EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS TO A RIGHT-FOCAL BOUNDARY VALUE PROBLEM ON TIME SCALES

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We are concerned with proving the existence of one or more than one positive solution of an n-point right-focal boundary value problem for the nonlinear dynamic equation $(-1)^{n-1}x^{\Delta^n}(t) = \lambda r(t) f(t, x^\sigma(t))$. We will also obtain criteria which lead to nonexistence of positive solutions. Here the independent variable t is in a time scale. We will use fixed point theorems for operators on a Banach space.

1. Introduction

Motivated by the work of Anderson [3] on discrete third-order three-point right-focal boundary value problems, in this paper we will study an nth-order n-point right-focal boundary problem on time scales. This paper also gives nonexistence and multiplicity results for positive solutions to the time scale boundary value problem

$$(-1)^{n-1}x^{\Delta^n}(t) = \lambda r(t) f(t, x^\sigma(t)) \quad \forall t \in [t_1, \rho(t_n)], \tag{1.1}$$

$$x(t_1) = x^\Delta(t_2) = \cdots = x^{\Delta^{n-1}}(t_n) = 0, \tag{1.2}$$

where $n \geq 2$, $t_1 < t_2 < \cdots < t_{n-1} < t_n$, λ is a real parameter, and $x = x(t)$ is a desired solution. The arguments are similar to those used in [9, 13].

In the third section we obtain multiplicity results for this problem with $\lambda = 1$. In the fourth section existence, nonexistence, and multiplicity results are given for the eigenvalue problem.

To understand this so-called dynamic equation (1.1) on a time scale **T**, we need some preliminary definitions.

Definition 1.1. Let **T** be a nonempty closed subset of $\mathbb{R}$ and define the forward jump operator $\sigma(t)$ at t for $t < \sup \mathbf{T}$ by

$$\sigma(t) := \inf \{s > t : s \in \mathbf{T}\} \tag{1.3}$$

and the backward jump operator $\rho(t)$ at t for $t > \inf \mathbf{T}$ by

$$\rho(t) := \sup \{s < t : s \in \mathbf{T}\} \tag{1.4}$$

for all $t \in \mathbf{T}$.

We assume throughout that $\mathbf{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$, we say t is left scattered. If $\sigma(t) = t$, we say t is right dense, while if $\rho(t) = t$, we say t is left dense.

We assume $\sigma^0(t) = t$, and for any integer $n > 0$, we have

$$\sigma^n(t) := \sigma(\sigma^{n-1}(t)). \tag{1.5}$$

Throughout this paper we make the blanket assumption that $a \leq b$ are points in $\mathbf{T}$.

Definition 1.2. Define the interval in $\mathbf{T}$:

$$[a,b] := \{t \in \mathbf{T} \text{ such that } a \leq t \leq b\}. \tag{1.6}$$

Other types of intervals are defined similarly.

We are concerned with calculus on time scales which is a unified approach to continuous and discrete calculus. In [4, 11], Aulbach and Hilger have initiated the development of this calculus. Since then, efforts have been made in the context of time scales, in establishing that some results for boundary value problems for ordinary differential equations and their discrete analogues are special cases of more general results on time scales; for a wide variety of problems addressed, see many references [1, 5, 6, 8–10, 14].

Definition 1.3. Assume $x : \mathbf{T} \to \mathbb{R}$ and fix $t \in \mathbf{T}$ such that $t < \sup \mathbf{T}$, then $x^\Delta(t)$ is defined to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| \left[x(\sigma(t)) - x(s) \right] - x^\Delta(t) \left[\sigma(t) - s \right] \right| < \epsilon \left| \sigma(t) - s \right| \tag{1.7}$$

for all $s \in U$. $x^\Delta(t)$ is called the delta derivative of x at t.

It can be shown that if $x : \mathbf{T} \to \mathbb{R}$ is continuous at $t \in \mathbf{T}$, $t < \sup \mathbf{T}$, and t is right scattered, then

$$x^\Delta(t) = \frac{x^\sigma(t) - x(t)}{\sigma(t) - t}. \tag{1.8}$$

Note, if $\mathbf{T} = \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, then

$$x^\Delta(t) = \Delta x(t) := x(t+1) - x(t). \tag{1.9}$$

Moreover, if $\mathbf{T} = \mathbb{R}$, then

$$x^\Delta(t) = x'(t). \tag{1.10}$$

Finally, for $n \geq 1$, define

$$x^{\Delta^n}(t) := \left\{ x^{\Delta}(t) \right\}^{\Delta^{n-1}} \tag{1.11}$$

assuming $x^{\Delta^0}(t) = x(t)$.

