ON THE GROWTH RATE OF GENERALIZED FIBONACCI NUMBERS

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Let $\alpha(t)$ be the limiting ratio of the generalized Fibonacci numbers produced by summing along lines of slope t through the natural arrayal of Pascal's triangle. We prove that $\alpha(t)^{\sqrt{3}+t}$ is an even function.

1. Overview

Pascal's triangle may be arranged in the Euclidean plane by associating the binomial coefficient $\binom{i}{j}$ with the point

$$\left(j-\frac{1}{2}i,-\frac{\sqrt{3}}{2}i\right)\in\mathbb{R}^2 \tag{1.1}$$

for all nonnegative integers i, j such that $j\le i$, as illustrated in Figure 1.1. The points in $\mathbb{R}^2$ associated with $\binom{i}{j}$, $\binom{i+1}{j}$, and $\binom{i+1}{j+1}$ form a unit equilateral triangle. This arrayal is called the *natural* arrayal of Pascal's triangle in $\mathbb{R}^2$.

For all $t\in\mathbb{R}:-\sqrt{3}<t<\sqrt{3}$ and nonnegative integers k, define $\mathcal{L}_k(t)$ to be the sum of all binomial coefficients associated with points in $\mathbb{R}^2$ which are on the line of slope t through the point in $\mathbb{R}^2$ associated with $\binom{k}{0}$. It is well known that $\{\mathcal{L}_k(\sqrt{3}/3)\}_{k=0}^{\infty}$ is the Fibonacci sequence $F_0,F_1,F_2,\ldots$, and $\{\mathcal{L}_k(-\sqrt{3}/3)\}_{k=0}^{\infty}$ is the sequence of every other Fibonacci number $F_0,F_2,F_4,\ldots$, as illustrated in Figure 1.1; for a fixed t, the sequence $\{\mathcal{L}_k(t)\}_{k=0}^{\infty}$ is called the *generalized Fibonacci sequence* induced by the slope t. Generalized Fibonacci numbers arise in many ways; for example, for any integers $a,b:1\le b\le a$, the number of ways to distribute a identical objects to any number of distinct recipients such that each recipient receives at least b objects is

$$\sum_{l=1}^{\infty}\binom{l-1+a-l\cdot b}{l-1}=\mathcal{L}_{a-b}\left(\frac{b-1}{b+1}\sqrt{3}\right). \tag{1.2}$$

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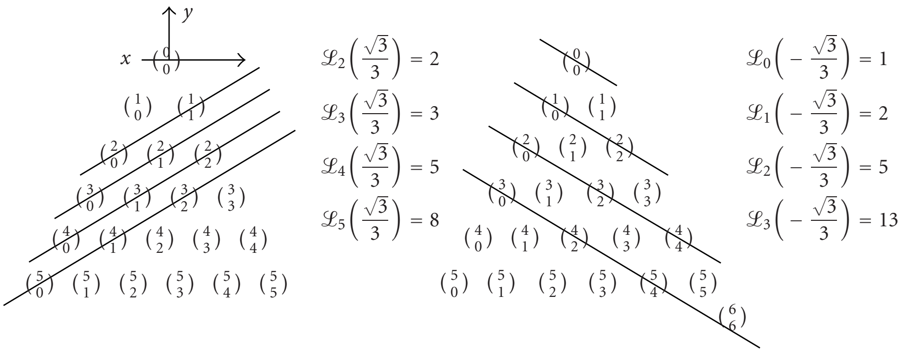

Figure 1.1. The natural arrayal of Pascal's triangle and Fibonacci numbers as line sums.

For all $t \in \mathbb{R} : -\sqrt{3} < t < \sqrt{3}$, we define $\alpha(t)$ to be the limiting ratio of the generalized Fibonacci sequence induced by the slope t; that is, $\alpha(t) := \lim_{k\to\infty} \mathscr{L}_{k+1}(t)/\mathscr{L}_k(t)$. The following is our main result.

