DELAY DYNAMIC EQUATIONS WITH STABILITY

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We first give conditions which guarantee that every solution of a first order linear delay dynamic equation for isolated time scales vanishes at infinity. Several interesting examples are given. In the last half of the paper, we give conditions under which the trivial solution of a nonlinear delay dynamic equation is asymptotically stable, for arbitrary time scales.

1. Preliminaries

The unification and extension of continuous calculus, discrete calculus, q-calculus, and indeed arbitrary real-number calculus to time-scale calculus, where a time scale is simply any nonempty closed set of real numbers, were first accomplished by Hilger in [4]. Since then, time-scale calculus has made steady inroads in explaining the interconnections that exist among the various calculuses, and in extending our understanding to a new, more general and overarching theory. The purpose of this work is to illustrate this new understanding by extending some continuous and discrete delay equations to certain time scales. Examples will include specific cases in differential equations, difference equations, q-difference equations, and harmonic-number equations. The definitions that follow here will serve as a short primer on the time-scale calculus; they can be found in [1, 2] and the references therein.

Definition 1.1. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup\mathbb{T}$ (resp., $\rho(t)$ at t for $t > \inf\mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}), \quad \forall t \in \mathbb{T}. \tag{1.1}$$

Also define $\sigma(\sup\mathbb{T}) = \sup\mathbb{T}$, if $\sup\mathbb{T} < \infty$, and $\rho(\inf\mathbb{T}) = \inf\mathbb{T}$, if $\inf\mathbb{T} > -\infty$. Define the graininess function $\mu : \mathbb{T} \to \mathbb{R}$ by $\mu(t) = \sigma(t) - t$.

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Throughout this work the assumption is made that $\mathbb{T}$ is unbounded above and has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also assume throughout that $a < b$ are points in $\mathbb{T}$ and define the time scale interval $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$. Other time scale intervals are defined similarly. The jump operators σ and ρ allow the classification of points in a time scale in the following way: if $\sigma(t) > t$ then call the point t right-scattered; while if $\rho(t) < t$ then we say t is left-scattered. If $\sigma(t) = t$ then call the point t right-dense; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$ then we say t is left-dense. We next define the so-called delta derivative. The novice could skip this definition and look at the results stated in Theorem 1.4. In particular in part (2) of Theorem 1.4 we see what the delta derivative is at right-scattered points and in part (3) of Theorem 1.4 we see that at right-dense points the derivative is similar to the definition given in calculus.

Definition 1.2. Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^{\Delta}(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t such that, for all $s \in U$,

$$\left|\left[y(\sigma(t)) - y(s)\right] - y^{\Delta}(t)\left[\sigma(t) - s\right]\right| \le \epsilon \left|\sigma(t) - s\right|. \tag{1.2}$$

Call $y^{\Delta}(t)$ the (delta) derivative of $y(t)$ at t.

Definition 1.3. If $F^{\Delta}(t) = f(t)$ then define the (Cauchy) delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a). \tag{1.3}$$

The following theorem is due to Hilger [4].

Theorem 1.4. *Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.*

(1) *If f is differentiable at t, then f is continuous at t.*

(2) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}. \tag{1.4}$$

(3) *If f is differentiable and t is right-dense, then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{1.5}$$

(4) *If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.*

Next we define the important concept of right-dense continuity. An important fact concerning right-dense continuity is that every right-dense continuous function has a delta antiderivative [1, Theorem 1.74]. This implies that the delta definite integral of any right-dense continuous function exists.

Definition 1.5. We say that $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (and write $f \in C_{\mathrm{rd}}(\mathbb{T};\mathbb{R})$) provided f is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

We say p is regressive provided $1+\mu(t)p(t) \neq 0$, $\forall t \in \mathbb{T}$. Let

$$\mathcal{R} := \{p \in C_{\mathrm{rd}}(\mathbb{T};\mathbb{R}) : 1+\mu(t)p(t) \neq 0,\ t \in \mathbb{T}\}. \tag{1.6}$$

Also, $p \in \mathcal{R}^+$ if and only if $p \in \mathcal{R}$ and $1+\mu(t)p(t) > 0$, $\forall t \in \mathbb{T}$. Then if $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$, one can define the generalized exponential function $e_p(t,t_0)$ to be the unique solution of the initial value problem

$$x^{\Delta} = p(t)x, \qquad x(t_0) = 1. \tag{1.7}$$

We will use many of the properties of this generalized exponential function $e_p(t,t_0)$ listed in Theorem 1.6.

Theorem 1.6 ([1, Theorem 2.36]). *If $p,q \in \mathcal{R}$ and $s,t \in \mathbb{T}$, then*

(1) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(2) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(3) $1/e_p(t,s) = e_{\ominus p}(t,s)$, *where* $\ominus p := -p/(1+\mu p)$;
(4) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;
(5) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(6) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$, *where* $p \oplus q := p+q+\mu pq$;
(7) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$.

2. Introduction to a delay dynamic equation

Since we are interested in the asymptotic properties of solutions we assume as mentioned earlier that our time scale $\mathbb{T}$ is unbounded above. Consider the delay dynamic equation

$$x^{\Delta}(t) = -a(t)x(\delta(t))\delta^{\Delta}(t), \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.1}$$

where the delay function $\delta : [t_0,\infty)_{\mathbb{T}} \to [\delta(t_0),\infty)_{\mathbb{T}}$ is strictly increasing and delta differentiable with $\delta(t) < t$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $\lim_{t\to\infty}\delta(t) = \infty$. For example, if $\mathbb{T} = [-m,\infty)$, and $\delta(t) := t-m$, $t \in [0,\infty)$, where $m > 0$, then (2.1) becomes the well-studied delay differential equation

$$x'(t) = -a(t)x(t-m). \tag{2.2}$$

If $\mathbb{T} = \{-m,-m+1,\dots,0,1,2,\dots\}$, and $\delta(t) := t-m$, $t \in \mathbb{N}_0$, where m is a positive integer, then (2.1) becomes

$$\Delta x(t) = -a(t)x(t-m), \tag{2.3}$$

where Δ is the forward difference operator defined by $\Delta x(t) = x(t+1)-x(t)$. If $\mathbb{T} = q^{\mathbb{N}_0} \cup \{q^{-1},q^{-2},\dots,q^{-m}\}$ where $q^{\mathbb{N}_0} := \{1,q,q^2,\dots\}$, $q > 1$, and $\delta(t) := (1/q^m)t$, $t \in q^{\mathbb{N}_0}$, where $m \in \mathbb{N}$, then (2.1) becomes the delay quantum equation

$$D_q x(t) = -\frac{1}{q^m}a(t)x\left(\frac{1}{q^m}t\right), \tag{2.4}$$

where

$$D_q x(t) := \frac{x(qt) - x(t)}{(q-1)t} \tag{2.5}$$

is the so-called quantum derivative studied in Kac and Cheung [5]. More examples will be given later. We will use the following three lemmas to prove Theorem 3.1.

