POSITIVE PERIODIC SOLUTIONS FOR NONLINEAR DIFFERENCE EQUATIONS VIA A CONTINUATION THEOREM

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Based on a continuation theorem of Mawhin, positive periodic solutions are found for difference equations of the form $y_{n+1} = y_n \exp(f(n, y_n, y_{n-1}, \dots, y_{n-k}))$, $n \in \mathbb{Z}$.

1. Introduction

There are several reasons for studying nonlinear difference equations of the form

$$y_{n+1} = y_n \exp\{f(n, y_n, y_{n-1}, \dots, y_{n-k})\}, \quad n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}, \tag{1.1}$$

where $f = f(t, u_0, u_1, \dots, u_k)$ is a real continuous function defined on $\mathbb{R}^{k+2}$ such that

$$f(t + \omega, u_0, \dots, u_k) = f(t, u_0, \dots, u_k), \quad (t, u_0, \dots, u_k) \in \mathbb{R}^{k+2}, \tag{1.2}$$

and ω is a positive integer. For one reason, the well-known equations

$$\begin{aligned} y_{n+1} &= \lambda y_n, \\ y_{n+1} &= \mu y_n (1 - y_n), \\ y_{n+1} &= y_n \exp\left\{\frac{\mu(1 - y_n)}{K}\right\}, \quad K > 0, \end{aligned} \tag{1.3}$$

are particular cases of (1.1). As another reason, (1.1) is intimately related to delay differential equations with piecewise constant independent arguments. To be more precise, let us recall that a solution of (1.1) is a real sequence of the form $\{y_n\}_{n \in \mathbb{Z}}$ which renders (1.1) into an identity after substitution. It is not difficult to see that solutions can be found when an appropriate function f is given. However, one interesting question is whether there are any solutions which are positive and ω-periodic, where a sequence $\{y_n\}_{n \in \mathbb{Z}}$ is said to be ω-periodic if $y_{n+\omega} = y_n$, for $n \in \mathbb{Z}$. Positive ω-periodic solutions of (1.1) are related to those of delay differential equations involving piecewise constant independent

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arguments:

$$y'(t) = y(t)f([t], y([t]), y([t-1]), y([t-2]), \ldots, y([t-k])), \quad t \in \mathbb{R}, \tag{1.4}$$

where $[x]$ is the greatest-integer function.

Such equations have been studied by several authors including Cooke and Wiener [5, 6], Shah and Wiener [9], Aftabizadeh et al. [1], Busenberg and Cooke [2], and so forth. Studies of such equations were motivated by the fact that they represent a hybrid of discrete and continuous dynamical systems and combine the properties of both differential and differential-difference equations. In particular, the following equation

$$y'(t) = ay(t)(1 - y([t])), \tag{1.5}$$

is in Carvalho and Cooke [3], where a is constant.

By a solution of (1.4), we mean a function $y(t)$ which is defined on $\mathbb{R}$ and which satisfies the following conditions [1]: (i) $y(t)$ is continuous on $\mathbb{R}$; (ii) the derivative $y'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points $[t] \in \mathbb{R}$, where one-sided derivatives exist; and (iii) (1.4) is satisfied on each interval $[n, n+1) \subset \mathbb{R}$ with integral endpoints.

Theorem 1.1. *Equation (1.1) has a positive ω-periodic solution if and only if (1.4) has a positive ω-periodic solution.*

Proof. Let $y(t)$ be a positive ω-periodic solution of (1.4). It is easy to see that for any $n \in \mathbb{Z}$,

$$y'(t) = y(t)f(n, y(n), y(n-1), \ldots, y(n-k)), \quad n \le t < n+1. \tag{1.6}$$

Integrating (1.6) from n to t, we have

$$y(t) = y(n)\exp((t-n)f(n, y(n), y(n-1), \ldots, y(n-k))). \tag{1.7}$$

Since $\lim_{t \to (n+1)^-} y(t) = y(n+1)$, we see further that

$$y(n+1) = y(n)\exp(f(n, y(n), y(n-1), \ldots, y(n-k))). \tag{1.8}$$

If we now let $y_n = y(n)$ for $n \in \mathbb{Z}$, then $\{y_n\}_{n\in\mathbb{Z}}$ is a positive ω-periodic solution of (1.1).