Theorem 1.1. *For all $t \in \mathbb{R} : -\sqrt{3} < t < \sqrt{3}$, it holds that $\alpha(t)^{\sqrt{3}+t} = \alpha(-t)^{\sqrt{3}-t}$.*

(Theorem 1.1 is easily and directly verified when $t = \pm\sqrt{3}/3$, since the rate of growth of the sequence of every other Fibonacci number is the square of the rate of growth of the Fibonacci sequence.)

Generalized Fibonacci numbers arising as line sums through Pascal's triangle were introduced by Dickinson [2], Harris and Styles [4], and Hochster [6], and have been presented extensively in the literature (see [1, 5, 7]). The classical setting has been the left-justified arrayal of Pascal's triangle, which we define in Section 2. In the setting of the left-justified arrayal, Harris and Styles (and, effectively, Dickinson) show that generalized Fibonacci numbers satisfy the difference equation (2.5) in Section 2, and thus have rate of growth as given in (2.6) of Section 2. Ferguson [3] investigated the roots of the polynomial in (2.6) when q is an integer.

Our contribution is to investigate this rate of growth as a function of the generating slope, to transfer the setting to the natural arrayal of Pascal's triangle, and, in Section 3, to prove Theorem 1.1. In Section 2, we review classical facts and correlate them to the natural arrayal of Pascal's triangle.

2. The Left-Justified Arrayal

It is sometimes easier to consider the *left-justified* arrayal of Pascal's triangle in $\mathbb{R}^2$, in which the binomial coefficient $\binom{i}{j}$ is associated with the point $(j, -i) \in \mathbb{R}^2$ for all nonnegative integers $i, j : j \le i$, as illustrated in Figure 2.1.

Consider any $q = n/d > -1$ such that n and d are relatively prime integers and d is positive. For all nonnegative integers k, define $L_k(q)$ to be the sum of all binomial coefficients associated with points in $\mathbb{R}^2$ on the line $y = qx - (1/d)k$ [this choice of y-intercept is such that every binomial coefficient $\binom{i}{j}$ is included in such a sum for some k]. Now, define $\beta(q) := \lim_{k\to\infty} L_{k+1}(q)/L_k(q)$ and also define $\gamma(q) := \lim_{k\to\infty} L_{(k+1)d}(q)/L_{kd}(q)$; we

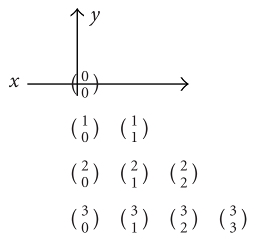

Figure 2.1. The left-justified arrayal of Pascal's triangle.

see from taking the limit of $L_{(k+1)d}(q)/L_{kd}(q) = \prod_{l=0}^{d-1} L_{kd+l+1}(q)/L_{kd+l}(q)$ as $k \to \infty$ that

$$\gamma(q) = [\beta(q)]^d. \tag{2.1}$$

Lines of slope $q = n/d$ in our left-justified arrayal correspond to lines of slope

$$\frac{(\sqrt{3}/2)n}{(1/2)n + d} = \frac{\sqrt{3}q}{q+2} \tag{2.2}$$

in the natural arrayal, and $\binom{k}{0}$ is a summand of $L_{kd}(q)$ and thus, for all nonnegative integers k, $L_{kd}(q) = \mathcal{L}_k(\sqrt{3}q/(q+2))$. Hence,

$$\gamma(q) = \alpha\left(\frac{\sqrt{3}q}{q+2}\right). \tag{2.3}$$

Defining $t := \sqrt{3}q/(q+2)$ (note that $-\sqrt{3} < t < \sqrt{3}$), we write q as a function of t, obtaining, by (2.3),

$$\alpha(t) = \gamma\left(\frac{2t}{\sqrt{3} - t}\right). \tag{2.4}$$

For the moment, suppose that q is positive. Using the identity $\binom{i}{j} + \binom{i}{j+1} = \binom{i+1}{j+1}$, there is a correspondence (see Endnote(I)) between the binomial coefficients summed in $L_k(q)$, those summed in $L_{n+k}(q)$, and those summed in $L_{n+d+k}(q)$ yielding the linear difference equation

$$L_{n+d+k}(q) - L_{n+k}(q) - L_k(q) = 0 \quad \text{for } k = 0, 1, 2, \ldots. \tag{2.5}$$