Definition 1.4. If $F^{\Delta}(t) = f(t)$, then define the integral of f by

$$\int_a^t f(\tau)\Delta\tau := F(t) - F(a). \tag{1.12}$$

Note that in the case $\mathbf{T} = \mathbb{R}$ we have

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt, \tag{1.13}$$

and in the case $\mathbf{T} = \mathbb{Z}$ we have

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k), \tag{1.14}$$

where $a, b \in \mathbf{T}$ with $a \leq b$.

2. Preliminaries

As in [2], we introduce the Taylor polynomials $h_j : \mathbf{T}^2 \to \mathbb{R}$, $j \in \mathbf{N}_0$, recursively defined as follows:

$$h_0(t,s) = 1 \quad \forall s, t \in \mathbf{T}, \tag{2.1}$$

$$h_{j+1}(t,s) = \int_s^t h_j(\tau,s)\Delta\tau \quad \forall s, t \in \mathbf{T}. \tag{2.2}$$

For integers $n \geq 2$ and for $i = 1, 2, \ldots, n-1$, define

$$u_{n,i}(t,s) \equiv u_{n,i}(t, s : t_1, t_2, \ldots, t_n), \tag{2.3}$$

with $t, s, t_j \in \mathbf{T}$ for $1 \leq j \leq n$, as follows:

$$u_{n,i}(t,s) := (-1)^{n+1} \begin{vmatrix} 0 & h_1(t,t_1) & h_2(t,t_1) & \ldots & h_{n-1}(t,t_1) \\ c_2(s,i) & 1 & h_1(t_2,t_1) & \ldots & h_{n-2}(t_2,t_1) \\ c_3(s,i) & 0 & 1 & \ldots & h_{n-3}(t_3,t_1) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ c_{n-1}(s,i) & 0 & 0 & \ldots & h_1(t_{n-1},t_1) \\ 1 & 0 & 0 & \ldots & 1 \end{vmatrix}, \tag{2.4}$$

where

$$c_j(s,i) := H(j-1-i)h_{n-j}(t_j,\sigma(s)), \tag{2.5}$$

for $j = 2, 3, \ldots, n-1$ and $i = 1, 2, \ldots, n-1$. Here

$$
H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}
\tag{2.6}
$$

is the usual Heaviside function, and $h_j(t,s)$ is as defined in (2.2). In addition, define

$$
v_{n,i}(t,s) := u_{n,i}(t,s) + (-1)^{n-1} h_{n-1}(t, \sigma(s)),
\tag{2.7}
$$

for integers $n \geq 2$ and for $i = 1, 2, \ldots, n-1$.

THEOREM 2.1 [2]. *For $u_{n,i}(t,s)$ as in (2.4) and $v_{n,i}(t,s)$ as in (2.7),*

$$
G_n(t,s : t_1, t_2, \ldots, t_n) = \begin{cases} s \in I_1 : \begin{cases} u_{n,1}(t,s) & \text{if } t \leq \sigma(s), \\ v_{n,1}(t,s) & \text{if } t \geq \sigma(s), \end{cases} \\[2ex] s \in I_2 : \begin{cases} u_{n,2}(t,s) & \text{if } t \leq \sigma(s), \\ v_{n,2}(t,s) & \text{if } t \geq \sigma(s), \end{cases} \\[2ex] \qquad\qquad \vdots \\[1ex] s \in I_{n-1} : \begin{cases} u_{n,n-1}(t,s) & \text{if } t \leq \sigma(s), \\ v_{n,n-1}(t,s) & \text{if } t \geq \sigma(s), \end{cases} \end{cases}
\tag{2.8}
$$

where $I_1 = [t_1, \rho(t_2)]$, and $I_i = [\rho(t_i), \rho(t_{i+1})]$ for $i = 1, 2, \ldots, n-1$, is Green's function for the homogeneous problem $(-1)^{n-1} x^{\Delta^{n-1}}(t) = 0$ satisfying the boundary conditions (1.2).