Conversely, let $\{y_n\}_{n\in\mathbb{Z}}$ be a positive ω-periodic solution of (1.1). Set $y(n) = y_n$, for $n \in \mathbb{Z}$, and let the function $y(t)$ on each interval $[n, n+1)$ be defined by (1.7). Then it is not difficult to check that this function is a positive ω-periodic solution of (1.4). The proof of Theorem 1.1 is complete. □

Therefore, once the existence of a positive ω-periodic solution of (1.1) can be demonstrated, we may then make immediate statements about the existence of positive ω-periodic solutions of (1.4).

There appear to be several techniques (see, e.g., [4, 8, 10]) which can help to answer such a question. Among these techniques are fixed point theorems such as that of Krasnolselskii, Leggett-Williams, and others; and topological methods such as degree theories.

Here we will invoke a continuation theorem of Mawhin for obtaining such solutions. More specifically, let X and Y be two Banach spaces and $L : \operatorname{Dom} L \subset X \to Y$ is a linear mapping and $N : X \to Y$ a continuous mapping [7, pages 39–40]. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$, and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L_{|\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$ there exist an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Theorem 1.2 (Mawhin's continuation theorem). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\bar{\Omega}$. Suppose*

(i) *for each $\lambda \in (0,1)$, $x \in \partial\Omega$, $Lx \neq \lambda Nx$;*
(ii) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$ and* $\deg(JQN, \Omega \cap \operatorname{Ker}, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \operatorname{dom} L$.

As a final remark in this section, note that if $\omega = 1$, then a positive ω-periodic solution of (1.1) is a constant sequence $\{c\}_{n\in\mathbb{Z}}$ that satisfies (1.1). Hence

$$f(n, c, \dots, c) = 0, \quad n \in \mathbb{Z}. \tag{1.9}$$

Conversely, if $c > 0$ such that $f(n, c, \dots, c) = 0$ for $n \in \mathbb{Z}$, then the constant sequence $\{c\}_{n\in\mathbb{Z}}$ is an ω-periodic solution of (1.1). For this reason, we will assume in the rest of our discussion that ω is an integer greater than or equal to 2.

2. Existence criteria

We will establish existence criteria based on combinations of the following conditions, where D and M are positive constants:

(a_1) $f(t, e^{x_0}, \dots, e^{x_k}) > 0$ for $t \in \mathbb{R}$ and $x_0, \dots, x_k \geq D$,
(a_2) $f(t, e^{x_0}, \dots, e^{x_k}) < 0$ for $t \in \mathbb{R}$ and $x_0, \dots, x_k \geq D$,
(b_1) $f(t, e^{x_0}, \dots, e^{x_k}) < 0$ for $t \in \mathbb{R}$ and $x_0, \dots, x_k \leq -D$,
(b_2) $f(t, e^{x_0}, \dots, e^{x_k}) > 0$ for $t \in \mathbb{R}$ and $x_0, \dots, x_k \leq -D$,
(c_1) $f(t, e^{x_0}, \dots, e^{x_k}) \geq -M$ for $(t, e^{x_0}, \dots, e^{x_k}) \in \mathbb{R}^{k+2}$,
(c_2) $f(t, e^{x_0}, \dots, e^{x_k}) \leq M$ for $(t, e^{x_0}, \dots, e^{x_k}) \in \mathbb{R}^{k+2}$.

Theorem 2.1. *Suppose either one of the following sets of conditions holds:*

(i) (a_1), (b_1), *and* (c_1), *or,*
(ii) *(a_2)*, (b_2), *and* (c_1), *or,*
(iii) (a_1), (b_1), *and* (c_2), *or*
(iv) (a_2), (b_2), *and* (c_2).

Then (1.1) has a positive ω-periodic solution.

We only give the proof in case (a_1), (b_1), and (c_1) hold, since the other cases can be treated in similar manners.