LEMMA 2.1 (chain rule). *Assume $\mathbb{T}$ is an isolated time scale, and $g(\sigma(t)) = \sigma(g(t))$ for $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$, then*

$$\left(\int_{t_0}^{g(t)} h(s)\Delta s \right)^{\Delta} = h(g(t)) g^{\Delta}(t). \tag{2.6}$$

Proof. Since t is right-scattered,

$$\begin{aligned}
\left(\int_{t_0}^{g(t)} h(s)\Delta s \right)^{\Delta} &= \frac{1}{\mu(t)} \left(\int_{t_0}^{g(\sigma(t))} h(s)\Delta s - \int_{t_0}^{g(t)} h(s)\Delta s \right) \\
&= \frac{1}{\mu(t)} \int_{g(t)}^{g(\sigma(t))} h(s)\Delta s \\
&= \frac{1}{\mu(t)} \int_{g(t)}^{\sigma(g(t))} h(s)\Delta s \\
&= \frac{1}{\mu(t)} h(g(t)) \left(\sigma(g(t)) - g(t) \right) \\
&= h(g(t)) \frac{g(\sigma(t)) - g(t)}{\mu(t)} \\
&= h(g(t)) g^{\Delta}(t).
\end{aligned} \tag{2.7}$$

□

LEMMA 2.2. *Assume $\mathbb{T}$ is an isolated time scale and the delay δ satisfies $\delta \circ \sigma = \sigma \circ \delta$, or $\mathbb{T} = \mathbb{R}$. Then the delay equation (2.1) is equivalent to the delay equation*

$$x^{\Delta}(t) = -a(\delta^{-1}(t))x(t) + \left(\int_{\delta(t)}^{t} a(\delta^{-1}(s))x(s)\Delta s \right)^{\Delta}. \tag{2.8}$$

Proof. Assume x is a solution of (2.8). Then using the chain rule (Lemma 2.1) for isolated time scales or the regular chain rule for $\mathbb{T} = \mathbb{R}$,

$$\begin{aligned}
x^{\Delta}(t) &= -a(\delta^{-1}(t))x(t) + \left(\int_{\delta(t)}^{t} a(\delta^{-1}(s))x(s)\Delta s \right)^{\Delta} \\
&= -a(\delta^{-1}(t))x(t) + a(\delta^{-1}(t))x(t) - a(t)x(\delta(t))\delta^{\Delta}(t) \\
&= -a(t)x(\delta(t))\delta^{\Delta}(t).
\end{aligned} \tag{2.9}$$

Hence x is a solution of (2.1). Reversing the above steps, we obtain the desired result. □

Lemma 2.3. *If x is a solution of (2.1) with initial function ψ, then*

$$
\begin{aligned}
x(t) = {} & e_{-a(\delta^{-1})}(t,t_0)\psi(t_0) + \int_{\delta(t)}^{t} a(\delta^{-1}(s))x(s)\Delta s \\
& - e_{-a(\delta^{-1})}(t,t_0)\int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s))\psi(s)\Delta s \\
& - \int_{t_0}^{t} \frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))} e_{-a(\delta^{-1})}(t,\tau)\left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s\right)\Delta\tau.
\end{aligned}
\tag{2.10}
$$

Proof. We use the variation of constants formula [1, page 77] for (2.8), to obtain

$$
x(t) = e_{-a(\delta^{-1})}(t,t_0)x(t_0) + \int_{t_0}^{t} e_{-a(\delta^{-1})}(t,\sigma(\tau))\left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s\right)^{\Delta_\tau}\Delta\tau. \tag{2.11}
$$

Using integration by parts [1, page 28],

$$
\begin{aligned}
x(t) = {} & e_{-a(\delta^{-1})}(t,t_0)x(t_0) + e_{-a(\delta^{-1})}(t,\tau)\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s \,\Big|_{t_0}^{t} \\
& - \int_{t_0}^{t} e^{\Delta_\tau}_{-a(\delta^{-1})}(t,\tau)\left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s\right)\Delta\tau.
\end{aligned}
\tag{2.12}
$$

It follows from Theorem 1.6 that

$$
\begin{aligned}
x(t) = {} & e_{-a(\delta^{-1})}(t,t_0)x(t_0) + \int_{\delta(t)}^{t} a(\delta^{-1}(s))x(s)\Delta s \\
& - e_{-a(\delta^{-1})}(t,t_0)\int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s))x(s)\Delta s \\
& - \int_{t_0}^{t} e^{\Delta_\tau}_{\ominus(-a(\delta^{-1}))}(\tau,t)\left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s\right)\Delta\tau \\
= {} & e_{-a(\delta^{-1})}(t,t_0)x(t_0) + \int_{\delta(t)}^{t} a(\delta^{-1}(s))x(s)\Delta s \\
& - e_{-a(\delta^{-1})}(t,t_0)\int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s))x(s)\Delta s \\
& - \int_{t_0}^{t} \ominus(-a(\delta^{-1}))(\tau)e_{\ominus(-a(\delta^{-1}))}(\tau,t)\left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s\right)\Delta\tau.
\end{aligned}
\tag{2.13}
$$

Finally, using Theorem 1.6 once again and $x(t)=\psi(t)$ for $t\in[\delta(t_0),t_0]$,

$$
\begin{aligned}
x(t) = {} & e_{-a(\delta^{-1})}(t,t_0)\psi(t_0) + \int_{\delta(t)}^{t} a(\delta^{-1}(s))x(s)\Delta s \\
& - e_{-a(\delta^{-1})}(t,t_0)\int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s))\psi(s)\Delta s \\
& - \int_{t_0}^{t} \frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))} e_{-a(\delta^{-1})}(t,\tau)\left(\int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))x(s)\Delta s\right)\Delta\tau.
\end{aligned}
\tag{2.14}
$$

□

3. Asymptotic properties of the delay equation

The results in this section generalize some of the results by Raffoul in [9]. Let $\psi : [\delta(t_0), t_0]_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous and let $x(t) := x(t, t_0, \psi)$ be the solution of (2.1) on $[t_0, \infty)_{\mathbb{T}}$ with $x(t) = \psi(t)$ on $[\delta(t_0), t_0]_{\mathbb{T}}$. Let $\|\phi\| = \sup|\phi(t)|$ for $t \in [\delta(t_0), \infty)_{\mathbb{T}}$, and define the Banach space $B = \{\phi \in C([\delta(t_0), \infty)_{\mathbb{T}} : \phi(t) \to 0 \text{ as } t \to \infty\}$, with

$$S := \left\{\phi \in B : \phi(t) = \psi(t)\ \forall t \in [\delta(t_0), t_0]_{\mathbb{T}}\right\}. \tag{3.1}$$

In the following we assume

$$e_{-a(\delta^{-1})}(t, t_0) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty, \tag{3.2}$$

and take $D : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ to be the function

$$\begin{aligned} D(t) := &\int_{t_0}^{t} \left(\left| \frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} \right| \left| e_{-a(\delta^{-1})}(t, \tau) \right| \int_{\delta(\tau)}^{\tau} \left| a(\delta^{-1}(s)) \right| \Delta s \right) \Delta\tau \\ &+ \int_{\delta(t)}^{t} \left| a(\delta^{-1}(s)) \right| \Delta s. \end{aligned} \tag{3.3}$$

