it is not difficult to show that A is completely continuous. Indeed, $A(B)$ is a bounded set for any bounded subset B of $X\times X$. Since $X\times X$ is made up of ω-periodic sequences, thus $A(B)$ is precompact. Consequently, A is completely continuous.

We will show that there exist r^*, r_* which satisfy $0 < r_* < r^*$ such that the fixed point index

$$i\big(A,(\Omega\times\Omega)_{r^*}\backslash(\Omega\times\Omega)_{r_*},\Omega\times\Omega\big) = 1. \tag{2.9}$$

To see this, we first infer from (2.4) that there exist $\beta > 0$ and $r_1 > 0$ such that

$$\widehat{h}_s f_2(s,x) \le \beta|x| \quad \text{for } |x| \le r_1,\ s\in\mathbb{Z}. \tag{2.10}$$

Let

$$\begin{gathered} 0<\varepsilon<\min\left\{1,\frac{\sigma}{2(1+M'\beta\omega)}\right\},\\ F_\eta(s;u,v) = \{s\le n\le s+\omega-1 : |u_n-v_n|\ge\eta\},\quad u,v\in\Omega. \end{gathered} \tag{2.11}$$

Then the number of elements in $F_{\varepsilon r}(s;u,v)$, denoted by #, satisfies

$$\#F_{\varepsilon r}(s;u,v) \ge \min\left\{\omega,\frac{\sigma}{2M'\beta}\right\}, \tag{2.12}$$

when $\|(u,v)\| = r \le r_1$ and $A_2(u,v) = v$. Indeed, if $|u_n - v_n| \ge \varepsilon r$ for any $n\in\mathbb{Z}$, then (2.12) is obvious. If there exists $n_1 \in \mathbb{Z}$ such that $|u_{n_1} - v_{n_1}| < \varepsilon r$, then $\|v\| \ge v_{n_1} > u_{n_1} - \varepsilon r \ge \sigma\|u\| - \varepsilon r$. Thus $\|v\| > (\sigma-\varepsilon)r$. Assume that $v_{n_2} = \|v\|$. Then from $A_2(u,v) = v$ and (2.10), we have

$$\begin{aligned} (\sigma-\varepsilon)r \le v_{n_2} &= \sum_{s=n_2}^{n_2+\omega-1}\widehat{G}(n_2,s)\widehat{h}_s f_2\big(s,u_{s-\tau(s)}-v_{s-\tau(s)}\big)\\ &\le M'\beta\Bigg(\sum_{s\in F_{\varepsilon r}(n_2;u,v)} + \sum_{s\in F(n_2)\backslash F_{\varepsilon r}(n_2;u,v)}\Bigg)\big|u_{s-\tau(s)}-v_{s-\tau(s)}\big|\\ &\le M'\beta r\big[\#F_{\varepsilon r}(n_2;u,v)+\varepsilon\#\big(F(n_2)\backslash F_{\varepsilon r}(n_2;u,v)\big)\big], \end{aligned} \tag{2.13}$$

where $F(n_2) = \{n\in\mathbb{Z} : n_2\le n\le n_2+\omega-1\}$. It is now not difficult to check that $\#F_{\varepsilon r}(s;u,v) \ge \sigma/2M'\beta$, that is, (2.12) holds.

Next choose α such that $\alpha \ge 1/ma\varepsilon$, where

$$a = \min\{\omega,\sigma\backslash(2M'\beta)\}. \tag{2.14}$$

Then in view of (2.3), there exists $r_* \le r_1$ such that

$$h_s f_1(s,x) \ge \alpha|x|, \quad \text{for } |x| \le r_*,\ s\in\mathbb{Z}. \tag{2.15}$$