We first need some basic tools. First of all, for any real sequence $\{u_n\}_{n\in\mathbb{Z}}$, we define a nonstandard "summation" operation

$$\bigoplus_{n=\alpha}^{\beta} u_n = \begin{cases} \sum_{n=\alpha}^{\beta} u_n, & \alpha \le \beta, \\ 0, & \beta = \alpha - 1, \\ -\sum_{n=\beta+1}^{\alpha-1} u_n, & \beta < \alpha - 1. \end{cases} \tag{2.1}$$

It is then easy to see if $\{x_n\}_{n\in\mathbb{Z}}$ is a ω-periodic solution of the following equation

$$x_n = x_0 + \bigoplus_{i=0}^{n-1} f\left(i, e^{x_i}, e^{x_{i-1}}, \ldots, e^{x_{i-k}}\right), \quad n \in \mathbb{Z}, \tag{2.2}$$

then $\{y_n\}_{n\in\mathbb{Z}} = \{e^{x_n}\}_{n\in\mathbb{Z}}$ is a positive ω-periodic solution of (1.1). We will therefore seek an ω-periodic solution of (2.2).

Let X_ω be the Banach space of all real ω-periodic sequences of the form $x = \{x_n\}_{n\in\mathbb{Z}}$, and endowed with the usual linear structure as well as the norm $\|x\|_1 = \max_{0\le i\le \omega-1} |x_i|$. Let Y_ω be the Banach space of all real sequences of the form $y = \{y_n\}_{n\in\mathbb{Z}} = \{n\alpha + h_n\}_{n\in\mathbb{Z}}$ such that $y_0 = 0$, where $\alpha \in \mathbb{R}$ and $\{h_n\}_{n\in\mathbb{Z}} \in X_\omega$, and endowed with the usual linear structure as well as the norm $\|y\|_2 = |\alpha| + \|h\|_1$. Let the zero element of X_ω and Y_ω be denoted by θ_1 and θ_2 respectively.

Define the mappings $L : X_\omega \to Y_\omega$ and $N : X_\omega \to Y_\omega$, respectively, by

$$(Lx)_n = x_n - x_0, \quad n \in \mathbb{Z}, \tag{2.3}$$

$$(Nx)_n = \bigoplus_{i=0}^{n-1} f\left(i, e^{x_i}, e^{x_{i-1}}, \ldots, e^{x_{i-k}}\right), \quad n \in \mathbb{Z}. \tag{2.4}$$

Let

$$\bar{h}_n = \bigoplus_{i=0}^{n-1} f\left(i, e^{x_i}, e^{x_{i-1}}, \ldots, e^{x_{i-k}}\right) - \frac{n}{\omega} \bigoplus_{i=0}^{\omega-1} f\left(i, e^{x_i}, e^{x_{i-1}}, \ldots, e^{x_{i-k}}\right), \quad n \in \mathbb{Z}. \tag{2.5}$$

Since $\bar{h} = \{\bar{h}_n\}_{n\in\mathbb{Z}} \in X_\omega$ and $\bar{h}_0 = 0$, N is a well-defined operator from X_ω to Y_ω. On the other hand, direct calculation leads to $\operatorname{Ker} L = \{x \in X_\omega \mid x_n = x_0,\ n \in \mathbb{Z},\ x_0 \in \mathbb{R}\}$ and $\operatorname{Im} L = X_\omega \cap Y_\omega$. Let us define $P : X_\omega \to X_\omega$ and $Q : Y_\omega \to Y_\omega$, respectively, by

$$(Px)_n = x_0, \quad n \in \mathbb{Z}, \text{ for } x = \{x_n\}_{n\in\mathbb{Z}} \in X_\omega, \tag{2.6}$$

$$(Qy)_n = n\alpha \quad \text{for } y = \{n\alpha + h_n\}_{n\in\mathbb{Z}} \in Y_\omega. \tag{2.7}$$

The operators P and Q are projections and $X_\omega = \operatorname{Ker} P \oplus \operatorname{Ker} L$, $Y_\omega = \operatorname{Im} L \oplus \operatorname{Im} Q$. It is easy to see that $\dim \operatorname{Ker} L = 1 = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$, and that

$$\operatorname{Im} L = \{y \in X_\omega \mid y_0 = 0\} \subset Y_\omega. \tag{2.8}$$

It follows that $\operatorname{Im} L$ is closed in Y_ω. Thus the following lemma is true.