To enable the use of the contraction mapping theorem, we in fact assume there exists $\alpha \in (0,1)$ such that

$$D(t) \leq \alpha, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.4}$$

Theorem 3.1. *Assume $\mathbb{T} = \mathbb{R}$ or $\mathbb{T}$ is an isolated time scale. If (3.2) and (3.4) hold and $\delta \circ \sigma = \sigma \circ \delta$, then every solution of (2.1) goes to zero at infinity.*

Proof. Assume $\mathbb{T}$ is an isolated time scale. Fix $\psi : [\delta(t_0), t_0] \to \mathbb{R}$ and define $P : S \to B$ by $(P\phi)(t) := \psi(t)$ for $t \leq t_0$ and for $t \geq t_0$,

$$\begin{aligned} (P\phi)(t) = &\ \psi(t_0) e_{-a(\delta^{-1})}(t, t_0) + \int_{\delta(t)}^{t} a(\delta^{-1}(s))\phi(s)\Delta s \\ &- e_{-a(\delta^{-1})}(t, t_0) \int_{\delta(t_0)}^{t_0} a(\delta^{-1}(s))\psi(s)\Delta s \\ &- \int_{t_0}^{t} \left(\frac{a(\delta^{-1}(\tau))}{1 - \mu(\tau)a(\delta^{-1}(\tau))} e_{-a(\delta^{-1})}(t, \tau) \int_{\delta(\tau)}^{\tau} a(\delta^{-1}(s))\phi(s)\Delta s \right) \Delta\tau. \end{aligned} \tag{3.5}$$

Then by Lemma 2.3, it suffices to show that P has a fixed point. We will use the contraction mapping theorem to show P has a fixed point. To show that $(P\phi)(t) \to 0$ as $t \to \infty$, note that the first and third terms on the right-hand side of $(P\phi)(t)$ go to zero by (3.2). From (3.3) and (3.4) and the fact that $\phi(t) \to 0$ as $t \to \infty$, we have that

$$|\phi(t)| \int_{\delta(t)}^{t} \left| a(\delta^{-1}(s)) \right| \Delta s \leq |\phi(t)|\alpha \longrightarrow 0, \quad t \longrightarrow \infty. \tag{3.6}$$

Let $\epsilon > 0$ be given and choose $t^* \in \mathbb{T}$ so that

$$\alpha\|\phi\|\,|e_{-a(\delta^{-1})}(t,T)| < \frac{\epsilon}{2}, \quad \forall t > t^*, \tag{3.7}$$

for some large $t^* > T$. For the same T it is possible to make

$$\alpha\|\phi\|_{[\delta(T),\infty)_{\mathbb{T}}} < \frac{\epsilon}{2}, \tag{3.8}$$

where $\|\phi\|_{[\delta(T),\infty)_{\mathbb{T}}} = \sup\{|\phi(t)|,\ t \in [\delta(T),\infty)_{\mathbb{T}}\}$. By (2.10) and (3.2), for $t \geq T$,

$$\begin{aligned}
&\int_{t_0}^{t}\left(\left|\frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))}\right|\,|e_{-a(\delta^{-1})}(t,\tau)|\int_{\delta(\tau)}^{\tau}|a(\delta^{-1}(s))\phi(s)|\,\Delta s\right)\Delta\tau\\
&\quad=\left(\int_{t_0}^{T}+\int_{T}^{t}\right)\left(\left|\frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))}\right|\,|e_{-a(\delta^{-1})}(t,\tau)|\right.\\
&\qquad\qquad\left.\times\int_{\delta(\tau)}^{\tau}|a(\delta^{-1}(s))\phi(s)|\,\Delta s\right)\Delta\tau\\
&\quad=\int_{t_0}^{T}\left(\left|\frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))}\right|\,|e_{-a(\delta^{-1})}(t,T)e_{-a(\delta^{-1})}(T,\tau)|\right.\\
&\qquad\qquad\left.\times\int_{\delta(\tau)}^{\tau}|a(\delta^{-1}(s))\phi(s)|\,\Delta s\right)\Delta\tau\\
&\qquad+\int_{T}^{t}\left(\left|\frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))}\right|\,|e_{-a(\delta^{-1})}(t,\tau)|\int_{\delta(\tau)}^{\tau}|a(\delta^{-1}(s))\phi(s)|\,\Delta s\right)\Delta\tau\\
&\quad\leq|e_{-a(\delta^{-1})}(t,T)|\,\|\phi\|\int_{t_0}^{T}\left(\left|\frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))}\right|\,|e_{-a(\delta^{-1})}(T,\tau)|\right.\\
&\qquad\qquad\left.\times\int_{\delta(\tau)}^{\tau}|a(\delta^{-1}(s))|\,\Delta s\right)\Delta\tau\\
&\qquad+\|\phi\|_{[\delta(T),\infty)_{\mathbb{T}}}\int_{T}^{t}\left(\left|\frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))}\right|\,|e_{-a(\delta^{-1})}(t,\tau)|\int_{\delta(\tau)}^{\tau}|a(\delta^{-1}(s))|\,\Delta s\right)\Delta\tau\\
&\quad\leq\alpha|e_{-a(\delta^{-1})}(t,T)|\,\|\phi\|+\alpha\|\phi\|_{[\delta(T),\infty)_{\mathbb{T}}}\\
&\quad<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}\tag{3.9}$$

Hence $(P\phi)(t) \to 0$ as $t \to \infty$ and therefore, P maps S into S. It remains to show that P is a contraction under the sup norm. Let $x, y \in S$. Then

$$
\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&\quad \le \int_{t_0}^{t} \left(\left| \frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t,\tau)| \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))|\,|x(s)-y(s)|\Delta s \right) \Delta\tau \\
&\qquad + \int_{\delta(t)}^{t} |a(\delta^{-1}(s))|\,|(x(s)-y(s))|\Delta s \\
&\quad \le \|x-y\| \Bigg[\int_{\delta(t)}^{t} |a(\delta^{-1}(s))|\Delta s \\
&\qquad\qquad + \int_{t_0}^{t} \left(\left| \frac{a(\delta^{-1}(\tau))}{1-\mu(\tau)a(\delta^{-1}(\tau))} \right| |e_{-a(\delta^{-1})}(t,\tau)| \int_{\delta(\tau)}^{\tau} |a(\delta^{-1}(s))|\Delta s \right) \Delta\tau \Bigg] \\
&\quad \le \alpha \|x-y\|.
\end{aligned}
\tag{3.10}
$$

Therefore, by the contraction mapping principle [6, page 300], P has a unique fixed point in S. This completes the proof in the isolated time scale case. See Raffoul [9] for the proof of the $\mathbb{T} = \mathbb{Z}$ case and a reference for a proof of the continuous case. □

Example 3.2. For any real number $q > 1$ and positive integer m, define

$$\mathbb{T} = \{q^{-m}, q^{-m+1}, \dots, q^{-1}, 1, q, q^2, \dots\}. \tag{3.11}$$

We show if $0 < c < q^m/2m(q-1)$, then for any initial function $\psi(t)$, $t \in [q^{-m}, 1]_{\mathbb{T}}$, the solution of the delay initial value problem

$$D_q x(t) = -\frac{1}{q^m}\frac{c}{t}x\left(\frac{1}{q^m}t\right), \quad t \in [1,\infty]_{\mathbb{T}}, \tag{3.12}$$

$$x(t) = \psi(t), \quad t \in [q^{-m}, 1]_{\mathbb{T}} \tag{3.13}$$

goes to zero as $t \to \infty$.