Set

$$H_n = \sum_{s=n}^{n+\omega-1} G(n,s), \quad n\in\mathbb{Z}. \tag{2.16}$$

Then $H = \{H_n\}_{n\in\mathbb{Z}} \in \Omega$, and for any $(u,v) \in \partial(\Omega\times\Omega)_{r_*}$ and $t \geq 0$, we assert that

$$(u,v) - A(u,v) \neq t(H,0). \tag{2.17}$$

To see this, assume to the contrary that there exist $(u^0,v^0) \in \partial(\Omega\times\Omega)_{r_*}$ and $t_0 \geq 0$ such that

$$u^0 - A_1(u^0,v^0) = t_0 H, \tag{2.18}$$

$$v^0 - A_2(u^0,v^0) = 0. \tag{2.19}$$

We may assume that $t_0 > 0$, for otherwise (u^0,v^0) is a fixed point of A. From (2.19), we know that (2.12) holds for the above ε. From (2.15), we have $u^0 \geq t_0 H$. Set $t^* = \sup\{t \mid u^0 \geq tH\}$. Then $t^* \geq t_0 > 0$. Furthermore, from (2.12), (2.15), and (2.18), we have

$$\begin{aligned}
u_n^0 &= t_0 H_n + A_1(u^0,v^0)_n \\
&= t_0 H_n + \sum_{s=n}^{n+\omega-1} G(n,s) h_s f_1\left(s, u^0_{s-\tau(s)} - v^0_{s-\tau(s)}\right) \\
&\geq t_0 H_n + \sum_{s-\tau(s)\in F_{\varepsilon r}(n-\tau(n);u,v)} G(n,s) h_s f_1\left(s, u^0_{s-\tau(s)} - v^0_{s-\tau(s)}\right) \\
&\geq t_0 H_n + \alpha \sum_{s-\tau(s)\in F_{\varepsilon r}(n-\tau(n);u,v)} G(n,s) \left| u^0_{s-\tau(s)} - v^0_{s-\tau(s)} \right| \\
&\geq t_0 H_n + m\alpha\varepsilon r \cdot \# F_{\varepsilon r}(n-\tau(n);u,v) \\
&\geq t_0 H_n + ma\alpha\varepsilon t^* H_n \\
&\geq (t_0 + t^*) H_n,
\end{aligned} \tag{2.20}$$

which is contrary to the definition of t^*. Thus (2.17) holds. Consequently (see, e.g., [2]),

$$i(A,(\Omega\times\Omega)_{r_*},\Omega\times\Omega) = 0. \tag{2.21}$$

Next, we will prove that there exists $r^* > 0$ such that

$$A(u,v) \ngeq (u,v) \quad \text{for } (u,v) \in \partial(\Omega\times\Omega)_{r^*}. \tag{2.22}$$

To see this, pick c such that $0 < c < \min\{\sigma/M\omega, \sigma/M'\omega\}$. In view of (2.5) and (2.6), there exists r_0 such that $h_s f_1(s,u) \leq cu$ for $u \geq r_0$ and $\widehat{h}_s f_2(s,v) \leq c|v|$ for $|v| \geq r_0$, where $s \in \mathbb{Z}$. Set

$$T_0 = \max\left\{ \sup_{0\leq u\leq r_0, s\in\mathbb{Z}} h_s f_1(s,u), \sup_{0\leq |v|\leq r_0, s\in\mathbb{Z}} \widehat{h}_s f_2(s,v) \right\}. \tag{2.23}$$

Then

$$h_s f_1(s,u) \le cu + T_0 \quad \text{for } u \ge 0, \tag{2.24}$$

$$\widehat{h}_s f_2(s,v) \le c|v| + T_0 \quad \text{for } v \in \mathbb{R}. \tag{2.25}$$

Take

$$r^* > \max\left\{r_*, r_0, \frac{\omega M T_0}{\sigma - cM\omega}, \frac{\omega M' T_0}{\sigma - cM'\omega}\right\}. \tag{2.26}$$