LEMMA 2.2 [2]. *For $s \in [t_1, \rho(t_2)]$ and $n \geq 2$,*

$$
G_n(t,s : t_1, t_2, \ldots, t_n) \begin{cases} < 0 & \text{if } t \in (-\infty, t_1), \\ > 0 & \text{if } t \in (t_1, \sigma^{n-1}(t_n)]. \end{cases}
\tag{2.9}
$$

THEOREM 2.3 [2]. *Let $u_{n,i}(t,s)$ and $v_{n,i}(t,s)$ be given as in (2.4) and (2.7), respectively. Assume for $n \geq 4$ that*

$$
v_{n-i,j-i+1}\left(\sigma^{n-i}(t_n), \sigma(s_j) : t_i, t_{i+1}, \ldots, t_{n-1}\right) > 0,
\tag{2.10}
$$

for $j \in \{2, 3, \ldots, n-2\}$ and $i = j-1, j-2, \ldots, 1$, and for $s_j \in [\rho(t_j), \rho(t_{j+1})]$. Then

$$
G_n(t,s : t_1, t_2, \ldots, t_n) \begin{cases} < 0 & \text{if } t \in (-\infty, t_1), \\ > 0 & \text{if } t \in (t_1, \sigma^{n-1}(t_n)], \end{cases}
\tag{2.11}
$$

for $s \in [t_1, \rho(t_n)]$ if n is even, or for $s \in [t_1, \rho(t_{n-1})]$ if n is odd. For odd $n \geq 3$, the additional assumption

$$u_{n,n-1}(\sigma^{n-1}(t_n)) + (-1)^{n-1} h_{n-1}(\sigma^{n-1}(t_n), t_n) > 0 \tag{2.12}$$

yields (2.11) for $s \in [\rho(t_{n-1}), \rho(t_n)]$ as well.

Throughout this paper, we assume that the time scale **T** is such that $\sigma(s)$ is delta differentiable for all $s \in$ **T**, t_1 is right-scattered, and hypotheses of Theorem 2.3 hold.

Furthermore, we have the following assumptions.

(H1) $r(s)$ is a nonnegative continuous function defined on $[t_1, \rho(t_n)]$ satisfying

$$0 < \int_{t_1}^{t_n} G_n(t,s)r(s)\Delta s < \infty, \tag{2.13}$$

for $t \in [\sigma(t_1), \sigma^{n-1}(t_n)]$.

(H2) $f : [t_1, \rho(t_n)] \times \mathbb{R} \to \mathbb{R}$ is such that $f(t,x) \geq 0$ for $x \in \mathbb{R}^+$ and continuous with respect to x, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

Let us set

$$M_n := \max G_n(t,s)r(s), \qquad m_n := \min G_n(t,s)r(s) \tag{2.14}$$

for $s \in [t_1, \rho(t_n)]$, $t \in [\sigma(t_1), \sigma^{n-1}(t_n)]$, and

$$A_{1_n} := \max_{t \in [\sigma(t_1), \sigma^{n-1}(t_n)]} \int_{t_1}^{t_n} G_n(t,s)r(s)\Delta s, \qquad A_{2_n} := \min_{t \in [\sigma(t_1), \sigma^{n-1}(t_n)]} \int_{t_1}^{t_n} G_n(t,s)r(s)\Delta s. \tag{2.15}$$

We refer to [7, 12] for a discussion of the fixed point index that we use below. In particular, we will make frequent use of the following lemma.

LEMMA 2.4. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume $r > 0$ and that $\Psi : \mathcal{P}_r \to \mathcal{P}$ is compact operator such that $\Psi x \neq x$ for $x \in \partial\mathcal{P}_r := \{x \in \mathcal{P} : \|x\| = r\}$. Then the following assertions hold.*

(i) *If $\|x\| \leq \|\Psi x\|$ for all $x \in \partial\mathcal{P}_r$, then $i(\Psi, \mathcal{P}_r, \mathcal{P}) = 0$.*

(ii) *If $\|x\| \geq \|\Psi x\|$ for all $x \in \partial\mathcal{P}_r$, then $i(\Psi, \mathcal{P}_r, P) = 1$.*

Thus, if there exists $r_1 > r_2 > 0$ such that condition (i) holds for $x \in \partial\mathcal{P}_{r_1}$ and (ii) holds for $x \in \partial\mathcal{P}_{r_2}$ (or (ii) and (i)), then, from the additivity properties of the index, we know that

$$i(\Psi, \mathcal{P}_{r_1}, \mathcal{P}) = i(\Psi, \mathcal{P}_{r_1} \setminus \mathrm{int}(\mathcal{P}_{r_2}), \mathcal{P}) + i(\Psi, \mathcal{P}_{r_2}, \mathcal{P}). \tag{2.16}$$

As a consequence of $i(\Psi, \mathcal{P}_{r_1} \setminus \mathrm{int}(\mathcal{P}_{r_2}), \mathcal{P}) \neq 0$, Ψ has a fixed point (nonzero) whose norm is between r_1 and r_2.

