ON THE SYSTEM OF RATIONAL DIFFERENCE EQUATIONS $x_{n+1} = f(x_n, y_{n-k}), y_{n+1} = f(y_n, x_{n-k})$

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We study the global asymptotic behavior of the positive solutions of the system of rational difference equations $x_{n+1} = f(x_n, y_{n-k})$, $y_{n+1} = f(y_n, x_{n-k})$, $n = 0,1,2,\ldots$, under appropriate assumptions, where $k \in \{1,2,\ldots\}$ and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in (0,+\infty)$. We give sufficient conditions under which every positive solution of this equation converges to a positive equilibrium. The main theorem in [1] is included in our result.

1. Introduction

Recently there has been published quite a lot of works concerning the behavior of positive solutions of systems of rational difference equations [2–7]. These results are not only valuable in their own right, but they can provide insight into their differential counterparts.

In [1], Camouzis and Papaschinopoulos studied the global asymptotic behavior of the positive solutions of the system of rational difference equations

$$\begin{aligned} x_{n+1} &= 1 + \frac{x_n}{y_{n-k}}, \\ y_{n+1} &= 1 + \frac{y_n}{x_{n-k}}, \end{aligned} \qquad n = 0,1,2,\ldots, \tag{1.1}$$

where $k \in \{1,2,\ldots\}$ and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0 \in (0,+\infty)$.

To be motivated by the above studies, in this paper, we consider the more general equation

$$\begin{aligned} x_{n+1} &= f(x_n, y_{n-k}), \\ y_{n+1} &= f(y_n, x_{n-k}), \end{aligned} \qquad n = 0,1,2,\ldots, \tag{1.2}$$

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where $k \in \{1,2,\dots\}$, the initial values $x_{-k}, x_{-k+1}, \dots, x_0, y_{-k}, y_{-k+1}, \dots, y_0 \in (0,+\infty)$ and f satisfies the following hypotheses.

(H_1) $f \in C(E \times E, (0,+\infty))$ with $a = \inf_{(u,v)\in E\times E} f(u,v) \in E$, where $E \in \{(0,+\infty), [0,+\infty)\}$.

(H_2) $f(u,v)$ is increasing in u and decreasing in v.

(H_3) There exists a decreasing function $g \in C((a,+\infty),(a,+\infty))$ such that

(i) For any $x > a$, $g(g(x)) = x$ and $x = f(x,g(x))$;

(ii) $\lim_{x\to a^+} g(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = a$.

A pair of sequences of positive real numbers $\{(x_n, y_n)\}_{n=-k}^{\infty}$ that satisfies (1.2) is a positive solution of (1.2). If a positive solution of (1.2) is a pair of positive constants (x, y), then (x, y) is called a positive equilibrium of (1.2). In this paper, our main result is the following theorem.

Theorem 1.1. *Assume that (H_1)–(H_3) hold. Then the following statements are true.*

(i) *Every pair of positive constant $(x, y) \in (a,+\infty)\times(a,+\infty)$ satisfying the equation*

$$y = g(x) \tag{1.3}$$

is a positive equilibrium of (1.2).

(ii) *Every positive solution of (1.2) converges to a positive equilibrium (x, y) of (1.2) satisfying (1.3) as $n \to \infty$.*

2. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. To do this we need the following lemma.

Lemma 2.1. *Let $\{(x_n, y_n)\}_{n=-k}^{\infty}$ be a positive solution of (1.2). Then there exists a real number $L \in (a,+\infty)$ with $L < g(L)$ such that $x_n, y_n \in [L, g(L)]$ for all $n \ge 1$. Furthermore, if $\limsup x_n = M$, $\liminf x_n = m$, $\limsup y_n = P$, $\liminf y_n = p$, then $M = g(p)$ and $P = g(m)$.*

Proof. From (H_1) and (H_2), we have

$$\begin{aligned} x_i &= f(x_{i-1}, y_{i-1-k}) > f(x_{i-1}, y_{i-1-k} + 1) \ge a, \\ y_i &= f(y_{i-1}, x_{i-1-k}) > f(y_{i-1}, x_{i-1-k} + 1) \ge a, \end{aligned} \quad \text{for every } 1 \le i \le k+1. \tag{2.1}$$

Since $\lim_{x\to a^+} g(x) = +\infty$, there exists $L \in (a,+\infty)$ with $L < g(L)$ such that

$$x_i, y_i \in [L, g(L)] \quad \text{for every } 1 \le i \le k+1. \tag{2.2}$$

It follows from (2.2) and (H_3) that

$$\begin{aligned} g(L) = f(g(L), L) &\ge x_{k+2} = f(x_{k+1}, y_1) \ge f(L, g(L)) = L, \\ g(L) = f(g(L), L) &\ge y_{k+2} = f(y_{k+1}, x_1) \ge f(L, g(L)) = L. \end{aligned} \tag{2.3}$$

Inductively, we have that $x_n, y_n \in [L, g(L)]$ for all $n \ge 1$.