We assert that (2.22) holds. In fact, let $\|(u,v)\| = r^*$ and $u \ge v$. Then

$$\begin{aligned}(A_1(u,v))_n &= \sum_{s=n}^{n+\omega-1} G(n,s) h_s f_1\left(s, u_{s-\tau(s)} - v_{s-\tau(s)}\right) \\ &\le \sum_{s=n}^{n+\omega-1} G(n,s)\left[c\left(u_{s-\tau(s)} - v_{s-\tau(s)}\right) + T_0\right] \\ &\le M r^* c\omega + M T_0 \omega \\ &< \sigma r^* < r^* = \|u\|\end{aligned} \tag{2.27}$$

by (2.24). Thus $A_1(u,v) \ngeq u$. That is, $A(u,v) \ngeq (u,v)$. If there exists $n_0 \in \mathbb{Z}$ such that $u_{n_0} < v_{n_0}$, then $\|v\| \ge \sigma r^*$. Hence, we have

$$\begin{aligned}A_2(u,v)_n &= \sum_{s=n}^{n+\omega-1} \widehat{G}(n,s) \widehat{h}_s f_2\left(s, u_{s-\tau(s)} - v_{s-\tau(s)}\right) \\ &\le \sum_{s=n}^{n+\omega-1} \widehat{G}(n,s)\left[c\left| u_{s-\tau(s)} - v_{s-\tau(s)}\right| + T_0\right] \\ &\le M' r^* c\omega + \omega M' T_0 \\ &< \sigma r^* \le \|v\|\end{aligned} \tag{2.28}$$

by (2.25). Thus $A_2(u,v) \ngeq v$. That is, $A(u,v) \ngeq (u,v)$.

From (2.22), we have

$$i\left(A, (\Omega\times\Omega)_{r^*}, \Omega\times\Omega\right) = 1, \tag{2.29}$$

and from (2.21) and (2.29), we have $i(A, (\Omega\times\Omega)_{r^*} \setminus (\Omega\times\Omega)_{r_*}, \Omega\times\Omega) = 1$ as required.

Thus, there exists $(u^*, v^*) \in (\Omega\times\Omega)_{r^*} \setminus (\Omega\times\Omega)_{r_*}$ such that $A(u^*, v^*) = (u^*, v^*)$. The proof is complete. □

3. Sublinear f_1 and f_2

It is possible to find periodic solutions of (1.3) without the assumptions (2.3) through (2.6). One such case arises when functions f_1 and f_2 satisfy the assumptions

$$f_1(n,x-y) \le a_n x + b_n, \quad x \geqslant 0,\ y \geqslant 0,\ n \in \mathbb{Z}, \tag{3.1}$$

$$f_2(n,x-y) \le c_n y + d_n(x), \quad x \geqslant 0,\ y \geqslant 0,\ n \in \mathbb{Z}, \tag{3.2}$$

where $a = \{a_n\}_{n\in\mathbb{Z}}$, $b = \{b_n\}_{n\in\mathbb{Z}}$, and $c = \{c_n\}$ are positive ω-periodic sequences, and for each $n \in \mathbb{Z}$, the function $d_n(x)$ is continuous, nonnegative, and $d_{n+\omega}(x) = d_n(x)$ for $x \ge 0$.

Let $\Omega_0 = \{u \in X \mid u \ge 0\}$. Define $K_1, K_2 : X \to X$ by

$$\begin{aligned} (K_1u)_n &= \sum_{s=n}^{n+\omega-1} G(n,s)h_s a_s u_{s-\tau(s)}, \quad u \in X, \\ (K_2u)_n &= \sum_{s=n}^{n+\omega-1} \widehat{G}(n,s)\widehat{h}_s c_s u_{s-\tau(s)}, \quad u \in X, \end{aligned} \tag{3.3}$$

respectively. Then under conditions (2.1), it is not difficult to show that K_1 and K_2 are completely continuous linear operators on X, and K_1, K_2 map Ω_0 into Ω_0.