Consider the Banach space of continuous functions on $[t_1, \sigma^{n-1}(t_n)]$ with the norm

$$\|x\| = \max\{|x(t)|, t \in [\sigma(t_1), \sigma^{n-1}(t_n)]\}, \tag{2.17}$$

and cone $\mathcal{P}$ in $\mathcal{B}$ given by

$$\mathcal{P} = \left\{ x \in \mathcal{B} : x(t) \geq 0,\ t \in \left[\sigma(t_1), \sigma^{n-1}(t_n)\right],\ \min_{t \in [\sigma(t_1), \sigma^{n-1}(t_n)]} x(t) \geq \frac{m_n}{M_n} \|x\| \right\}. \qquad (2.18)$$

By Theorem 2.1, solving the BVP (1.1)–(1.2) is equivalent to solving the following integral equation in $\mathcal{P}$:

$$x(t) = \lambda \int_{t_1}^{t_n} G_n(t,s) r(s) f\left(s, x^\sigma(s)\right) \Delta s, \quad t \in \left[t_1, \sigma^{n-1}(t_n)\right], \qquad (2.19)$$

and consequently, it is equivalent to finding fixed points of the operator $\Psi_{n_\lambda} : \mathcal{B} \to \mathcal{B}$ defined by

$$\Psi_{n_\lambda} x(t) := \lambda \int_{t_1}^{t_n} G_n(t,s) r(s) f\left(s, x^\sigma(s)\right) \Delta s, \quad t \in \left[t_1, \sigma^{n-1}(t_n)\right]. \qquad (2.20)$$

First, we prove that for every $\lambda > 0$ given, this operator maps the cone $\mathcal{P}$ in itself.

LEMMA 2.5. *Let $\lambda > 0$ be given. Under the hypotheses (H1) and (H2), the operator Ψ_{n_λ} is a compact operator such that $\Psi_{n_\lambda}(\mathcal{P}) \subset \mathcal{P}$.*

Proof. That Ψ_{n_λ} is a compact operator follows by Arzela-Ascoli's theorem. Next, for all $x \in \mathcal{P}$, by (H1), (H2), and the positivity property of the Green function, we have from (2.20), $\Psi_{n_\lambda} x(t) \geq 0$ for all $t \in [\sigma(t_1), \sigma^{n-1}(t_n)]$. If $x \in \mathcal{P}$, then

$$
\begin{aligned}
\min_{t \in [\sigma(t_1), \sigma^{n-1}(t_n)]} \Psi_{n_\lambda} x(t) &\geq \lambda m_n \int_{t_1}^{t_n} f\left(s, x^\sigma(s)\right) \Delta s \\
&\geq \lambda \frac{m_n}{M_n} \int_{t_1}^{t_n} \left\{ \max_{t \in [\sigma(t_1), \sigma^{n-1}(t_n)]} G_n(t,s) r(s) \right\} f\left(s, x^\sigma(s)\right) \Delta s \\
&\geq \lambda \frac{m_n}{M_n} \max_{t \in [\sigma(t_1), \sigma^{n-1}(t_n)]} \int_{t_1}^{t_n} G_n(t,s) r(s) f\left(s, x^\sigma(s)\right) \Delta s \\
&= \frac{m_n}{M_n} \|\Psi_{n_\lambda} x\|.
\end{aligned} \qquad (2.21)
$$

Therefore, $\Psi_{n_\lambda} x \in \mathcal{P}$. $\square$

3. Noneigenvalue problem

In this section we study the existence of at least two positive solutions to the following BVP:

$$
\begin{aligned}
(-1)^{n-1} x^{\Delta^n}(t) &= r(t) f\left(t, x^\sigma(t)\right) \quad \forall t \in \left[t_1, \rho(t_n)\right], \\
x(t_1) &= x^\Delta(t_2) = \cdots = x^{\Delta^{n-1}}(t_n) = 0,
\end{aligned} \qquad (3.1)
$$

which is problem (1.1)-(1.2) with $\lambda = 1$. As an application, we also give an example to demonstrate our result.

THEOREM 3.1. *The boundary value problem (3.1) has at least two positive solutions, x_1 and x_2, if (H1) and (H2) are satisfied and, in addition, both of the following hold.*

(H3) *There exists $0 < k < R < +\infty$ such that*

$$f(t,x) > \frac{M_n}{m_n^2(t_n - t_1)}x \quad \forall x \in [0,k] \bigcup [R,+\infty], t \in [t_1, \rho(t_n)]. \tag{3.2}$$

(H4) *There exists $p > 0$ such that*

$$f(t,x) < \frac{p}{M_n(t_n - t_1)} \quad \forall x \in [0,p], \, t \in [t_1, \rho(t_n)], \tag{3.3}$$

where M_n and m_n are given as (2.14). Moreover, $0 < \|x_1\| < p < \|x_2\|$.