Let $\limsup x_n = M$, $\liminf x_n = m$, $\limsup y_n = P$, $\liminf y_n = p$, then there exist sequences $l_n \geq 1$ and $s_n \geq 1$ such that

$$\lim_{n\to\infty} x_{l_n} = M, \qquad \lim_{n\to\infty} y_{s_n} = p. \tag{2.4}$$

Without loss of generality, we may assume (by taking a subsequence) that there exist $A,D \in [m,M]$ and $B,C \in [p,P]$ such that

$$\begin{aligned} \lim_{n\to\infty} x_{l_n-1} &= A, \\ \lim_{n\to\infty} y_{l_n-k-1} &= B, \\ \lim_{n\to\infty} y_{s_n-1} &= C, \\ \lim_{n\to\infty} x_{s_n-k-1} &= D. \end{aligned} \tag{2.5}$$

Thus, from (1.2), (H_2) and (H_3), we have

$$\begin{aligned} f(M,g(M)) = M = f(A,B) \leq f(M,p), \\ f(p,g(p)) = p = f(C,D) \geq f(p,M), \end{aligned} \tag{2.6}$$

from which it follows that

$$g(M) \geq p, \qquad g(p) \leq M. \tag{2.7}$$

By (H_3), we obtain

$$p = g(g(p)) \geq g(M). \tag{2.8}$$

Therefore, $M = g(p)$. By the symmetry, we have also $P = g(m)$. Lemma 2.1 is proven. □

Proof of Theorem 1.1.

(i) Is obvious.

(ii) Let $\{(x_n, y_n)\}_{n=-k}^{\infty}$ be a positive solution of (1.2) with the initial conditions $x_0, x_{-1},\ldots,x_{-k}, y_0, y_{-1},\ldots, y_{-k} \in (0,+\infty)$. By Lemma 2.1, we have that

$$\begin{aligned} a < \liminf x_n = g(P) \leq \limsup x_n = M < +\infty, \\ a < \liminf y_n = g(M) \leq \limsup y_n = P < +\infty. \end{aligned} \tag{2.9}$$

Without loss of generality, we may assume (by taking a subsequence) that there exists a sequence $l_n \geq 4k$ such that

$$\begin{aligned} &\lim_{n\to\infty} x_{l_n} = M, \\ &\lim_{n\to\infty} x_{l_n-j} = M_j \in [g(P),M], \quad \text{for } j \in \{1,2,\ldots,3k+1\}, \\ &\lim_{n\to\infty} y_{l_n-j} = P_j \in [g(M),P], \quad \text{for } j \in \{1,2,\cdots,3k+1\}. \end{aligned} \tag{2.10}$$

From (1.2), (2.10) and (H_3), we have

$$f(M,g(M)) = M = f(M_1,P_{k+1}) \le f(M_1,g(M)) \le f(M,g(M)), \tag{2.11}$$

from which it follows that

$$M_1 = M, \qquad P_{k+1} = g(M). \tag{2.12}$$

In a similar fashion, we may obtain that

$$f(M,g(M)) = M = M_1 = f(M_2,P_{k+2}) \le f(M_2,g(M)) \le f(M,g(M)), \tag{2.13}$$

from which it follows that

$$M_2 = M, \qquad P_{k+2} = g(M). \tag{2.14}$$

Inductively, we have that

$$\begin{aligned} M_j &= M, \\ P_{k+j} &= g(M), \end{aligned} \quad \text{for } j \in \{1,2,\dots,2k+1\}, \tag{2.15}$$

from which it follows that

$$\begin{aligned} \lim_{n\to\infty} x_{l_n-j} &= M, \quad \text{for } j \in \{0,1,\dots,2k+1\}, \\ \lim_{n\to\infty} y_{l_n-j} &= g(M), \quad \text{for } j \in \{k+1,\dots,3k+1\}. \end{aligned} \tag{2.16}$$

In view (2.16), for any $0 < \varepsilon < M - a$, there exists some $l_s \ge 4k$ such that

$$\begin{aligned} M-\varepsilon < x_{l_s-j} &< M+\varepsilon, \quad \text{if } j \in \{0,1,\dots,2k+1\}, \\ g(M+\varepsilon) < y_{l_s-j} &< g(M-\varepsilon), \quad \text{if } j \in \{k+1,\dots,2k+1\}. \end{aligned} \tag{2.17}$$

From (1.2) and (2.17), we have

$$y_{l_s-k} = f(y_{l_s-k-1},x_{l_s-2k-1}) < f(g(M-\varepsilon),M-\varepsilon) = g(M-\varepsilon). \tag{2.18}$$