Theorem 3.1. *In addition to the assumptions imposed on the functions G, $\widehat{G}$, h, $\widehat{h}$, f_1, and f_2 in Section 1, suppose that f_1 and f_2 satisfy (3.1) and (3.2). Suppose further that the operators defined by (3.3) satisfy $\rho(K_1) < 1$ and $\rho(K_2) < 1$. Then (1.3) has at least one periodic solution.*

Proof. Note that $\Omega_0 \times \Omega_0$ is a normal solid cone of $X \times X$. Let A_1, A_2, and A be the same operators in the proof of Theorem 2.1. Set

$$g_n = \sum_{s=n}^{n+\omega-1} G(n,s)h_s b_s, \quad n \in \mathbb{Z}. \tag{3.4}$$

Then $g = \{g_n\}_{n\in\mathbb{Z}} \in \Omega_0$. $\rho(K_1) < 1$ implies that $(I-K_1)^{-1}$ exists and that

$$(I-K_1)^{-1} = I + K_1 + K_1^2 + \dots. \tag{3.5}$$

Thus, we have $(I-K_1)^{-1}(\Omega_0) \subset \Omega_0$ and it is increasing. Then $u - K_1u \le g$ for $u \in X$ implies that $u \le (I-K_1)^{-1}g$. Let

$$r_0 = \max_{s\in[0,\omega]} (I-K_1)^{-1} g_s, \tag{3.6}$$

we get that $u \le K_1u + g$ for any $u \in \Omega_0$, which satisfies $\|u\| \le r_0$.

Let $d^* = \max\{d_n(x) \mid n \in \mathbb{Z}, 0 \le x \le r_0\}$. Then from (3.2), we have

$$f_2(n, x-y) \le c_n y + d^*, \quad y \geqslant 0,\ 0 \le x \le r_0,\ n \in \mathbb{Z}. \tag{3.7}$$

Let

$$q_n = d^* \sum_{s=n}^{n+\omega-1} \widehat{G}(n,s)\widehat{h}_s, \quad n \in \mathbb{Z}. \tag{3.8}$$

Then $q = \{q_n\}_{n\in\mathbb{Z}} \in \Omega_0$ and $A_2(u,v) \le K_2(v) + q$. If for any $(u,v) \in X \times X$, there exists $\lambda_0 \in [0,1]$ such that $v = \lambda_0 A_2(u,v)$, then, we have

$$|v| = \lambda_0 \left| A_2(u,v) \right| \le \left| A_2(u,v) \right| \le K_2(|v|) + q. \tag{3.9}$$

Note that if $|v| \in \Omega_0$ and $\rho(K_2) < 1$, we have $|v| \le (I - K_1)^{-1} q$. Choose

$$r^* > \max\left\{r_0, \left\|(I-K_1)^{-1} q\right\|\right\}. \tag{3.10}$$

Then for any open set $\Psi \subset \Omega_0 \times \Omega_0$ that satisfies $\Psi \supset (\Omega_0 \times \Omega_0)_{r^*}$, $A_2(u,v) \ne \mu v$ for $(u,v) \in \partial\Psi$ and $\mu \geqslant 1$.

Consequently,

$$A(u,v) \ne \mu(u,v) \tag{3.11}$$

for any $(u,v) \in \Omega_0 \times \Omega_0$, $\|(u,v)\| = r^*$, and $\mu \geqslant 1$. Indeed, if there exist $(u^0,v^0) \in \Omega_0 \times \Omega_0$, $\|(u^0,v^0)\| = r^*$, and $\mu_0 \geqslant 1$ such that $A(u^0,v^0) = \mu_0(u^0,v^0)$, then from $A_2(u^0,v^0) = \mu_0 v^0$, $r^* > r_0$, and (3.2), we have $\|u\| > r_0$. But from (3.1), we know that $u_n \le \mu_0 u_n = (A_1(u,v))_n \le K_1 u_n + g_n$, this is contrary to the fact that $\|u\| \le r_0$ as shown above.

