SMOOTH AND DISCRETE SYSTEMS—ALGEBRAIC, ANALYTIC, AND GEOMETRICAL REPRESENTATIONS

FRANTIŠEK NEUMAN

What is a differential equation? Certain objects may have different, sometimes equivalent representations. By using algebraic and geometrical methods as well as discrete relations, different representations of objects mainly given as analytic relations, differential equations can be considered. Some representations may be suitable when given data are not sufficiently smooth, or their derivatives are difficult to obtain in a sufficient accuracy; other ones might be better for expressing conditions on qualitative behaviour of their solution spaces. Here, an overview of old and recent results and mainly new approaches to problems concerning smooth and discrete representations based on analytic, algebraic, and geometrical tools is presented.

1. Motivation

When considering certain objects, we may represent them in different, often equivalent ways. For example, graphs can be viewed as collections of vertices (points) and edges (arcs), or as matrices of incidence expressing in their entries (a_{ij}) the number of (oriented) edges going from one vertex (i) to the other one (j).

Another example of different representations are matrices: we may look at them as centroaffine mappings of m-dimensional vector space to n-dimensional one, or as $n \times m$ entries, or coefficients of the above mappings in particular coordinate systems of the vector spaces, placed at lattice points of rectangles.

Still there is another example. Some differential equations can be considered in the form

$$y' = f(x, y), \tag{1.1}$$

with the initial condition $y(x_0) = y_0$. For continuous f satisfying Lipschitz condition, we get the unique solution of (1.1). The solution space of (1.1) is a set of differentiable functions satisfying (1.1) and depending on one constant, the initial value y_0.

Under weaker conditions, the Carathéodory theory considers the relation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, dt \qquad (1.2)$$

instead of (1.1). Its solution space coincides with that of (1.1) if the above, stronger conditions are satisfied, see, for example, [6, Chapter IV, paragraph 6, page 198].

However, no derivatives occur in relation (1.2) and still it is common to speak about it as a differential equation. The reason is perhaps the fact that (1.2) has the same (or a wider) solution space as (1.1). This leads to the idea of considering *the solution space as a representative of the corresponding equation*.

The following problems occur. How many objects, relations, and equations correspond to a given set of solutions? If they are several ones, might it be that some of them are better than others, for example, because of simple numerical verification of their validity? What is a differential equation? How can we use formulas involving functions with derivatives when our functions are not differentiable, or they have no derivatives of sufficiently high order? Say, because the given experimental (discrete) data do not admit evaluating expressions needed in a formula. What is the connection between differential and difference equation? On this subject see the monograph [2] which includes very interesting material.

Still there is one more example of this nature. Let $\mathscr{D}$ denote the set of all real differentiable functions defined on the reals, $f : \mathbb{R} \to \mathbb{R}$. Consider the decomposition of $\mathscr{D}$ into classes of functions such that two elements f_1 and f_2 belong to the same class if and only if they differ by a constant, that is, $f_1(x) - f_2(x) = \text{const}$ for all $x \in \mathbb{R}$.

Evidently, we have a criterion for two functions f_1, f_2 belonging to the same class, namely, their first derivatives are identical, $f_1' = f_2'$. However, if we consider the set of *all* real continuous functions defined on $\mathbb{R}$, then this criterion is not applicable because some functions need not have derivatives, and more general situations can be considered when functions have no smooth properties at all. Here is a simple answer: *two functions f_1 and f_2 are from the same class of the above decomposition $\mathscr{D}$ if and only if their difference has the first derivative which is identically zero:*

$$\left(f_1(x) - f_2(x) \right)' \equiv 0 \quad \text{on } \mathbb{R}. \qquad (1.3)$$

These considerations lead to the following question. How can we deal with conditions or formulas in which derivatives occur, but the entrance data are not sufficiently smooth, or even do not satisfy any regularity condition?

We will show how algebraic means can help in some situations and enable us to formulate conditions in a discrete form, more adequate for experimental data and often even suitable for quick verification on computers.