To obtain (3.12) from (2.1), take $a(t) = c/t$ and $\delta(t) = q^{-m}t$ which implies $a(\delta^{-1}(t)) = c/q^m t$ and $\delta^{\Delta}(t) = q^{-m}$. To use Theorem 3.1, we verify that conditions (3.2) and (3.4) hold. Note that

$$e_{-a(\delta^{-1})}(t,1) = \prod_{s \in [1,t)_{\mathbb{T}}} [1 - s(q-1)a(q^m s)] = (1 - q^{-m}c(q-1))^n \tag{3.14}$$

for $t = q^n$. If $c \in (0, q^m/2m(q-1))$, then $c \in (0, 2q^m/(q-1))$ so that $1 - q^{-m}c(q-1) \in (-1,1)$ and

$$\lim_{t\to\infty} e_{-a(\delta^{-1})}(t,1) = \lim_{n\to\infty} (1 - q^{-m}c(q-1))^n = 0. \tag{3.15}$$

Thus, (3.2) is satisfied. Now consider $D(t)$ as defined in (3.3). We seek $\alpha \in (0,1)$ such that $D(t) \le \alpha$, $\forall t \in [1,\infty)_{\mathbb{T}}$. Here we have $t_0 = 1$, $\mu(t) = (q-1)t$, and

$$e_{-a(\delta^{-1})}(t,\tau) = \left(1 - q^{-m}c(q-1)\right)^{n-k} \tag{3.16}$$

for $t = q^n$, $\tau = q^k$ with $k < n$. For the second integral in $D(t)$, note that

$$\int_u^{qu} f(\zeta)\Delta\zeta = (qu-u)f(u), \tag{3.17}$$

whence

$$\begin{aligned}\int_{\delta(t)}^{t} a(\delta^{-1}(s))\Delta s &= \left(\int_{q^{-m}t}^{q^{-m+1}t} + \int_{q^{-m+1}t}^{q^{-m+2}t} + \cdots + \int_{q^{-1}t}^{t}\right)\left(\frac{c}{q^m s}\right)\Delta s \\ &= \left(\frac{mc}{q^m(q^{-m}t)}q^{-m}t\right)(q-1) \\ &= \frac{mc}{q^m}(q-1),\end{aligned} \tag{3.18}$$

which is independent of t. It follows that

$$\begin{aligned}D(t) &= \frac{mc}{q^m}(q-1) + \frac{mc}{q^m}(q-1)\int_1^t \left(\frac{c}{q^m\tau}\cdot\frac{1}{1-(q-1)\tau c/q^m\tau}e_{-a(\delta^{-1})}(t,\tau)\right)\Delta\tau \\ &= \frac{mc}{q^m}(q-1) + \frac{mc}{q^m}(q-1)\sum_{k=0}^{n-1}\frac{c}{q^m q^k - q^k(q-1)c}\left(1-q^{-m}c(q-1)\right)^{n-k}(q-1)q^k \\ &= \frac{mc}{q^m}(q-1) + \frac{mc}{q^m}\sum_{k=0}^{n-1}\frac{c(q-1)^2}{q^m-(q-1)c}\left(1-q^{-m}c(q-1)\right)^{n-k} \\ &= \frac{mc}{q^m}(q-1) + \frac{mc^2q^{-m}(q-1)^2}{q^m - c(q-1)}\,\frac{1-q^{-m}c(q-1)}{-q^{-m}(q-1)c}\left(\left(1-q^{-m}c(q-1)\right)^n - 1\right) \\ &= \frac{mc(q-1)}{q^m} + \frac{mcq^{-m}(q-1)}{1-cq^{-m}(q-1)}\left(1-q^{-m}c(q-1)\right)\left(1-\left(1-q^{-m}c(q-1)\right)^n\right).\end{aligned} \tag{3.19}$$

Consequently,

$$D(t) = \frac{mc(q-1)}{q^m}\left[2 - \left(1-q^{-m}c(q-1)\right)^n\right] < \frac{2mc(q-1)}{q^m}, \quad \forall t = q^n \in [1,\infty)_{\mathbb{T}}. \tag{3.20}$$

Since $0 < c < q^m/2m(q-1)$, by taking $\alpha := 2mc(q-1)/q^m$ condition (3.4) is satisfied by

$$D(t) < \alpha < 1, \quad \forall t \in [1,\infty)_{\mathbb{T}}. \tag{3.21}$$

Thus (3.2) and (3.4) are met, so that by Theorem 3.1, the solution of the IVP (3.12), (3.13) goes to zero as $t \to \infty$.

Example 3.3. Consider the time scale of harmonic numbers

$$\mathbb{T} = \{H_{-m}, H_{-m+1}, \dots, H_0, H_1, \dots\} \tag{3.22}$$

for some $m \in \mathbb{N}$, where $H_0 := 0$, $H_n := \sum_{j=1}^{n}(1/j)$ and $H_{-n} := -H_n$ for $n \in \mathbb{N}$. We will show that if

$$0 < c < \frac{H_m}{2m}, \tag{3.23}$$

then for any initial function $\psi(t)$, $t \in [H_{-m}, 0]_{\mathbb{T}}$, the solution of the delay initial value problem

$$\Delta_n x(H_n) = -\frac{(n-m+1)c}{H_m} x(H_{n-m})\Delta_n H_{n-m}, \quad n \in \mathbb{N}_0, \tag{3.24}$$

$$x(H_n) = \psi(H_n), \quad n = 0, -1, \dots, -m, \tag{3.25}$$

goes to zero as $t \to \infty$.

To get (3.24) from (2.1), take

$$a(t) = a(H_n) = \frac{(n-m+1)c}{H_m}, \qquad \delta(t) = \delta(H_n) = H_{n-m}. \tag{3.26}$$

It follows that

$$e_{-a(\delta^{-1})}(H_n, 0) = \left(1 - \frac{c}{H_m}\right)^n \quad \text{for } n \in \mathbb{N}_0. \tag{3.27}$$