Lemma 2.2. *The mapping L defined by (2.3) L is a Fredholm mapping of index zero.*

Next we recall that a subset S of a Banach space X is relatively compact if, and only if, for each $\varepsilon > 0$, it has a finite ε-net.

Lemma 2.3. *A subset S of X_ω is relatively compact if and only if S is bounded.*

Proof. It is easy to see that if S is relatively compact in X_ω, then S is bounded. Conversely, if the subset S of X_ω is bounded, then there is a subset

$$\Gamma := \{x \in X_\omega \mid \|x\|_1 \le H\}, \tag{2.9}$$

where H is a positive constant, such that $S \subset \Gamma$. It suffices to show that Γ is relatively compact in X_ω. Note that for each $\varepsilon > 0$, we may choose numbers $y_0 < y_1 < \cdots < y_l$ such that $y_0 = -H$, $y_l = H$ and $y_{i+1} - y_i < \varepsilon$ for $i = 0,\dots,l-1$. Then

$$\{v = \{v_n\}_{n\in\mathbb{Z}} \in X_\omega \mid v_j \in \{y_0, y_1,\dots,y_{l-1}\},\ j = 0,\dots,\omega-1\} \tag{2.10}$$

is a finite ε-net of Γ. This completes the proof. □

Lemma 2.4. *Let L and N be defined by (2.3) and (2.4), respectively. Suppose Ω is an open bounded subset of X_ω. Then N is L-compact on $\overline{\Omega}$.*

Proof. From (2.4), (2.5), and (2.7), we see that for any $x = \{x_n\}_{n\in\mathbb{Z}} \in \overline{\Omega}$,

$$(QNx)_n = \frac{n}{\omega}\bigoplus_{i=0}^{\omega-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right), \quad n \in \mathbb{Z}. \tag{2.11}$$

Thus

$$\|QNx\|_2 = \left\|\frac{n}{\omega}\bigoplus_{i=0}^{\omega-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right)\right\|_2 = \frac{1}{\omega}\left|\sum_{i=0}^{\omega-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right)\right|, \tag{2.12}$$

so that $QN(\overline{\Omega})$ is bounded. We denote the inverse of the mapping $L|_{\operatorname{Dom} L\cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ by K_P. Direct calculations lead to

$$\left(K_P(I-Q)Nx\right)_n = \bigoplus_{i=0}^{n-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right) - \frac{n}{\omega}\bigoplus_{i=0}^{\omega-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right). \tag{2.13}$$

It is easy to see that

$$\left\|K_P(I-Q)Nx\right\|_1 \le 2\left|\bigoplus_{i=0}^{\omega-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right)\right|. \tag{2.14}$$

Noting that $\overline{\Omega}$ is a closed and bounded subset of X_ω and f is continuous on $\mathbb{R}^{k+2}$, relation (2.14) implies that $K_P(I-Q)N(\overline{\Omega})$ is bounded in X_ω. In view of Lemma 2.3, $K_P(I-Q)N(\overline{\Omega})$ is relatively compact in X_ω. Since the closure of a relatively compact set is relatively compact, $\overline{K_P(I-Q)N(\overline{\Omega})}$ is relatively compact in X_ω and hence N is L-compact on $\overline{\Omega}$. This completes the proof. □

Now, we consider the following equation

$$x_n - x_0 = \lambda \bigoplus_{i=0}^{n-1} f\left(i, e^{x_i}, e^{x_{i-1}}, \ldots, e^{x_{i-k}}\right), \quad n \in \mathbb{Z}, \tag{2.15}$$

where $\lambda \in (0,1)$.