2. Ordinary differential equations

2.1. Analytic approach—smooth representations. Having a set of certain functions depending on one or more constants, we may think about its representation: an expression

invariantly attached to this set, a relation, all solutions forming exactly the given set. Differential equations occur often in such cases; might it be because (if it is possible, i.e., if required derivatives exist) it is easy.

Examples 2.1. (i) Solution space: $y(x) = \{c \cdot x; \, x \in \mathbb{R}, \, c \in \mathbb{R} \text{ const}\}$.

A procedure of obtaining an invariant for the whole set is *an* elimination of the constant c, for example, by differentiation:

$$\frac{d}{dx} : y'(x) = c \Longrightarrow y(x) = y'(x) \cdot x \quad \text{or} \quad y' = \frac{y}{x}, \tag{2.1}$$

a differential equation.

(ii) Solution space: $\{y(x) = 1/(x - c)\}$:

$$y' = \frac{-1}{(x - c)^2} \Longrightarrow y' = -y^2. \tag{2.2}$$

(iii) $y(x) = \{c_1 \sin x + c_2 \cos x\} \Rightarrow y'' + y = 0$.

(iv) Linear differential equations of the nth order. Solution space:

$$\{y(x) = c_1 y_1(x) + \cdots + c_n y_n(x); \, x \in I \subseteq \mathbb{R}\}, \tag{2.3}$$

with linearly independent $y_i \in C^n(I)$, with the nonvanishing Wronskian

$$\det \begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \neq 0. \tag{2.4}$$

Since

$$\det \begin{pmatrix} y_1 & \cdots & y_n & y \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} & y^{(n-1)} \\ y_1^{(n)} & \cdots & y_n^{(n)} & y^{(n)} \end{pmatrix} = 0, \tag{2.5}$$

the last relation is a nonsingular nth-order linear differential equation with continuous coefficients:

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_0(x) y = 0 \quad \text{on } I. \tag{2.6}$$

We have seen that *differential equations are representations of solution spaces obtained after elimination of parameters (constants) by means of differentiation.*

What can we do when it is impossible because required derivatives do not exist, or Wronskian is vanishing somewhere, or the definition set of the solution space is discrete? Are there other ways of elimination of constants?

2.2. Algebraic approach—discrete representations. The linear independence is an algebraic property not requiring any kind of smoothness. n functions $f_1, \ldots, f_n; f_i : M \to \mathbb{R}$ (or $\mathbb{C}$) are defined as linearly independent (on M) if (and only if) the relation

$$c_1 f_1 + \cdots + c_n f_n = 0 \quad \text{on } M \text{ (i.e., } \equiv 0) \tag{2.7}$$

is satisfied just for $c_1 = \cdots = c_n = 0$.

Examples 2.2. (i)

$$f_1(x) = \begin{cases} 0 & \text{for } -1 \leqq x < 0, \\ x & \text{for } 0 \leqq x \leqq 1, \end{cases} \qquad f_2(x) = \begin{cases} -x & \text{for } -1 \leqq x < 0, \\ 0 & \text{for } 0 \leqq x \leqq 1. \end{cases} \tag{2.8}$$

Functions f_1, f_2 are linearly independent of the interval $[-1,1]$:

$$0 = c_1 f_1(-1) + c_2 f_2(-1) = c_2, \qquad 0 = c_1 f_1(1) + c_2 f_2(1) = c_1. \tag{2.9}$$

$\{c_1 f_1 + c_2 f_2\}$ is the 2-dimensional solution space. Where is a differential equation?

(ii) $y_1, \ldots, y_n \in C^{n-1}$, but $y_1, \ldots, y_n \notin C^n$ and still nonvanishing Wronskian; they are linearly independent. Where is a differential equation?