If we restrict $c \in (0, 2H_m)$,

$$\lim_{t\to\infty} e_{-a(\delta^{-1})}(t, 0) = \lim_{n\to\infty}\left(1 - \frac{c}{H_m}\right)^n = 0, \tag{3.28}$$

satisfying (3.2). Simplifying (3.3),

$$\begin{aligned}
D(t) &= \int_0^{H_n}\left(\frac{(\tau+1)c}{H_m}\frac{1}{1-(\tau+1)c/(\tau+1)H_m}\left(1-\frac{c}{H_m}\right)^{n-\tau}\int_{H_{\tau-m}}^{H_\tau}\frac{(s+1)c}{H_m}\Delta s\right)\Delta\tau \\
&\quad + \int_{H_{n-m}}^{H_n}\frac{(s+1)c}{H_m}\Delta s \\
&= \frac{cm}{H_m} + \frac{c^2 m}{H_m}\cdot\frac{1}{H_m - c}\sum_{\tau=0}^{n-1}\left(1-\frac{c}{H_m}\right)^{n-\tau} \\
&= \frac{cm}{H_m} + \frac{c^2 m}{H_m(H_m-c)}\left(\frac{H_m-c}{H_m}\right)\left(\frac{-H_m}{c}\right)\left[\left(1-\frac{c}{H_m}\right)^n - 1\right] \\
&= \frac{cm}{H_m}\left[2 - \left(1-\frac{c}{H_m}\right)^n\right] < \frac{2cm}{H_m}
\end{aligned} \tag{3.29}$$

for all $t = H_n \in [0,\infty)_{\mathbb{T}}$. By choosing $c \in (0, H_m/2m)$, $D(t) < \alpha := 2cm/H_m < 1$, $\forall t = H_n \in [0,\infty)_{\mathbb{T}}$, satisfying (3.4). Thus by Theorem 3.1, for any given initial function ψ, the solution of the IVP (2.1), (3.25) goes to zero as $t = H_n$ goes to infinity.

Example 3.4. Let $\mathbb{T} = \{-mh,\dots,-h,0,h,2h,\dots\}$ where $m \in \mathbb{N}$, then (2.1) becomes

$$\Delta_h x(t) = -cx(t-hm), \quad t \in h\mathbb{N}, \tag{3.30}$$

where $\Delta_h x(t) := (x(t+h) - x(t))/h$. Our results give that if

$$0 < c < \frac{1}{2mh}, \tag{3.31}$$

then all solutions go to zero as $t \to \infty$.

In this case, $a(t) = c$, $\delta(t) = t - mh$, and $\delta^{-1}(t) = t + mh$. It can be shown that

$$e_{-a(\delta^{-1})}(t,0) = (1-ch)^{t/h}. \tag{3.32}$$

Note for $0 < ch < 2$, condition (3.2) is satisfied because

$$e_{-a(\delta^{-1})}(t,0) \longrightarrow 0 \tag{3.33}$$

as $t \to \infty$. It also can be shown that

$$D(t) = mhc + mhc\left(1 - |1-ch|^{t/h}\right) \le 2mhc \tag{3.34}$$

if $0 < ch < 1$. Hence (3.4) holds if $0 < c < 1/2mh$. Therefore, we use Theorem 3.1 to conclude that all solutions of (3.30) go to zero as $t \to \infty$. It can be shown that if $c \le 0$, then there is $\lambda \ge 1$ such that $x(t) = \lambda^{ht}$ is a solution of (3.30). However, $x(t)$ does not approach zero as $t \to \infty$. Hence our lower estimate for c is sharp. If $m = 1$, then it can be shown that all solutions of (3.30) go to zero if $0 < c < 1/h$. If $m = 2$, our example shows that if $0 < c < 1/4h$ all solutions go to zero as $t \to \infty$. It can be shown that if $c = 1/2h$, then there is a solution that does not go to zero as $t \to \infty$. Next we give an elementary example where $\mathbb{T}$ is the real interval $[-m,\infty)$, $m > 0$. Delay differential equations have been studied extensively; for example, see [8].

Example 3.5. Let $\mathbb{T} = [-m,\infty)$. For the delay differential equation

$$x'(t) = -cx(t-m), \quad t \in [0,\infty), \tag{3.35}$$

if $0 < c < 1/2m$, then all solutions $x(t)$ approach zero as $t \to \infty$.

Note that (2.1) reduces to (3.35) if $a(t) = c$, $\delta(t) = t - m$, and $\mathbb{T} = [-m,\infty)$. It can be shown that

$$e_{-a(\delta^{-1})}(t,\tau) = e^{-c(t-\tau)}, \qquad D(t) = cm\left(2 - e^{-ct}\right). \tag{3.36}$$

Note that $c > 0$ implies $e_{-a(\delta^{-1})}(t,0) \to 0$ as $t \to \infty$ and $D(t) \le 2cm < 1$ if $c < 1/2m$. Our result follows from Theorem 3.1. Also note that if $c = \pi/2m$, then $x(t) = \sin((\pi/2m)t)$ is a solution that does not approach zero as $t \to \infty$. It is well known that if $0 < c < \pi/2m$, then all solutions of (3.35) approach zero as $t \to \infty$.

4. Asymptotic stability of a nonlinear delay dynamic equation

In this section we consider, on arbitrary time scales, the nonlinear delay dynamic equation

$$x^{\Delta}(t) = -\int_{\delta(t)}^{t}\left(\sum_{i=1}^{n} f_i(t,x(s))\right)\Delta s, \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{4.1}$$

where $f_i(t,x)$ for each fixed $t \in \mathbb{T}$ is continuous with respect to x. In addition, we always suppose

(H1) $xf_i(t,x) \geq 0$ and $\sum_{i=1}^{n} f_i(t,x) = 0 \Leftrightarrow x = 0$, $t \in [t_0,\infty)_{\mathbb{T}}$,

(H2) $\delta : \mathbb{T} \to \mathbb{T}$ is continuous and nondecreasing, with $\delta(t) \leq t$ and $\lim_{t\to\infty}\delta(t) = \infty$.

The initial condition associated with (4.1) takes the form

$$x(t) = \psi(t), \quad t \in [\delta(t_0),t_0],\ \psi \text{ is rd-continuous on } [\delta(t_0),t_0]. \tag{4.2}$$

Equation (4.1) is studied extensively in [7] in the case when $\mathbb{T} = \mathbb{R}$; indeed many of our techniques in this section are motivated by those in [7]. See also a related discussion in [3].

Theorem 4.1. *Assume there exists $M > 0$ such that for $|c| \leq M$,*

(i) $\int_{\delta(t)}^{\sigma(t)}\sum_{i=1}^{n}(1/c)f_i(\tau,c)(\tau-\delta(\tau))\Delta\tau \leq \xi < 1$, $c \neq 0$, *for* $t \in [t_1,\infty)_{\mathbb{T}}$;

(ii) $\int_{t_0}^{\infty}\sum_{i=1}^{n}|f_i(\tau,c)|(\tau-\delta(\tau))\Delta\tau = \infty$;

(iii) $|\sum_{i=1}^{n} f_i(\tau,c)| \leq \sum_{i=1}^{n} a_i(\tau)|c|$ *where the a_i are rd-continuous and nonnegative for* $\tau \geq t_0$;

(iv) $|\sum_{i=1}^{n} f_i(\tau,x)| \leq \sum_{i=1}^{n} f_i(\tau,y)$ *for* $|x| \leq y \leq M$, $\tau \in [\delta(t_0),\infty)_{\mathbb{T}}$.

Then for any $0 < \epsilon \leq M$, there is an $\eta(\epsilon) > 0$ such that for any rd-continuous initial function ψ with $\|\psi\|_{[\delta(t_0),t_0]_{\mathbb{T}}} < \eta(\epsilon)$, the solution x of (4.1), (4.2) satisfies

$$|x(t)| < \epsilon \quad \forall t \in [\delta(t_0),\infty)_{\mathbb{T}}, \quad \lim_{t\to\infty} x(t) = 0. \tag{4.3}$$

In other words, the trivial solution of (4.1) is asymptotically stable.