Proof. Let $x \in \partial \mathcal{P}_k$. From condition (H3), we have

$$\begin{aligned} \|A_{1_n}\| &= \max_{t \in [\sigma(t_1), \sigma^{n-1}(t_n)]} \int_{t_1}^{t_n} G_n(t,s)r(s)f(s,x^\sigma(s))\Delta s \\ &> m_n \frac{M_n}{m_n^2(t_n - t_1)} \int_{t_1}^{t_n} x^\sigma(s)\Delta s \\ &\geq \frac{M_n}{m_n(t_n - t_1)} \frac{m_n}{M_n} \|x\| \int_{t_1}^{t_n} \Delta s = \|x\|. \end{aligned} \tag{3.4}$$

If $x \in \partial \mathcal{P}_{R_1}$, $R_1 \geq M_n/m_n R$, we have

$$\min_{t \in [\sigma(t_1), \sigma^{n-1}(t_n)]} x(t) \geq \frac{m_n}{M_n} \|x\| = \frac{m_n}{M_n} R_1 \geq R. \tag{3.5}$$

Hence $x(s) \geq R$ for all $s \in [t_1, \rho(t_n)]$. Therefore using condition (H3) again, we arrive at the same conclusion.

Now, from (H4), if $\|x\| = p$,

$$\begin{aligned} \|A_{1_n}\| &= \max_{t \in [\sigma(t_1), \sigma^{n-1}(t_n)]} \int_{t_1}^{t_n} G_n(t,s)r(s)f(s,x^\sigma(s))\Delta s \\ &\leq \int_{t_1}^{t_n} M_n f(s,x^\sigma(s))\Delta s < \|x\|. \end{aligned} \tag{3.6}$$

Since we can choose $k > 0$ small enough and R_1 sufficiently large so that $k < p < R_1$, we assure the existence of two solutions: $x_1 \in \mathcal{P}_p \setminus \text{int}(\mathcal{P}_k)$ and $x_2 \in \mathcal{P}_{R_1} \setminus \text{int}(\mathcal{P}_p)$. □

Example 3.2. We illustrate Theorem 3.1 with specific time scale

$$\mathbf{T} = \mathbf{T}_c = \{c^m : m \in \mathbb{Z}\} \cup \{0\}, \tag{3.7}$$

where $c > 1$ and the following specific parameter values for $n = 3$. Let $c = 11/10$, $t_1 = 1$, $t_2 = (11/10)^3$, and $t_3 = (11/10)^4$. Bohner and Peterson [6] show that

$$h_j(t,s) = \prod_{\nu=0}^{j-1} \frac{t - c^\nu s}{\sum_{\mu=0}^{\nu} c^\mu} \tag{3.8}$$

for all $s, t \in \mathbf{T}$. Using this formula, we have

$G_3(t,s)$

$$
= \begin{cases}
s \in \left[1, \left(\frac{11}{10}\right)^2\right] : \begin{cases}
(t-1)\left(\frac{11}{10}s - 1\right) & \text{if } t \le \frac{11}{10}s, \\
(t-1)\left(\frac{11}{10}s - 1\right) + \frac{10}{21}\left(t - \frac{11}{10}s\right)\left(t - \frac{121}{100}s\right) & \text{if } t \ge \frac{11}{10}s,
\end{cases} \\[20pt]
s \in \left[\left(\frac{11}{10}\right)^2, \left(\frac{11}{10}\right)^3\right] : \begin{cases}
\frac{331}{1000}(t-1) & \text{if } t \le \frac{11}{10}s, \\
\frac{331}{1000}(t-1) + \frac{10}{21}\left(t - \frac{11}{10}s\right)\left(t - \frac{121}{100}s\right) & \text{if } t \ge \frac{11}{10}s.
\end{cases}
\end{cases}
$$
(3.9)

If $r(s) = s$, then $m_3 = \min G(t,s)r(s) = 10^{-2}$, $M_3 = \max G(t,s)r(s) = 3870659646821 \cdot 10^{-13}$ for $t \in [11/10, (11/10)^6], s \in [1, (11/10)^3]$.