(iii) $y_1, y_2 \in C^1$, $y_1, y_2 \notin C^2$, Wronskian identically, are still linearly independent, like, for example,

$$f_1(x) = \begin{cases} 0 & \text{for } -1 \leqq x < 0, \\ x^2 & \text{for } 0 \leqq x \leqq 1, \end{cases} \qquad f_2(x) = \begin{cases} x^2 & \text{for } -1 \leqq x < 0, \\ 0 & \text{for } 0 \leqq x \leqq 1. \end{cases} \tag{2.10}$$

Functions f_1, f_2 are linearly independent of the interval $[-1,1]$. Where is a differential equation?

Fortunately, we have Curtiss' result [4].

PROPOSITION 2.3. *n functions $y_1, \ldots, y_n : M \to \mathbb{R}$, $M \subset \mathbb{R}$, are linearly dependent (on M) if and only if*

$$\det \begin{pmatrix} y_1(x_1) & \cdots & y_n(x_1) \\ \vdots & \vdots & \vdots \\ y_1(x_n) & \cdots & y_n(x_n) \end{pmatrix} = 0 \quad \forall (x_1, \ldots, x_n) \in M^n. \tag{2.11}$$

Proof. The proof was given in [4], see also [1, page 229]. □

With respect to this result, we have also another way to characterize the n-dimensional space (2.3).

Proposition 2.4. *The condition*

$$\det \begin{pmatrix} y_1(x_1) & \cdots & y_n(x_1) & y(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ y_1(x_n) & \cdots & y_n(x_n) & y(x_n) \\ y_1(x) & \cdots & y_n(x) & y(x) \end{pmatrix} = 0 \quad \forall (x_1,\ldots,x_n,x) \in I^{n+1} \tag{2.12}$$

is satisfied just for functions in (2.3).

It means that the relation (2.12) can be considered as a representation of the solution space (2.3), suitable also in cases when the differential equation (2.6) is not applicable, neither derivatives nor integrals occur in (2.12).

Proof. The proof is a direct consequence of Proposition 2.3. □

Example 2.5. (i) For $y_1 : M \to \mathbb{R}$, $y_1(x_1) \neq 0$, $\{c_1 y_1\}$ is a 1-dimensional vector space.
    Due to (2.12), we have

$$\det \begin{pmatrix} y_1(x_1) & y(x_1) \\ y_1(x) & y(x) \end{pmatrix} = 0, \tag{2.13}$$

that gives $y_1(x_1)y(x) - y(x_1)y_1(x) = 0$, or

$$y(x) = \frac{y(x_1)}{y_1(x_1)} \cdot y_1(x) = c_1 y_1(x), \tag{2.14}$$

where $y(x_1)/y_1(x_1) =: c_1 = \text{const}$.

2.3. Geometrical approach—zeros of solutions. The essence of this approach is based on another *representation* of a linear differential equation by its n-tuple of linearly independent solutions $\mathbf{y}(x) = (y_1(x),\ldots,y_n(x))^T$ considered as a *curve* in n-dimensional Euclidean space $\mathbb{E}_n$, with the independent variable x as the parameter and the column vector $y_1(x),\ldots,y_n(x)$ forming the coordinates of the curve (M^T denotes the transpose of the matrix M). We note that this kind of considerations was started by Borůvka [3] for the second-order linear differential equations.
    Define the n-tuple $\mathbf{v} = (v_1,\ldots,v_n)^T$ in the Euclidean space $\mathbb{E}_n$ by

$$\mathbf{v}(x) := \frac{\mathbf{y}(x)}{\|\mathbf{y}(x)\|}, \tag{2.15}$$

where $\|\cdot\|$ denotes the Euclidean norm. It was shown (see [11]) that $\mathbf{v} \in C^n(I)$, $\mathbf{v} : I \to \mathbb{E}_n$, and the Wronskian of $\mathbf{v}$, $W[\mathbf{v}] := \det(\mathbf{v},\mathbf{v}',\ldots,\mathbf{v}^{(n-1)})$, is nonvanishing on I. Of course, $\|\mathbf{v}(x)\| = 1$, that is, $\mathbf{v}(x) \in \mathbb{S}_{n-1}$, where $\mathbb{S}_{n-1}$ denotes the unit sphere in $\mathbb{E}_n$. Evidently, we can consider the differential equation which has this $\mathbf{v}$ as its n-tuple of linearly independent solutions.
    The idea leading to geometrical description of distribution of zeros is based on two readings of the following relation:

$$\mathbf{c}^T \cdot \mathbf{y}(x_0) = c_1 y_1(x_0) + \cdots + c_n y_n(x_0) = 0. \tag{2.16}$$