Proof. Let $\epsilon \in (0,M]$ and $t_1 \in [t_0,\infty)_{\mathbb{T}}$ be given. Also let $p(t) := \sum_{i=1}^{n} a_i(t)(t-\delta(t))$; by (H2) and (iii), $p \in \mathcal{R}^+$. Define $\eta(\epsilon) := \epsilon/e_p(t_1,t_0)$ and take an initial function ψ with $\|\psi\|_{[\delta(t_0),t_0]_{\mathbb{T}}} < \eta(\epsilon)$. Integrating (4.1) from t_0 to $t \in [t_0,t_1]_{\mathbb{T}}$, we get

$$x(t) = x(t_0) - \int_{t_0}^{t}\left(\int_{\delta(\tau)}^{\tau}\sum_{i=1}^{n} f_i(\tau,x(s))\Delta s\right)\Delta\tau. \tag{4.4}$$

Taking absolute values,

$$|x(t)| \leq |x(t_0)| + \int_{t_0}^{t}\left(\int_{\delta(\tau)}^{\tau}\left|\sum_{i=1}^{n} f_i(\tau,x(s))\right|\Delta s\right)\Delta\tau. \tag{4.5}$$

By property (iii),

$$|x(t)| \le |x(t_0)| + \int_{t_0}^{t}\left(\int_{\delta(\tau)}^{\tau}\sum_{i=1}^{n} a_i(\tau)|x(s)|\Delta s\right)\Delta\tau. \tag{4.6}$$

Let $\widetilde{x}(t) := \max\{|x(s)| : s \in [\delta(t_0), t]\}$. Then rewriting (4.6), we have

$$\widetilde{x}(t) \le \widetilde{x}(t_0) + \int_{t_0}^{t}\widetilde{x}(\tau)\sum_{i=1}^{n} a_i(\tau)(\tau - \delta(\tau))\Delta\tau. \tag{4.7}$$

By Gronwall's inequality [1, page 257],

$$\widetilde{x}(t) \le \widetilde{x}(t_0)e_p(t,t_0), \quad \forall t \in [t_0,\infty)_{\mathbb{T}}. \tag{4.8}$$

Therefore by the definition of $\eta(\epsilon)$ and the choice of ψ, it follows that $\widetilde{x}(t_0) < \eta(\epsilon)$ and $|x(t)| \le \widetilde{x}(t) < \epsilon$ for $t \in [t_0,t_1]_{\mathbb{T}}$. Suppose there exists $t^* \ge t_1$ such that $x(t^*) = \epsilon$ and $x^{\Delta}(t^*) \ge 0$. Note the case when $x(t^*) = -\epsilon$ and $x^{\Delta}(t^*) \le 0$ is similar. By (4.1), there exists $\bar{t} \in [\delta(t^*), t^*)_{\mathbb{T}}$ such that $x(\bar{t}) \le 0$. Integrating (4.1) from $\bar{t}$ to t^*, we obtain

$$\epsilon - x(\bar{t}) = -\int_{\bar{t}}^{t^*}\left(\int_{\delta(\tau)}^{\tau}\sum_{i=1}^{n} f_i(\tau,x(s))\Delta s\right)\Delta\tau. \tag{4.9}$$

Since $x(\bar{t}) \le 0$,

$$\epsilon \le \int_{\delta(t^*)}^{t^*}\left(\int_{\delta(\tau)}^{\tau}\left|\sum_{i=1}^{n} f_i(\tau,x(s))\right|\Delta s\right)\Delta\tau. \tag{4.10}$$

By (H1) and (iv),

$$\epsilon \le \int_{\delta(t^*)}^{t^*}\left(\int_{\delta(\tau)}^{\tau}\sum_{i=1}^{n}|f_i(\tau,\epsilon)|\Delta s\right)\Delta\tau, \tag{4.11}$$

so that

$$1 \le \frac{1}{\epsilon}\int_{\delta(t^*)}^{t^*}(\tau - \delta(\tau))\sum_{i=1}^{n} f_i(\tau,\epsilon)\Delta\tau, \tag{4.12}$$

which is a contradiction of (i). Therefore no such t^* exists. Now suppose there exists $t^* \in [t_1,\infty)_{\mathbb{T}}$ such that t^* is right scattered, $|x(t)| \le \epsilon$ for all $t \in [\delta(t_0), t^*)_{\mathbb{T}}$, and $x(t^*) \in (-\epsilon,\epsilon)$ but $|x(\sigma(t^*))| \ge \epsilon$. Without loss of generality, assume

$$x(\sigma(t^*)) \ge \epsilon. \tag{4.13}$$

By (4.1),

$$\int_{\delta(t^*)}^{t^*}\left(\sum_{i=1}^{n} f_i(t^*,x(s))\right)\Delta s < 0. \tag{4.14}$$

Therefore by (H1) and (4.1), there exists $\bar{t} \in [\delta(t^*), t^*)_{\mathbb{T}}$ such that $x(\bar{t}) \leq 0$. Integrate (4.1) from $\bar{t}$ to $\sigma(t^*)$ and use (4.13) to see that

$$\epsilon \leq x(\sigma(t^*)) \leq x(\sigma(t^*)) - x(\bar{t}) = -\int_{\bar{t}}^{\sigma(t^*)} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^{n} f_i(\tau, x(s)) \Delta s \right) \Delta\tau. \tag{4.15}$$

Thus, comparing the extremities,

$$\epsilon \leq \int_{\delta(t^*)}^{\sigma(t^*)} \left(\int_{\delta(\tau)}^{\tau} \left| \sum_{i=1}^{n} f_i(\tau, x(s)) \right| \Delta s \right) \Delta\tau \leq \int_{\delta(t^*)}^{\sigma(t^*)} \sum_{i=1}^{n} f_i(\tau, \epsilon)(\tau - \delta(\tau)) \Delta\tau; \tag{4.16}$$

division by ϵ yields

$$1 \leq \frac{1}{\epsilon} \int_{\delta(t^*)}^{\sigma(t^*)} \sum_{i=1}^{n} f_i(\tau, \epsilon)(\tau - \delta(\tau)) \Delta\tau, \tag{4.17}$$

which also contradicts (i) and provides the desired result. Now we show the limit of $x(t)$ goes to zero as $t \to \infty$.