Thus $i(A, (\Omega_0 \times \Omega_0)_{r^*}, \Omega_0 \times \Omega_0) = 1$, which shows that there exists $(u^*,v^*) \in (\Omega_0 \times \Omega_0)_{r^*}$ such that $A(u^*,v^*) = (u^*,v^*)$. The proof is complete. □

Theorem 3.2. *In addition to the assumptions imposed on the functions G, $\widehat{G}$, h, $\widehat{h}$, f_1, and f_2 in Section 1, suppose that f_1 and f_2 satisfy*

$$\begin{aligned} f_1(n, x-y) &\le a_n y + b_n(x), \quad x \geqslant 0,\ y \geqslant 0,\ n \in \mathbb{Z}, \\ f_2(n, x-y) &\le c_n x + d_n, \quad x \geqslant 0,\ y \geqslant 0,\ n \in \mathbb{Z}, \end{aligned} \tag{3.12}$$

where $a = \{a_n\}_{n\in\mathbb{Z}}$, $b = \{b_n\}_{n\in\mathbb{Z}}$, and $c = \{c_n\}$ are positive ω-periodic sequences, and for each $n \in \mathbb{Z}$, $b_n = b_n(x)$ is continuous, nonnegative, and $b_{n+\omega}(x) = b_n(x)$ for $x \ge 0$. Suppose further that the operators defined by (3.3) satisfy $\rho(K_1) < 1$ and $\rho(K_2) < 1$. Then (1.3) has at least one periodic solution.

The proof is similar to that of Theorem 3.1 and hence omitted.

4. Applications

We now turn to the existence of nontrivial periodic solutions for the delay difference equation

$$x_{n+1} = a_n x_n + h_n f_1(n, x_{n-\tau(n)}) - \hat{h}_n f_2(n, x_{n-\tau(n)}), \quad n \in \mathbb{Z}, \tag{4.1}$$

where $\{h_n\}_{n\in\mathbb{Z}}$ and $\{\hat{h}_n\}_{n\in\mathbb{Z}}$ are positive ω-periodic sequences, $\{\tau(n)\}_{n\in\mathbb{Z}}$ is an integer-valued ω-periodic sequence, and f_1, f_2 are real continuous functions which satisfy $f_1(n+\omega, u) = f_1(n,u)$ and $f_2(n+\omega, u) = f_2(n,u)$ for any $u \in \mathbb{R}^1$ and $n \in \mathbb{Z}$.

We proceed formerly from (4.1) and obtain

$$\Delta\left\{x_n \prod_{k=q}^{n-1} \frac{1}{a_k}\right\} = \prod_{k=q}^{n} \frac{1}{a_k}\left[h_n f_1(n, x_{n-\tau(n)}) - \hat{h}_n f_2(n, x_{n-\tau(n)})\right]. \tag{4.2}$$

Then summing the above formal equation from n to $n+\omega$-1, we obtain

$$x_n = \sum_{s=n}^{n+\omega-1} G(n,s)\left[h_s f_1(s, x_{s-\tau(s)}) - \hat{h}_s f_2(s, x_{s-\tau(s)})\right], \quad n \in \mathbb{Z}, \tag{4.3}$$

where

$$G(n,s) = \left(\prod_{k=n}^{s} \frac{1}{a_k}\right)\left(\prod_{k=0}^{\omega-1} \frac{1}{a_k} - 1\right)^{-1}, \quad n, s \in \mathbb{Z}, \tag{4.4}$$

which is positive if $\{a_n\}_{n\in\mathbb{Z}}$ is a positive ω-periodic sequence which satisfies $\prod_{s=0}^{\omega-1} a_s^{-1} > 1$.