Lemma 2.5. *Suppose* (a_1), (b_1), *and* (c_1) *are satisfied. Then for any ω-periodic solution $x = \{x_n\}_{n\in\mathbb{Z}}$ of (2.15),*

$$\|x\|_1 = \max_{0\le i\le \omega-1} \left|x_i\right| \le D + 4\omega M. \tag{2.16}$$

Proof. Let $x = \{x_n\}_{n\in\mathbb{Z}}$ be a ω-periodic solution $x = \{x_n\}_{n\in\mathbb{Z}}$ of (2.15). Then

$$\bigoplus_{i=0}^{\omega-1} f\left(i, e^{x_i}, e^{x_{i-1}}, \ldots, e^{x_{i-k}}\right) = 0. \tag{2.17}$$

If we write

$$G_n^+ = \max\left\{f\left(n, e^{x_n}, e^{x_{n-1}}, \ldots, e^{x_{n-k}}\right), 0\right\}, \quad n \in \mathbb{Z}, \tag{2.18}$$

$$G_n^- = \max\left\{-f\left(n, e^{x_n}, e^{x_{n-1}}, \ldots, e^{x_{n-k}}\right), 0\right\}, \quad n \in \mathbb{Z}, \tag{2.19}$$

then $\{G_n^+\}_{n\in\mathbb{Z}}$ and $\{G_n^-\}_{n\in\mathbb{Z}}$ are nonnegative real sequences and

$$f\left(n, e^{x_n}, e^{x_{n-1}}, \ldots, e^{x_{n-k}}\right) = G_n^+ - G_n^-, \quad n \in \mathbb{Z}, \tag{2.20}$$

as well as

$$\left|f\left(n, e^{x_n}, e^{x_{n-1}}, \ldots, e^{x_{n-k}}\right)\right| = G_n^+ + G_n^-, \quad n \in \mathbb{Z}. \tag{2.21}$$

In view of (c_1) and (2.19), we have

$$\left|G_n^-\right| = G_n^- \le M, \quad n \in \mathbb{Z}. \tag{2.22}$$

Thus

$$\bigoplus_{i=0}^{\omega-1} G_i^- \le \omega M, \tag{2.23}$$

and in view of (2.17), (2.20), and (2.23),

$$\bigoplus_{i=0}^{\omega-1} G_i^+ = \bigoplus_{i=0}^{\omega-1} G_i^- \le \omega M. \tag{2.24}$$

By (2.21) and (2.24), we know that

$$\bigoplus_{i=0}^{\omega-1} \left| f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right) \right| \le 2\omega M. \tag{2.25}$$

Let $x_\alpha = \max_{0\le i\le \omega-1} x_i$ and $x_\beta = \min_{0\le i\le\omega-1} x_i$, where $0 \le \alpha, \beta \le \omega - 1$. By (2.15), we have

$$\begin{aligned} x_\alpha - x_\beta = \left| x_\alpha - x_\beta \right| &= \lambda \left| \bigoplus_{i=0}^{\alpha-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right) - \bigoplus_{i=0}^{\beta-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right) \right| \\ &\le 2 \bigoplus_{i=0}^{\omega-1} \left| f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right) \right| \le 4\omega M. \end{aligned} \tag{2.26}$$

If there is some x_l, $0 \le l \le \omega - 1$, such that $|x_l| < D$, then in view of (2.15) and (2.25), for any $n \in \{0,1,\dots,\omega-1\}$, we have

$$\begin{aligned} \left| x_n \right| &= \left| x_l \right| + \left| x_n - x_l \right| \\ &\le D + \left| \bigoplus_{i=0}^{n-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right) - \bigoplus_{i=0}^{l-1} f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right) \right| \\ &\le D + 2 \bigoplus_{i=0}^{\omega-1} \left| f\left(i,e^{x_i},e^{x_{i-1}},\dots,e^{x_{i-k}}\right) \right| \\ &\le D + 4\omega M. \end{aligned} \tag{2.27}$$

Otherwise, by (a_1), (b_1), and (2.17), $x_\alpha \geqslant D$ and $x_\beta \le -D$. From (2.26), we have

$$\begin{aligned} x_\alpha &\le x_\beta + 4\omega M \le -D + 4\omega M, \\ x_\beta &\ge x_\alpha - 4\omega M \ge D - 4\omega M. \end{aligned} \tag{2.28}$$

It follows that

$$D - 4\omega M \le x_\beta \le x_n \le x_\alpha \le -D + 4\omega M, \quad 0 \le n \le \omega - 1, \tag{2.29}$$

or

$$\left| x_n \right| \le D + 4\omega M, \quad 0 \le n \le \omega - 1. \tag{2.30}$$