Case 1. Let x be a nonoscillatory function. Assume $\xi < 1$ but there exists a solution x of (4.1) such that

$$\lim_{t \to \infty} x(t) \neq 0. \tag{4.18}$$

Since x is nonoscillatory, there exists $T_1 > t_0$ such that $x(t)x(T_1) > 0$ for all $t \in [T_1, \infty)_{\mathbb{T}}$. Without loss of generality, assume $x(t) > 0$ for all $t \in [T_1, \infty)_{\mathbb{T}}$. From (4.1) and (H1), $x^{\Delta}(t) < 0$. Hence there exists a constant $x^* > 0$ such that

$$\lim_{t \to \infty} x(t) = x^* \tag{4.19}$$

and there exists $T^* \in \mathbb{T}$ such that for all $t \in [T_1, \infty)_{\mathbb{T}}$, $x(t) \in [x^*, 3x^*/2]$. Integrate (4.1) from $\delta^{-1}(T^*)$ to t to obtain

$$x(\delta^{-1}(T^*)) - x(t) = \int_{\delta^{-1}(T^*)}^{t} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^{n} f_i(\tau, x(s)) \Delta s \right) \Delta\tau. \tag{4.20}$$

For $t \geq \delta^{-1}(T^*)$,

$$x(\delta^{-1}(T^*)) - x(t) \leq \frac{x^*}{2}, \tag{4.21}$$

but

$$\begin{aligned} \int_{\delta^{-1}(T^*)}^{t} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^{n} f_i(\tau, x(s)) \Delta s \right) \Delta\tau &\geq \int_{\delta^{-1}(T^*)}^{t} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^{n} f_i(\tau, x^*) \Delta s \right) \Delta\tau \\ &= \int_{\delta^{-1}(T^*)}^{t} \left(\sum_{i=1}^{n} f_i(\tau, x^*)(\tau - \delta(\tau)) \right) \Delta\tau, \end{aligned} \tag{4.22}$$

which goes to ∞ as $t \to \infty$ by (H1) and properties (ii) and (iv). This contradicts (4.20) and (4.21), so when x is nonoscillatory, the limit of $x(t)$ goes to zero as $t \to \infty$.

Case 2. Now assume x is oscillatory. Pick $\epsilon \in (0,M]$ and $T_2 > t_0$ such that

$$\int_{\delta(t)}^{t} \left(\sum_{i=1}^{n} \frac{1}{c} f_i(\tau,c)(\tau - \delta(\tau)) \right) \Delta\tau < \xi \tag{4.23}$$

for $0 < |c| \le \epsilon$ and $\forall t \in [T_2,\infty)_{\mathbb{T}}$. Assume

$$\bar{x} := \limsup_{t\to\infty} |x(t)| \neq 0. \tag{4.24}$$

Without loss of generality, we will assume

$$\limsup_{t\to\infty} x(t) = \bar{x} \neq 0. \tag{4.25}$$

Since the solution is oscillatory, there exists a sequence $\{t_j\}_{j=1}^{\infty}$ in $\mathbb{T}$ such that $\lim_{j\to\infty} t_j = \infty$, $|x(\sigma(t_j))| \to \bar{x} \le \epsilon$ as $j \to \infty$, and $x^{\Delta}(t_j) \ge 0$. Let $\beta := (\bar{x}/2)(1-\xi)$ and $T_3 > T_2$ such that $|x(t)| < \beta + \bar{x}$ for $t \in [T_3,\infty)_{\mathbb{T}}$. Further assume the t_j were chosen such that $\delta(t_j) > T_3$ and $x(\sigma(t_j)) > \bar{x} - \beta$. From (4.1) and the choice of t_j, we see that

$$0 \le x^{\Delta}(t_j) = -\int_{\delta(t_j)}^{t_j} \left(\sum_{i=1}^{n} f_i(t_j, x(s)) \right) \Delta s. \tag{4.26}$$

By (H1), there exists $t^* \in [\delta(t_j), t_j]_{\mathbb{T}}$ such that $x(t^*) \le 0$. Then

$$\bar{x} - \beta < x(\sigma(t_j)) \le x(\sigma(t_j)) - x(t^*) = -\int_{t^*}^{\sigma(t_j)} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^{n} f_i(\tau, x(s)) \Delta s \right) \Delta\tau$$

$$\begin{aligned} \bar{x} - \beta &< \int_{t^*}^{\sigma(t_j)} \left(\int_{\delta(\tau)}^{\tau} \left| \sum_{i=1}^{n} f_i(\tau, x(s)) \right| \Delta s \right) \Delta\tau \\ &\le \int_{t^*}^{\sigma(t_j)} \sum_{i=1}^{n} f_i(\tau, \beta+\bar{x})(\tau - \delta(\tau)) \Delta\tau \\ &\le \int_{\delta(t_j)}^{\sigma(t_j)} \sum_{i=1}^{n} f_i(\tau, \beta+\bar{x})(\tau - \delta(\tau)) \Delta\tau \\ &< (\beta + \bar{x})\xi \end{aligned} \tag{4.27}$$

by (i). But then $\beta > ((1-\xi)/(1+\xi))\bar{x}$, a contradiction of our selection of β. Therefore $\bar{x} = 0$, in other words, the solution goes to zero as $t \to \infty$. □

Example 4.2. Let $\mathbb{T} = \mathbb{Z}$, the set of integers, the delay function $\delta(t) := t - m$ for some positive integer m, $n = 1$ and $f_1(t,x) := kx$ for some constant k. Then the four conditions of Theorem 4.1 are met if $0 < k \le \xi/m(m+1)$ for any $\xi \in (0,1)$, whereby the trivial solution of (4.1) is asymptotically stable.

For $\mathbb{T} = \mathbb{Z}$ and f_1 as above, (4.1) becomes the (delay) difference equation

$$x(t+1) = x(t) - \sum_{s=t-m}^{t-1} kx(s), \quad t \in \mathbb{Z}. \tag{4.28}$$

Fix $\xi \in (0,1)$, $m \in \mathbb{N}$, and $0 < k \leq \xi/m(m+1)$. To check (i), note that

$$\sum_{\tau=t-m}^{t} \frac{1}{c} kc(\tau - (\tau - m)) = km(m+1) \leq \xi < 1, \quad t \in \mathbb{Z}. \tag{4.29}$$

In (ii), $\sum_{\tau=t_0}^{\infty} k|c|m = \infty$. For (iii), $|f_1(\tau,c)| = k|c|$, so we take $a_1(\tau) \equiv k > 0$, which is rd-continuous. Finally, (iv) is met as $|f_1(\tau,x)| = k|x| \leq ky = f_1(\tau,y)$ for $|x| \leq y \leq M$ with $\tau \in [t_0,\infty)_{\mathbb{T}}$. Therefore the trivial solution of (4.1) is asymptotically stable by Theorem 4.1.