It is not difficult to check that any ω-periodic sequence $\{x_n\}_{n\in\mathbb{Z}}$ that satisfies (4.3) is also an ω-periodic solution of (4.1). Furthermore, note that

$$\begin{gathered} G(n,n) = \left(\frac{1}{a_n}\right)\left(\prod_{k=0}^{\omega-1} \frac{1}{a_k} - 1\right)^{-1} = G(n+\omega, n+\omega), \\ G(n, n+\omega-1) = \left(\prod_{k=0}^{\omega-1} \frac{1}{a_k}\right)\left(\prod_{k=0}^{\omega-1} \frac{1}{a_k} - 1\right)^{-1} = G(0, \omega-1), \\ 0 < N \equiv \min_{n\le i\le n+\omega-1} G(n,s) \le G(n,s) \le \max_{n\le i\le n+\omega-1} G(n,i) \equiv M, \quad n \le s \le n+\omega-1. \end{gathered} \tag{4.5}$$

Theorem 4.1. *Suppose that $\{h_n\}_{n\in\mathbb{Z}}$ and $\{\hat{h}_n\}_{n\in\mathbb{Z}}$ are positive ω-periodic sequences, $\{\tau(n)\}_{n\in\mathbb{Z}}$ is an integer-valued ω-periodic sequence, and f_1, f_2 are nonnegative continuous functions which satisfy $f_1(n+\omega, u) = f_1(n,u)$ and $f_2(n+\omega,u) = f_2(n,u)$ for any $u \in \mathbb{R}^1$ and $n \in \mathbb{Z}$. Suppose further that $\{a_n\}_{n\in\mathbb{Z}}$ is a real sequence which satisfies $\prod_{s=0}^{\omega-1} a_s^{-1} > 1$. If f_1 and f_2 satisfy the additional conditions $f_1(n,0) = 0 = f_2(n,0)$ for $n \in \mathbb{Z}$ as well as (2.3), (2.4), (2.5), and (2.6) uniformly with respect to all $n \in \mathbb{Z}$, then (4.1) has at least a nontrivial periodic solution.*

Indeed, let A_1, A_2, and A be defined as in the proof of Theorem 2.1. Then from Theorem 2.1, we know that there exists $(u^*,v^*) \neq (0,0)$, such that $A(u^*,v^*) = (u^*,v^*)$, that is,

$$\begin{aligned} u_n^* &= \sum_{s=n}^{n+\omega-1} G(n,s)h_s f_1\left(s, u^*_{s-\tau(s)} - v^*_{s-\tau(s)}\right), \\ v_n^* &= \sum_{s=n}^{n+\omega-1} G(n,s)\widehat{h}_s f_2\left(s, u^*_{s-\tau(s)} - v^*_{s-\tau(s)}\right). \end{aligned} \tag{4.6}$$

Since $f_1(n,0) = 0 = f_2(n,0)$ for $n \in \mathbb{Z}$, we know that $u^* \neq v^*$. (Indeed, if $u^* = v^*$, then $u^* = v^* = 0$, which is contrary to the fact that $(u^*,v^*) \neq (0,0)$.) Thus $u^* - v^*$ is a nontrivial periodic solution of (4.3), and also a nontrivial periodic solution of (4.1).

Next, we illustrate Theorem 3.1 by considering the delay difference equations

$$x_{n+1} = a_n x_n + f\left(n, x_{n-\tau(n)}\right), \quad n \in \mathbb{Z}, \tag{4.7}$$

where $\{a_n\}_{n\in\mathbb{Z}}$ is a positive ω-periodic sequence but $\prod_{s=0}^{\omega-1} a_s^{-1} > 1$, $\{\tau(n)\}_{n\in\mathbb{Z}}$ is integer-valued ω-periodic sequence, $f(n,u)$ is a real continuous function, and $f(n+\omega,u) = f(n,u)$ for any $u \in \mathbb{R}$ and $n \in \mathbb{Z}$.