The *first* meaning is a solution $\mathbf{c}^T \cdot \mathbf{y}(x)$ has a zero at x_0. The *second*, equivalent reading gives the hyperplane

$$c_1 \eta_1 + \cdots + c_n \eta_n = 0 \tag{2.17}$$

intersects the curve $\mathbf{y}(x)$ at a point $\mathbf{y}(x_0)$ of parameter x_0. This is the reasoning for the following assertion.

PROPOSITION 2.6. *Let coordinates of $\mathbf{y}$ be linearly independent solutions of (2.6). If $\mathbf{y}$ is considered as a curve in n-dimensional Euclidean space and $\mathbf{v}$ is the central projection of $\mathbf{y}$ onto the unit sphere (without a change of parameterization), then parameters of intersections of $\mathbf{v}$ with great circles correspond to zeros of solutions of (2.6); multiplicities of zeros occur as orders of contacts plus 1.*

Proof. The proof in detail and further results of this nature can be found in [11].   □

By using this method, we can see, simply by drawing a curve $\mathbf{v}$ on a sphere, what is possible and what is impossible in distribution of zeros without lengthy and sometimes tiresome ϵ, δ calculations. Only $\mathbf{v}$ must be sufficiently smooth, that is, of the class C^n for the nth-order equations and its Wronskian $\det(\mathbf{v}, \mathbf{v}', \dots, \mathbf{v}^{(n-1)})$ has to be nonvanishing at each point. As examples we mention the Sturm separation theorem for the second-order equations, equations of the third order with all oscillatory solutions (Sansone's result), or an equation of the third order with just 1-dimensional subset of oscillatory solutions that cannot occur for equations with constant coefficients. Compare oscillation results in [11] and those described in Swanson's monograph [14].

Remark 2.7. Other applications of this geometrical representation can be found in [11]. There, one can find also constructions of global canonical forms, structure of transformations, together with results obtained by Cartan's moving-frame-of-reference method.

Remark 2.8. The coordinates of the curve $\mathbf{y}$ (or $\mathbf{v}$) need not be of the class C^n. A lot of constructions can be done when only smoothness of the class C^{n-1} is supposed, or even C^0 is sometimes sufficient.

3. Partial differential equations—decomposition of functions

Throughout the history of mathematics, there are attempts to decompose objects of higher orders into objects of lower orders and simpler structures. Examples can be found in factorization of polynomials in different fields and in decomposition of operators of different kind, including differential operators.

There have occurred questions regarding representation of functions of several variables in terms of finite sums of products of factor functions in less number of variables. One of these questions is closely related to the 13th problem of Hilbert [8] and concerns the solvability of algebraic equations.

For functions of several variables, a problem of this kind has occurred when d'Alembert [5] considered scalar functions h of two variables that can be expressed in

the form

$$h(x, y) = f(x) \cdot g(y). \tag{3.1}$$

3.1. Analytic approach—d'Alembert equation. For sufficiently smooth functions h of the form (3.1), d'Alembert [5] proved that h has to satisfy the following partial differential equation:

$$\frac{\partial^2 \log h}{\partial x \partial y} = 0, \tag{3.2}$$

known today as *d'Alembert equation.*

For the case when more terms on the right-hand side of (3.1) are admitted, that is, if

$$h(x, y) = \sum_{k=1}^{n} f_k(x) \cdot g_k(y), \tag{3.3}$$

Stéphanos (see [13]) presented the following necessary condition in the section *Arithmetics and Algebra* at the Third International Congress of Mathematicians in Heidelberg.