Lemma 4.3. *Assume x is a global solution for (4.1), (4.2). Then either x is bounded or x is oscillatory.*

Proof. If x is nonoscillatory, then there exists $T > 0$ such that for $t > T$, x does not change sign. Without loss of generality, we suppose $x(t) > 0$ for $t > T$. By (H1), $\sum_{i=1}^{n} f_i(t,x(t)) > 0$ for $t > T$, which together with (4.1) yields that $x^{\Delta}(t) < 0$ for large t. Therefore, x is strictly decreasing on each interval, so by continuity, x is bounded. □

Theorem 4.4. *Assume there exists $M > 0$ and $t_1 \geq t_0$ such that for $|c| \geq M$,*

(i) $\int_{\delta(t)}^{\sigma(t)} \sum_{i=1}^{n}(1/c) f_i(\tau,c)(\tau - \delta(\tau))\Delta\tau \leq \xi < 1$, *for $t \in [t_1,\infty)_{\mathbb{T}}$;*

(ii) $|\sum_{i=1}^{n} f_i(\tau,c)| \leq \sum_{i=1}^{n} a_i(\tau)|c|$ *where the a_i are rd-continuous and nonnegative for $\tau \geq t_0$;*

(iii) $|\sum_{i=1}^{n} f_i(t,u)| \leq \sum_{i=1}^{n} f_i(t,y)$ *for $|u| \leq y$, $t \in [\delta(t_0),\infty)_{\mathbb{T}}$.*

Then every solution x of (4.1), (4.2) for bounded initial function ψ is bounded and satisfies

$$\limsup_{t\to\infty} |x(t)| \leq M. \tag{4.30}$$

Proof. Appealing to assumption (ii), $x^{\Delta}(t)$ exists and is finite for each $t \in [t_0,\infty)_{\mathbb{T}}$, so that solutions of (4.1), (4.2) are global. Suppose x is an unbounded solution of (4.1), (4.2) with bounded initial function ψ. By Lemma 4.3, x is also oscillatory. As in Case 2 of the proof of Theorem 4.1, without loss of generality there exists a sequence $\{t_j\}_{j=1}^{\infty}$ in $\mathbb{T}$ such that $\lim_{j\to\infty} t_j = \infty$, $M \leq x(\sigma(t_j))$ with $|x(s)| \leq x(\sigma(t_j))$ for all $s \in [\delta^2(t_j),t_j]$, and $x(\sigma(t_j)) \to \infty$ as $j \to \infty$. Moreover there exist corresponding $t_j^* \in [\delta(t_j),t_j]_{\mathbb{T}}$ satisfying $x(t_j^*) \leq 0$. Then

$$x(\sigma(t_j)) \leq x(\sigma(t_j)) - x(t_j^*) = -\int_{t_j^*}^{\sigma(t_j)} \left(\int_{\delta(\tau)}^{\tau} \sum_{i=1}^{n} f_i(\tau,x(s))\Delta s \right) \Delta\tau, \tag{4.31}$$

so that by (iii) we have

$$
\begin{aligned}
x(\sigma(t_j)) &\le \int_{t_j^*}^{\sigma(t_j)} \left(\int_{\delta(\tau)}^{\tau} \left| \sum_{i=1}^{n} f_i(\tau, x(s)) \right| \Delta s \right) \Delta\tau \\
&\le \int_{\delta(t_j)}^{\sigma(t_j)} \int_{\delta(\tau)}^{\tau} \sum_{i=1}^{n} f_i(\tau, x(\sigma(t_j))) \Delta s \Delta\tau.
\end{aligned} \tag{4.32}
$$

But then

$$
\xi < 1 \le \int_{\delta(t_j)}^{\sigma(t_j)} \frac{\tau - \delta(\tau)}{x(\sigma(t_j))} \sum_{i=1}^{n} f_i(\tau, x(\sigma(t_j))) \Delta\tau, \tag{4.33}
$$

a contradiction of (i). Therefore x must be bounded. To prove the last assertion of the theorem, suppose

$$
\limsup_{t\to\infty} |x(t)| = \bar{M} > M. \tag{4.34}
$$

As in the proof of Theorem 4.1 there are two cases to consider, nonoscillatory and oscillatory. Assuming the former leads to a contradiction; in the latter case, there exists a sequence $\{\bar{t}_j\}_{j=1}^{\infty}$ as before, which would likewise lead to a contradiction, whereby the conclusion of the theorem holds. □

Finally we consider the general nonlinear delay dynamic equation

$$
x^{\Delta}(t) = -\int_{\delta(t)}^{t} \left(\sum_{i=1}^{n} f_i(t, x(s)) \right) \Delta_s g(t,s), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.35}
$$

where for simplicity we define

$$
\int_a^b f(t,s) \Delta_s g(t,s) := \int_a^b f(t,s) g^{\Delta_s}(t,s) \Delta s \tag{4.36}
$$

and assume for each fixed $t \in \mathbb{T}$ that $f(t,s)g^{\Delta_s}(t,s)$ is rd-continuous. Then it is straightforward to generalize Theorem 4.1 to get the following result.

Theorem 4.5. *Assume there exists $M > 0$ such that for $|c| \le M$,*

(i) $\int_{\delta(t)}^{\sigma(t)} \sum_{i=1}^{n} (1/c) f_i(\tau, c)(g(\tau,\tau) - g(\tau, \delta(\tau))) \Delta\tau \le \xi < 1$, $c \ne 0$, *for* $t \in [t_1, \infty)_{\mathbb{T}}$;

(ii) $\int_{t_0}^{\infty} \sum_{i=1}^{n} |f_i(\tau, c)| (g(\tau,\tau) - g(\tau, \delta(\tau))) \Delta\tau = \infty$;

(iii) $|\sum_{i=1}^{n} f_i(\tau, c)| \le \sum_{i=1}^{n} a_i(\tau)|c|$ *where the* a_i *are rd-continuous and nonnegative for* $\tau \ge t_0$;

(iv) $|\sum_{i=1}^{n} f_i(\tau, x)| \le \sum_{i=1}^{n} f_i(\tau, y)$ *for* $|x| \le y \le M$, $\tau \in [\delta(t_0), \infty)_{\mathbb{T}}$;

(v) $g(t,t) > g(t, \delta(t))$, $t \in [t_0, \infty)_{\mathbb{T}}$.

Then the trivial solution of (4.35) is asymptotically stable.

Example 4.6. Equation (2.1) is a special case of (4.35) if $\mathbb{T}$ is an isolated time scale.

If $\mathbb{T}$ is an isolated time scale, then the delay dynamic equation (4.35) with $f_1(t,x) = a(t)\delta^\Delta(t)x$ and $f_i(t,x) = 0$, $2 \le i \le n$ becomes

$$x^\Delta(t) = -\int_{\tau(t)}^{t} a(t)\delta^\Delta(t)x(s)\Delta_s g(t,s), \tag{4.37}$$

where $\tau(t) \le \delta(t) < t$ are delay functions and g is (the Heaviside function) defined by

$$g(t,s) = \begin{cases} 0, & \tau(t) \le s \le \delta(t), \\ 1, & s > \delta(t). \end{cases} \tag{4.38}$$

Then (the Dirac delta function)

$$g^{\Delta_s}(t,s) = \begin{cases} \dfrac{1}{\mu(s)}, & s = \delta(t), \\ 0, & \text{otherwise}. \end{cases} \tag{4.39}$$

Using the above we obtain

$$\begin{aligned} x^\Delta(t) &= -\int_{\tau(t)}^{t} a(t)\delta^\Delta(t)x(s)\Delta_s g(t,s) \\ &= -a(t)\delta^\Delta(t)\int_{\tau(t)}^{t} x(s)g^{\Delta_s}(t,s)\Delta s \\ &= -a(t)\delta^\Delta(t)\int_{\delta(t)}^{\sigma(\delta(t))} x(s)g^{\Delta_s}(t,s)\Delta s \\ &= -a(t)\delta^\Delta(t)x(\delta(t))\frac{1}{\mu(\delta(t))}\mu(\delta(t)) \\ &= -a(t)\delta^\Delta(t)x(\delta(t)). \end{aligned} \tag{4.40}$$

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