Let $k = 1/18000$, $p = 1/5$, $R = 2/5$, let

$$
f(t,x) = \begin{cases}
2.10^4 k \sin \dfrac{t\pi}{6} & \text{if } x \in [0,k], \\[8pt]
L(x) \sin \dfrac{t\pi}{6} & \text{if } x \in [k, p], \\[8pt]
K(x) \sin \dfrac{t\pi}{6} & \text{if } x \in [p, R], \\[8pt]
2.10^4 \dfrac{k+1}{R} x \sin \dfrac{t\pi}{6} & \text{if } x \in [R, +\infty),
\end{cases}
$$
(3.10)

where

$$
L(x) = 1 + \frac{p-x}{p-k}[2.10^4 k - 1],
$$
$$
K(x) = 1 + \frac{p-x}{p-R}[2.10^4 (k+1) - 1].
$$
(3.11)

Note that f is continuous and nonnegative valued for $x \ge 0$.

For $t \in [1, (11/10)^3]$ and $x \in [0,k] \cup [R, \infty)$, $f(t,x) > (8340, 14144, \dots)x$. Indeed; for $x \in [0,k]$, $f(t,x) = 2.10^4 k \sin(t\pi/6) \ge 10^4 k > (8340, 14144, \dots)x$, for $x \in [R, \infty)$, $f(t,x) = 2.10^4((k+1)/R)x \sin(t\pi/6) \ge 10^4((k+1)/R)x = (900050/36)x > (8340, 14144, \dots)x$. So (H3) is verified.

For $x \in [0,k]$, $f(t,x) = 2.10^4 k \sin(t\pi/6) < 2.10^4 k < p/0, 17963731420896261$.

For $x \in [k,p]$, $f(t,x) = 1 + ((p-x)/(p-k))[2.10^4 k - 1] \sin(t\pi/6) \le 2.10^4 k \sin(t\pi/6) < 2.10^4 k < p/0, 17963731420896261$. Hence it verifies the (H4).

We conclude from Theorem 3.1 that for these parameter values, (3.1) for $n = 3$ has at least two positive solutions, x_1 and x_2 such that $0 < \|x_1\| < 1/5 < \|x_2\|$.

4. Eigenvalue problem

Define the nonnegative extended real numbers f_0, f^0, f_∞, and f^∞ by

$$f_0 := \liminf_{x \to 0^+} \min_{t \in [t_1, \rho(t_n)]} \frac{f(t,x)}{x}, \qquad f^0 := \limsup_{x \to 0^+} \max_{t \in [t_1, \rho(t_n)]} \frac{f(t,x)}{x},$$

$$f_\infty := \liminf_{x \to \infty} \min_{t \in [t_1, \rho(t_n)]} \frac{f(t,x)}{x}, \qquad f^\infty := \limsup_{x \to \infty} \max_{t \in [t_1, \rho(t_n)]} \frac{f(t,x)}{x}, \qquad (4.1)$$

respectively.

These numbers can be regarded as generalized super or sublinear conditions on the function $f(t,x)$ at $x = 0$ and $x = \infty$. Thus, if $f_0 = f^0 = 0(+\infty)$, then $f(t,x)$ is superlinear (sublinear) at $x = 0$ and if $f_\infty = f^\infty = 0(+\infty)$, then $f(t,x)$ is sublinear (superlinear) at $x = +\infty$.

First, we obtain an existence result for λ belonging to a given interval.

THEOREM 4.1. *If (H1)-(H2) hold and either*
(a) $M_n/(m_n A_{2_n} f_0) < \lambda < 1/(A_{1_n} f^\infty)$, *or*
(b) $M_n/(m_n A_{2_n} f_\infty) < \lambda < 1/(A_{1_n} f^0)$
is satisfied, where M_n, m_n, A_{1_n}, and A_{2_n} are given as in (2.14) and (2.15), then the eigenvalue problem (1.1)-(1.2) has at least one positive solution.

Proof. Assume (a) holds. First we consider $f_0 < \infty$. Since

$$\frac{M_n}{m_n A_{2_n} f_0} < \lambda, \qquad (4.2)$$

there is an $\epsilon > 0$ so that

$$\lambda(f_0 - \epsilon) \frac{m_n}{M_n} A_{2_n} \geq 1. \qquad (4.3)$$

Using the definition of f_0, there is an $r_1 > 0$, sufficiently small, so that

$$f_0 - \epsilon < \min_{t \in [t_1, \rho(t_n)]} \frac{f(t,x)}{x} \qquad (4.4)$$

for $0 < x \leq r_1$.