Also (1.2), (2.17) and (2.18) implies

$$x_{l_s+1} = f(x_{l_s},y_{l_s-k}) > f(M-\varepsilon,g(M-\varepsilon)) = M-\varepsilon. \tag{2.19}$$

Inductively, it follows that

$$\begin{aligned} y_{l_s+n-k} &< g(M-\varepsilon) \quad \forall n \ge 0, \\ x_{l_s+n} &> M-\varepsilon \quad \forall n \ge 0. \end{aligned} \tag{2.20}$$

Since $\limsup x_n = M$ and $\liminf y_n = g(M)$, we have

$$\lim_{n\to\infty} x_n = M, \quad \lim_{n\to\infty} y_n = g(M). \tag{2.21}$$

Thus $\lim_{n\to\infty}(x_n, y_n) = (M,P)$ with $P = g(M)$. Theorem 1.1 is proven. □

3. Examples

To illustrate the applicability of Theorem 1.1, we present the following examples.

Example 3.1. Consider equation

$$\begin{aligned} x_{n+1} &= \frac{p+x_n}{1+y_{n-k}}, \\ y_{n+1} &= \frac{p+y_n}{1+x_{n-k}}, \end{aligned} \quad n = 0,1,\dots, \tag{3.1}$$

where $k \in \{1,2,\cdots\}$, the initial conditions $x_{-k}, x_{-k+1},\dots,x_0, y_{-k}, y_{-k+1},\dots,y_0 \in (0,+\infty)$ and $p \in (0,+\infty)$. Let $E = [0,+\infty)$ and

$$f(x,y) = \frac{p+x}{1+y} \quad (x \geq 0, y \geq 0), \qquad g(x) = \frac{p}{x} \quad (x > 0). \tag{3.2}$$

It is easy to verify that (H_1)–(H_3) hold for (3.1). It follows from Theorem 1.1 that

(i) every pair of positive constant $(x,y) \in (0,+\infty)\times(0,+\infty)$ satisfying the equation

$$xy = p \tag{3.3}$$

is a positive equilibrium of (3.1).

(ii) every positive solution of (3.1) converges to a positive equilibrium (x,y) of (3.1) satisfying (3.3) as $n \to \infty$.

Example 3.2. Consider equation

$$\begin{aligned} x_{n+1} &= 1+\frac{x_n}{y_{n-k}}, \\ y_{n+1} &= 1+\frac{y_n}{x_{n-k}}, \end{aligned} \quad n = 0,1,\dots, \tag{3.4}$$

where $k \in \{1,2,\dots\}$ and the initial conditions $x_{-k}, x_{-k+1},\dots,x_0, y_{-k}, y_{-k+1},\dots,y_0 \in (0,+\infty)$. Let $E = (0,+\infty)$ and

$$f(x,y) = 1+\frac{x}{y} \quad (x > 0, y > 0), \qquad g(x) = \frac{x}{x-1} \quad (x > 1). \tag{3.5}$$

It is easy to verify that (H_1)–(H_3) hold for (3.4). It follows from Theorem 1.1 that

(i) every pair of positive constant $(x,y) \in (1,+\infty)\times(1,+\infty)$ satisfying the equation

$$xy = x+y \tag{3.6}$$

is a positive equilibrium of (3.4);

(ii) every positive solution of (3.4) converges to a positive equilibrium (x,y) of (3.4) satisfying (3.6) as $n \to \infty$.

Example 3.3. Consider equation

$$x_{n+1} = p + \frac{A + x_n}{q + y_{n-k}}, \qquad y_{n+1} = p + \frac{A + y_n}{q + x_{n-k}}, \qquad n = 0,1,\ldots, \tag{3.7}$$

where $k \in \{1,2,\ldots\}$, the initial conditions $x_{-k}, x_{-k+1},\ldots,x_0, y_{-k}, y_{-k+1},\ldots,y_0 \in (0,+\infty)$, $A \in (0,+\infty)$ and $p,q \in [0,1]$ with $p+q = 1$. Let $E = (0,+\infty)$ if $p > 0$ and $E = [0,+\infty)$ if $p = 0$ and

$$f(x,y) = p + \frac{A+x}{q+y}, \tag{3.8}$$

then $a = \inf_{(x,y)\in E\times E} f(x,y) = p$. Let $g(x) = (pq + px + A)/(x - p)$ $(x > p)$. It is easy to verify that (H_1)–(H_3) hold for (3.7). It follows from Theorem 1.1 that

(i) every pair of positive constant $(x,y) \in (p,+\infty)\times(p,+\infty)$ satisfying the equation

$$xy = pq + px + py + A \tag{3.9}$$

is a positive equilibrium of (3.7);

(ii) every positive solution of (3.7) converges to a positive equilibrium (x,y) of (3.7) satisfying (3.9) as $n \to \infty$

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