It is not hard to verify that the associated auxiliary polynomial $x^{n+d} - x^n - 1$ has distinct roots, say $\lambda_1, \lambda_2, \ldots, \lambda_{n+d}$, and the initial conditions ensure nonzero constants $c_1, c_2, \ldots, c_{n+d} \in \mathbb{R}$ in the expansion $L_k(q) = \sum_{l=1}^{n+d} c_l \lambda_l^k$, $k = 0, 1, 2, \ldots$ (these constants c_l are given explicitly in [2]). Among these roots, $\lambda_1, \lambda_2, \ldots, \lambda_{n+d}$, there is a unique positive root, and this root is also the root of maximum modulus (see Endnote(II)). Thus, $\beta(q)$ is the unique positive root of $x^{n+d} - x^n - 1$ and, substituting (2.1) into this,

$$\gamma(q) \text{ is the unique positive root of } x^{q+1} - x^q - 1. \tag{2.6}$$

If, instead, $-1 < q \leq 0$ (i.e., $d > -n \geq 0$), then similar analysis yields the linear difference equation

$$L_{d+k}(q) - L_{-n+k}(q) - L_k(q) = 0 \quad \text{for } k = 0, 1, 2, \ldots, \tag{2.7}$$

and $\beta(q)$ is the unique positive root of the auxiliary polynomial $x^d - x^{-n} - 1$. Multiplying the equation $x^d - x^{-n} - 1 = 0$ by x^n yields that $\beta(q)$ is the unique positive root of the (now nonpolynomial) $x^{n+d} - x^n - 1$ and thus, by (2.1), statement (2.6) holds for nonpositive q's as well.

3. Proof of Theorem 1.1

Let $\sqrt{3}\mathbb{Q}_{(-1,1)}$ denote the set $\{t \in (-\sqrt{3}, \sqrt{3}) : t/\sqrt{3} \text{ is rational}\}$, where $(-\sqrt{3}, \sqrt{3})$ is the open interval of real numbers from $-\sqrt{3}$ to $\sqrt{3}$.

Proposition 3.1. *The function $\alpha(t)$ is continuous on the set $\sqrt{3}\mathbb{Q}_{(-1,1)}$. The function $\alpha(t)$ is identically* 1 *on the set $(-\sqrt{3}, \sqrt{3}) \setminus \sqrt{3}\mathbb{Q}_{(-1,1)}$.*

Proof. As noted before, the slope $t \in (-\sqrt{3}, \sqrt{3})$ in the natural arrayal corresponds to the slope $q = 2t/(\sqrt{3} - t) > -1$ in the left-justified arrayal, and such q is rational if and only if $t/\sqrt{3}$ is rational. If such q is not rational, $\{\mathcal{L}_k(t)\}_{k=0}^{\infty}$ is just the sequence $\{\binom{k}{0}\}_{k=0}^{\infty}$, and $\alpha(t) = 1$.