This completes the proof. □

We now turn to the proof of Theorem 2.1. Let L, N, P and Q be defined by (2.3), (2.4), (2.6), and (2.7), respectively. Set

$$\Omega = \left\{ x \in X_\omega \mid \|x\|_1 < \overline{D} \right\}, \tag{2.31}$$

where $\overline{D}$ is a fixed number which satisfies $\overline{D} > D + 4\omega M$. It is easy to see that Ω is an open and bounded subset of X_ω. Furthermore, in view of Lemma 2.2 and Lemma 2.4, L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. Noting that $\overline{D} > D + 4\omega M$, by Lemma 2.5, for each $\lambda \in (0,1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$. Next, note that a sequence $x = \{x_n\}_{n\in\mathbb{Z}} \in \partial\Omega \cap \operatorname{Ker} L$ must be constant: $\{x_n\}_{n\in\mathbb{Z}} = \{\overline{D}\}_{n\in\mathbb{Z}}$ or $\{x_n\}_{n\in\mathbb{Z}} = \{-\overline{D}\}_{n\in\mathbb{Z}}$. Hence by ($a_1$), ($b_1$), and (2.11),

$$(QNx)_n = \frac{n}{\omega}\bigoplus_{i=0}^{\omega-1} f(i, e^{x_0}, \dots, e^{x_0}), \quad n \in \mathbb{Z}, \tag{2.32}$$

so

$$QNx \neq \theta_2. \tag{2.33}$$

The isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is defined by $(J(n\alpha))_n = \alpha$, for $\alpha \in \mathbb{R}$, $n \in \mathbb{Z}$. Then

$$(JQNx)_n = \frac{1}{\omega}\bigoplus_{i=0}^{\omega-1} f(i, e^{x_0}, \dots, e^{x_0}) \neq 0, \quad n \in \mathbb{Z}. \tag{2.34}$$

In particular, we see that if $\{x_n\}_{n\in\mathbb{Z}} = \{\overline{D}\}_{n\in\mathbb{Z}}$, then

$$(JQNx)_n = \frac{1}{\omega}\bigoplus_{i=0}^{\omega-1} f\left(i, e^{\overline{D}}, \dots, e^{\overline{D}}\right) > 0, \quad n \in \mathbb{Z}, \tag{2.35}$$

and if $\{x_n\}_{n\in\mathbb{Z}} = \{-\overline{D}\}_{n\in\mathbb{Z}}$, then

$$(JQNx)_n = \frac{1}{\omega}\bigoplus_{i=0}^{\omega-1} f\left(i, e^{-\overline{D}}, \dots, e^{-\overline{D}}\right) < 0, \quad n \in \mathbb{Z}. \tag{2.36}$$

Consider the mapping

$$H(x,s) = sx + (1-s)JQNx, \quad 0 \le s \le 1. \tag{2.37}$$

From (2.35) and (2.37), for each $s \in [0,1]$ and $\{x_n\}_{n\in\mathbb{Z}} = \{\overline{D}\}_{n\in\mathbb{Z}}$, we have

$$\big(H(x,s)\big)_n = s\overline{D} + (1-s)\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1} f\left(i, e^{\overline{D}}, \dots, e^{\overline{D}}\right) > 0, \quad n \in \mathbb{Z}. \tag{2.38}$$

Similarly, from (2.36) and (2.37), for each $s \in [0,1]$ and $\{x_n\}_{n\in\mathbb{Z}} = \{-\overline{D}\}_{n\in\mathbb{Z}}$, we have

$$\big(H(x,s)\big)_n = -s\overline{D} + (1-s)\frac{1}{\omega}\bigoplus_{i=0}^{\omega-1} f\left(i, e^{-\overline{D}}, \dots, e^{-\overline{D}}\right) < 0, \quad n \in \mathbb{Z}. \tag{2.39}$$

By (2.38) and (2.39), $H(x,s)$ is a homotopy. This shows that

$$\deg\left(JQNx, \Omega \cap \operatorname{Ker} L, \theta_1\right) = \deg\left(-x, \Omega \cap \operatorname{Ker} L, \theta_1\right) \neq 0. \tag{2.40}$$

By Theorem 1.2, we see that equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$. In other words, (2.2) has an ω-periodic solution $x = \{x_n\}_{n\in\mathbb{Z}}$, and hence $\{e^{x_n}\}_{n\in\mathbb{Z}}$ is a positive ω-periodic solution of (1.1).