It follows that $f(t,x) > (f_0 - \epsilon)x$ for $0 < x \leq r_1$, $t \in [t_1, \rho(t_n)]$.
Assume that $x \in \partial \mathcal{P}_{r_1}$, then

$$\begin{aligned}
\Psi_{n\lambda} x(t) &= \lambda \int_{t_1}^{t_n} G_n(t,s) r(s) f(s, x^\sigma(s)) \Delta s \\
&> \lambda(f_0 - \epsilon) \int_{t_1}^{t_n} G_n(t,s) r(s) x^\sigma(s) \Delta s \\
&\geq \lambda(f_0 - \epsilon) \frac{m_n}{M_n} \|x\| A_{2_n} \geq \|x\|.
\end{aligned} \qquad (4.5)$$

Next, we consider the case $f_0 = \infty$. Choose $K > 0$ sufficiently large so that

$$\lambda K \frac{m_n}{M_n} A_{2_n} \geq 1 \tag{4.6}$$

for any $t \in [t_1, \sigma^{n-1}(t_n)]$.

So there exists $r_1 > 0$ so that $f(t,x) > Kx$ for $0 < x \leq r_1$.

Assume that $x \in \partial \mathcal{P}_{r_1}$, then

$$\Psi_{n_\lambda} x(t) > \lambda K \int_{t_1}^{t_n} G_n(t,s) r(s) x^\sigma(s) \Delta s \geq \lambda K \frac{m_n}{M_n} \|x\| A_{2_n} \geq \|x\|. \tag{4.7}$$

Finally, we use the assumption

$$\lambda < \frac{1}{A_{1_n} f^\infty}. \tag{4.8}$$

Pick an $\epsilon_1 > 0$ so that

$$\lambda (f^\infty + \epsilon_1) A_{1_n} \leq 1. \tag{4.9}$$

Using the definition of f^∞, there is an $r > r_1$ sufficiently large, so that

$$\max_{t \in [t_1, \rho(t_n)]} \frac{f(t,x)}{x} < f^\infty + \epsilon_1, \tag{4.10}$$

for $x \geq r$.

It follows that $f(t,x) < (f^\infty + \epsilon_1)x$ for $x \geq r$.

We now show that there is an $r_2 \geq r$ such that if $x \in \partial \mathcal{P}_{r_2}$, then $\|\Psi_{n_\lambda} x\| < \|x\|$.

Pick $r_2 \geq rM_n/m_n > r_1$. Now assume $x \in \partial \mathcal{P}_{r_2}$ and consider

$$\Psi_{n_\lambda} x(t) < \lambda (f^\infty + \epsilon_1) \int_{t_1}^{t_n} G_n(t,s) r(s) x^\sigma(s) \Delta s \leq \lambda (f^\infty + \epsilon_1) A_{1_n} \|x\| \leq \|x\|. \tag{4.11}$$

Therefore, by Lemma 2.4, Ψ_{n_λ} has a fixed point x with $r_1 < \|x\| < r_2$. This shows that condition (a) yields the existence of a positive solution of the eigenvalue problem (1.1)-(1.2). This completes the proof of the theorem. □

The proof of part (b) is similar.

Our next results give criteria for the existence of one, more than one, or no positive solutions of the eigenvalue problem (1.1)-(1.2) in terms of the superlinear or sublinear behavior of $f(t,x)$. For the next three theorems, in addition to the assumptions (H1) and (H2) we assume.

(H5) $f(t,x) > 0$ on $[t_1, \rho(t_n)] \times \mathbb{R}^+$.

THEOREM 4.2. *If hypotheses (H1), (H2), and (H5) are satisfied, then the following assertions hold.*

(a) *If $f_0 = \infty$ or $f_\infty = \infty$, then there is a $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$ the eigenvalue problem (1.1)-(1.2) has a positive solution.*

(b) *If $f^0 = 0$ or $f^\infty = 0$, then there is a $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ the eigenvalue problem (1.1)-(1.2) has a positive solution.*

Proof of part (a). Let $r > 0$ be given. From conditions (H2) and (H5) we can define

$$L := \max \left\{ f(t,x) : (t,x) \in [t_1, \rho(t_n)] \times [0,r] \right\} > 0. \tag{4.12}$$

Then if $x \in \partial \mathcal{P}_r$, it follows that

$$\Psi_{n_\lambda} x(t) \leq \lambda L \int_{t_1}^{t_n} G_n(t,s) r(s) \Delta s \leq \lambda L A_{1_n}. \tag{4.13}$$

It follows that we can pick $\lambda_0 > 0$ sufficiently small so that for all $0 < \lambda \leq \lambda_0$,

$$\|\Psi_{n_\lambda} x\| \leq \|x\| \tag{4.14}$$

for all $x \in \partial \mathcal{P}_r$.