The existence of positive periodic solutions for (4.7) have been studied extensively by a number of authors (see, e.g., [1, 3, 5, 7, 8, 9]). Here, we proceed formerly from (4.7) and obtain

$$\Delta\left\{x_n \prod_{k=q}^{n-1} \frac{1}{a_k}\right\} = \prod_{k=q}^{n} \frac{1}{a_k} f\left(n, x_{n-\tau(n)}\right). \tag{4.8}$$

Then summing the above formal equation from n to $n+\omega$-1, we obtain

$$x_n = \sum_{s=n}^{n+\omega-1} G(n,s) f\left(s, x_{s-\tau(s)}\right), \quad n \in \mathbb{Z}, \tag{4.9}$$

where

$$G(n,s) = \left(\prod_{k=n}^{s} \frac{1}{a_k}\right)\left(\prod_{k=0}^{\omega-1} \frac{1}{a_k} - 1\right)^{-1}. \tag{4.10}$$

Set $\lambda_0 = (\prod_{k=0}^{\omega-1}(1/a_k) - 1)$, then $G(n,s) = (1/\lambda_0)(\prod_{k=n}^{s}(1/a_k))$. It is not difficult to check that any ω-periodic sequence $\{x_n\}_{n\in\mathbb{Z}}$ that satisfies (4.9) is also an ω-periodic solution of (4.7).

Choose

$$\begin{aligned} f(n,x) &= \lambda \sin x + p_n, \\ f_1(n,x) &= \lambda \frac{|\sin x| + \sin x}{2} + p_n, \\ f_2(n,x) &= \lambda \frac{|\sin x| - \sin x}{2}, \end{aligned} \tag{4.11}$$

where $\lambda > 0$ and $\{p_n\}$ is a positive ω-periodic sequence. Then $f_1(n, x-y) \le \lambda x + 2\lambda + p_n$ and $f_2(n, x-y) \le \lambda y + 2\lambda$ for $x, y \geqslant 0$. Set

$$(K_i u)_n = \lambda \sum_{s=n}^{n+\omega-1} G(n,s) u_{s-\tau(s)}, \quad i = 1,2, \tag{4.12}$$

then

$$\begin{aligned} \|K_i u\| &= \max_{0\le n\le \omega-1} \left| \lambda \sum_{s=n}^{n+\omega-1} G(n,s) u_{s-\tau(s)} \right| \\ &= \max_{0\le n\le \omega-1} \left| \frac{\lambda}{\lambda_0} \sum_{s=n}^{n+\omega-1} \left(\prod_{k=n}^{s} \frac{1}{a_k} \right) u_{s-\tau(s)} \right| \\ &\le \max_{0\le n\le \omega-1} \left| \frac{\lambda}{\lambda_0} \|u\| \sum_{s=n}^{n+\omega-1} \prod_{k=n}^{s} \frac{1}{a_k} \right| \\ &= \frac{\lambda}{\lambda_0} \|u\| \max_{0\le n\le \omega-1} \sum_{s=n}^{n+\omega-1} \left(\prod_{k=n}^{s} \frac{1}{a_k} \right) \end{aligned} \tag{4.13}$$

for $i = 1,2$. Thus

$$\|K_i\| \le \frac{\lambda}{\lambda_0} \max_{0\le n\le \omega-1} \sum_{s=n}^{n+\omega-1} \left(\prod_{k=n}^{s} \frac{1}{a_k} \right), \quad i = 1,2. \tag{4.14}$$

Since $\rho(K_i) \le \|K_i\|$, thus $\rho(K_i) \le \|K_i\| < 1$ for

$$\lambda < \lambda_0 \left[\max_{0\le n\le \omega-1} \sum_{s=n}^{n+\omega-1} \left(\prod_{k=n}^{s} \frac{1}{a_k} \right) \right]^{-1}. \tag{4.15}$$

Under this condition, Theorem 3.1 asserts that (4.7) has at least one periodic solution. Note that 0 is not its solution. Thus, our periodic solution is nontrivial.

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Guang Zhang: Department of Mathematics, Qingdao Technological University, Qingdao, Shandong 266033, China
E-mail address: shzhg@qtech.edu.cn

Shugui Kang: Department of Mathematics, Yanbei Normal University, Datong, Shanxi 037000, China
E-mail address: dtkangshugui@yahoo.com.cn

Sui Sun Cheng: Department of Mathematics, National Tsing Hua University, Hsinchu 300, Taiwan
E-mail address: sscheng@math.nthu.edu.tw