On the other hand, for $q \in \mathbb{Q} : q > -1$, (2.6) may be solved to yield that $q = -\ln(\gamma(q) - 1)/\ln(\gamma(q))$; the continuity of the inverse function of γ implies the continuity of γ. By (2.4) and for t such that $q = 2t/(\sqrt{3} - t)$ is rational, α is the composition of γ and another continuous function, thus α is continuous on $\sqrt{3}\mathbb{Q}_{(-1,1)}$. □

Lemma 3.2. *For any rational number $q > -1$, $\gamma(q)^{q+1} = \gamma(-q/(q+1))$.*

Proof. Suppose $q = n/d > -1$ such that n and d are relatively prime integers and d is positive. Note that $-n/(n+d) > -1$ and observe that $\beta(n/d)$ and $\beta(-n/(n+d))$ are each the unique positive root of $x^{n+d} - x^n - 1$, which is also the unique positive root of $x^d - x^{-n} - 1$ and so, in particular,

$$\beta\left(\frac{n}{d}\right) = \beta\left(\frac{-n}{n+d}\right). \tag{3.1}$$

Thus, by (2.1) and (3.1), we have

$$\gamma\left(\frac{n}{d}\right)^{n+d} = \beta\left(\frac{n}{d}\right)^{d(n+d)} = \beta\left(\frac{-n}{n+d}\right)^{d(n+d)} = \gamma\left(\frac{-n}{n+d}\right)^{d}. \tag{3.2}$$

Taking the dth root of (3.2) and simplifying yields the desired result. □

We next prove our main result, Theorem 1.1, which states that for all $t \in \mathbb{R}$ such that $-\sqrt{3} < t < \sqrt{3}$ we have $\alpha(t)^{\sqrt{3}+t} = \alpha(-t)^{\sqrt{3}-t}$.

Proof of Theorem 1.1. For all $t \notin \sqrt{3}\mathbb{Q}_{(-1,1)}$, we have, by Proposition 3.1, $\alpha(t) = 1$, in which case the result is trivial. For all $t \in \sqrt{3}\mathbb{Q}_{(-1,1)}$, we have that $2t/(\sqrt{3}-t)$ is rational and greater than -1. Thus,

$$\begin{aligned}
\alpha(t)^{\sqrt{3}+t} &= \gamma\left(\frac{2t}{\sqrt{3}-t}\right)^{\sqrt{3}+t} \quad \text{(by (2.4))} \\
&= \left(\left[\gamma\left(\frac{2t}{\sqrt{3}-t}\right)\right]^{2t/(\sqrt{3}-t)+1}\right)^{\sqrt{3}-t} \\
&= \gamma\left(-\frac{2t/(\sqrt{3}-t)}{2t/(\sqrt{3}-t)+1}\right)^{\sqrt{3}-t} \quad \text{(by Lemma 3.2)} \\
&= \gamma\left(\frac{2(-t)}{\sqrt{3}-(-t)}\right)^{\sqrt{3}-t} \\
&= \alpha(-t)^{\sqrt{3}-t} \quad \text{(by (2.4))}.
\end{aligned} \tag{3.3}$$

□

Endnotes. (I) Since consecutive line sums in the sequence of line sums differ in y-intercept by $1/d$, the line sum including $\binom{i}{j+1}$ and the line sum including $\binom{i+1}{j+1}$, which have y-intercepts that differ by 1, must be d line sums apart. Symmetric reasoning dictates that the line sum including $\binom{i}{j}$ and the line sum including $\binom{i}{j+1}$ are n line sums apart in the sequence of line sums.

(II) To sketch some details, the nonzero roots of $(d/dx)(x^{n+d}-x^n-1)$ are the dth roots of $n/(n+d)$, exactly one of which is positive and none of which are roots of $x^{n+d}-x^n-1$, hence the roots of $x^{n+d}-x^n-1$ are distinct and (considering a few basic features of this polynomial on the positive real line) exactly one is positive, call it λ. If $\tilde{\lambda}$ is any other root of $x^{n+d}-x^n-1$ besides λ, then $1 = |\tilde{\lambda}^{n+d}-\tilde{\lambda}^n| \geq |\tilde{\lambda}|^{n+d}-|\tilde{\lambda}|^n$ and, because equality does not hold in this triangle inequality, we have $|\tilde{\lambda}|^{n+d}-|\tilde{\lambda}|^n-1<0$, implying $|\tilde{\lambda}|<\lambda$.

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