Fix $\lambda \leq \lambda_0$. Choose $T > 0$ sufficiently large so that

$$\lambda \frac{m_n}{M_n} T A_{2_n} \geq 1. \tag{4.15}$$

Since $f_0 = \infty$, there exists $s < r$ such that

$$\min_{t \in [t_1, \rho(t_n)]} \frac{f(t,x)}{x} > T \tag{4.16}$$

for $0 < x \leq s$. Hence, we have that

$$f(t,x) > Tx \quad \text{for } t \in [t_1, \rho(t_n)], \ 0 < x \leq s. \tag{4.17}$$

Now, let $x \in \partial \mathcal{P}_s$. In this case,

$$\Psi_{n_\lambda} x(t) > \lambda T \int_{t_1}^{t_n} G_n(t,s) r(s) x^\sigma(s) \Delta s \geq \lambda T \frac{m_n}{M_n} \|x\| A_{2_n} \geq \|x\| \tag{4.18}$$

for $t \in [\sigma(t_1), \sigma^{n-1}(t_n)]$. Hence we have shown that if $x \in \partial \mathcal{P}_s$, then $\|\Psi_{n_\lambda} x\| \geq \|x\|$.

It follows from Lemma 2.4 that the operator Ψ_{n_λ} has a fixed point.

When $f_\infty = \infty$, there is a $w > r$ such that

$$\min_{t \in [t_1, \rho(t_n)]} \frac{f(t,x)}{x} > T \tag{4.19}$$

for $x \geq \omega$.

It follows that $f(t,x) > Tx$ for $t \in [t_1, \rho(t_n)]$, $x \geq \omega$.

Let $\omega_0 := \omega M_n / m_n$. Next if $x \in \partial \mathcal{P}_{\omega_0}$, then we show that $\|\Psi_{n_\lambda} x\| > \|x\|$. In fact,

$$\Psi_{n_\lambda} x(t) > \lambda T \int_{t_1}^{t_n} G_n(t,s) r(s) x^\sigma(s) \Delta s \geq \lambda T \frac{m_n}{M_n} \|x\| A_{2_n} \geq \|x\| \tag{4.20}$$

for $t \in [t_1, \sigma^{n-1}(t_n)]$. This completes the proof of part (a). $\qquad\square$

Part (b) holds in an analogous way.

Similar to the proof of Theorem 4.2, we get the next result.

Theorem 4.3. *Under the hypotheses of Theorem 4.2, the following assertions hold.*
   (a) *If $f_0 = f_\infty = \infty$, then there is a $\lambda_0 > 0$ such that for all $0 < \lambda \le \lambda_0$, the eigenvalue problem (1.1)-(1.2) has two positive solutions.*
   (b) *If $f^0 = f^\infty = 0$, then there is a $\lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$, the eigenvalue problem (1.1)-(1.2) has two positive solutions.*

Now, we give a nonexistence result as follows.

Theorem 4.4. *Under the hypotheses of Theorem 4.2, the following assertions hold.*
   (a) *If there is a constant $c > 0$ such that $f(t,x) \ge cx$ for $x \ge 0$, then there is a $\lambda_0 > 0$ such that the eigenvalue problem (1.1)-(1.2) has no positive solutions for $\lambda \ge \lambda_0$.*
   (b) *If there is a constant $c > 0$ such that $f(t,x) \le cx$ for $x \ge 0$, then there is a $\lambda_0 > 0$ such that the eigenvalue problem (1.1)-(1.2) has no positive solutions for $0 < \lambda \le \lambda_0$.*

Proof of part (b). Assume there is constant $c > 0$ such that $f(t,x) \le cx$ for $x \ge 0$. Assume $x(t)$ is a positive solution of the eigenvalue problem (1.1)-(1.2). We will show that for λ sufficiently small and positive that this leads to a contradiction. Since $\Psi_{n_\lambda} x(t) = x(t)$ for $t \in [t_1, \sigma^{n-1}(t_n)]$,

$$
\begin{aligned}
x(t) &= \lambda \int_{t_1}^{t_n} G_n(t,s) r(s) f\left(s, x^\sigma(s)\right) \Delta s \le c\lambda \int_{t_1}^{t_n} G_n(t,s) r(s) x^\sigma(s) \Delta s \\
&\le c\lambda \|x\| \int_{t_1}^{t_n} G_n(t,s) r(s) \Delta s \le c\lambda A_{1_n} \|x\|
\end{aligned}
\tag{4.21}
$$

for $t \in [t_1, \sigma^{n-1}(t_n)]$. Pick λ_0 sufficiently small so that for $0 < \lambda \le \lambda_0$,

$$
c\lambda A_{1_n} < 1,
\tag{4.22}
$$

then we have $x(t) < \|x\|$ for $t \in [t_1, \sigma^{n-1}(t_n)]$ which is a contradiction.

The proof of part *(a)* is similar. □

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