Corollary 2.6. *Under the same assumption of Theorem 1.1, (1.4) has a positive ω-periodic solution.*

3. Examples

Consider the difference equation

$$y_{n+1} = y_n \exp\left(r(n) \left(\frac{a(n) - y_{n-k}}{a(n) + c(n)r(n)y_{n-k}} \right)^{\delta} \right), \quad n \in \mathbb{Z}, \tag{3.1}$$

and the semi-discrete "food-limited" population model of

$$y'(t) = y(t)r([t]) \left(\frac{a([t]) - y([t-k])}{a([t]) + c([t])r([t])y([t-k])} \right)^{\delta}, \quad t \in \mathbb{R}. \tag{3.2}$$

In (3.1) or (3.2), r, a, and c belong to $C(\mathbb{R},(0,\infty))$, and $r(t+\omega) = r(t)$, $a(t+\omega) = a(t)$, $c(t+\omega) = c(t)$ and δ is a positive odd integer. Letting

$$\begin{gathered} M = \max_{0 \le t \le \omega} r(t), \\ f(t,u_0,u_1,\dots,u_k) = r(t) \left(\frac{a(t) - u_k}{a(t) + c(t)r(t)u_k} \right)^{\delta}, \\ D = \max_{0 \le t \le \omega} \left| \ln a(t) \right| + \varepsilon_0, \quad \varepsilon_0 > 0. \end{gathered} \tag{3.3}$$

It is easy to verify that the conditions (a_2), (b_2), and (c_1) are satisfied. By Theorem 2.1 and Corollary 2.6, we know that (3.1) and (3.2) have positive ω-periodic solutions.

As another example, consider the semi-discrete Michaelis-Menton model

$$y'(t) = y(t)r([t]) \left(1 - \sum_{i=0}^{k} \frac{a_i([t])y([t-i])}{1 + c_i([t])y([t-i])} \right), \quad t \in \mathbb{R}, \tag{3.4}$$

and its associated difference equation

$$y_{n+1} = y_n \exp\left(r(n) \left(1 - \sum_{i=0}^{k} \frac{a_i(n)y_{n-i}}{1 + c_i(n)y_{n-i}} \right) \right), \quad n \in \mathbb{Z}. \tag{3.5}$$

In (3.4) and (3.5), r, a_i, and c_i belong to $C(\mathbb{R},(0,\infty))$, $r(t+\omega) = r(t)$, $a_i(t+\omega) = a_i(t)$ and $c_i(t+\omega) = c_i(t)$ for $i = 0,1,\dots,k$ and $t \in \mathbb{R}$, and $\sum_{i=0}^{k} a_i(t)/c_i(t) > 1$. Letting

$$f(t,u_0,u_1,\dots,u_k) = r(t) \left(1 - \sum_{i=0}^{k} \frac{a_i(t)u_i}{1 + c_i(t)u_i} \right), \tag{3.6}$$

then

$$f(t,e^{x_0},e^{x_1},\dots,e^{x_k}) = r(t) \left(1 - \sum_{i=0}^{k} \frac{a_i(t)e^{x_i}}{1 + c_i(t)e^{x_i}} \right). \tag{3.7}$$

Since

$$\lim_{x_0,\dots,x_k\to+\infty}\ \min_{0\le t\le\omega}\sum_{i=0}^{k}\frac{a_i(t)e^{x_i}}{1+c_i(t)e^{x_i}}>1,$$
$$\lim_{x_0,\dots,x_k\to-\infty}\ \max_{0\le t\le\omega}\sum_{i=0}^{k}\frac{a_i(t)e^{x_i}}{1+c_i(t)e^{x_i}}=0, \tag{3.8}$$

we can choose $M = \max_{0\le t\le\omega} r(t)$ and some positive number D such that conditions (a_2), (b_2), and (c_1) are satisfied. By Theorem 2.1 and Corollary 2.6, (3.4), and (3.5) have positive ω-periodic solution.

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