Functions (3.3) form the space of solutions of the partial differential equation ($h_x = \partial h/\partial x$):

$$\det D_n(h) := \det \begin{pmatrix} h & h_y & \cdots & h_{y^n} \\ h_x & h_{xy} & \cdots & h_{xy^n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{x^n} & h_{x^n y} & \cdots & h_{x^n y^n} \end{pmatrix} = 0. \tag{3.4}$$

A necessary and sufficient condition reads as follows.

PROPOSITION 3.1. *A function $h : I \times J \in \mathbb{R}$, having continuous derivatives $h_{x^i y^j}$ for $i, j \leq n$, can be written in the form (3.3) on $I \times J$ with $f_k \in C^n(I)$, $g_k \in C^n(J)$, $k = 1, \ldots, n$, and*

$$\det \left(f_k^{(j)}(x) \right) \neq 0 \quad \text{for } x \in I, \qquad \det \left(g_k^{(j)}(y) \right) \neq 0 \quad \text{for } y \in J \tag{3.5}$$

if and only if

$$\det D_n(h) \equiv 0, \quad \det D_{n-1}(h) \text{ is nonvanishing on } I \times J. \tag{3.6}$$

Moreover, if (3.6) is satisfied, then there exist $f_k \in C^n(I)$ and $g_k \in C^n(J)$, $k = 1, \ldots, n$, such that (3.3) and (3.5) hold and all decompositions of h of the form

$$h(x, y) = \sum_{k=1}^{n} \bar{f}_k(x) \bar{g}_k(y) \tag{3.7}$$

are exactly those for which

$$(\bar{f}_1,\ldots,\bar{f}_n) = (f_1,\ldots,f_n) \cdot C^T, \qquad (\bar{g}_1,\ldots,\bar{g}_n) = (g_1,\ldots,g_n) \cdot C^{-1}, \qquad (3.8)$$

C being an arbitrary regular constant matrix.

Proof. The proof was given in [10] (the result announced in [9]). □

Remark 3.2. We note that instead of ordinary differential equations for the case when a finite number of constants has to be eliminated, we have a partial differential equation for elimination of functions f_k, g_k.

3.2. Algebraic approach—discrete conditions. However, there is again a problem concerning sufficient smoothness. Determinants of the type (3.4) are really not very suitable for experimental data. Fortunately, we have in [9] also the sufficient and necessary condition for the case when h is not sufficiently smooth and even discontinuous.

PROPOSITION 3.3. *For arbitrary sets X and Y (intervals, discrete ones, etc.), a function $h : X \times Y \to \mathbb{R}$ (or $\mathbb{C}$) is of the form (3.3) with linearly independent sets $\{f_k\}_{k=1}^n$ and $\{g_k\}_{k=1}^n$ if and only if the maximal rank of the matrices*

$$\begin{pmatrix} h(x_1,y_1) & h(x_1,y_2) & \cdots & h(x_1,y_{n+1}) \\ h(x_2,y_1) & h(x_2,y_2) & \cdots & h(x_2,y_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ h(x_{n+1},y_1) & h(x_{n+1},y_2) & \cdots & h(x_{n+1},y_{n+1}) \end{pmatrix} \qquad (3.9)$$

is $\mathbf{n}$ for all $x_i \in X$ and $y_j \in Y$.

Proof. The proof is given in [10]; see also [12] for continuation in this research. □

Problem 3.4. Falmagne [7] asked about conditions on a function $h : X \times Y \to \mathbb{R}$ which guarantee the representation

$$h(x,y) = \varphi\left(\sum_{k=1}^n f_k(x) \cdot g_k(y)\right) \qquad (3.10)$$

for all $x \in X$ and $y \in Y$, where X and Y are arbitrary sets and an unknown function $\varphi : \mathbb{R} \to \mathbb{R}$ is strictly monotonic. The answer for $\varphi = \mathrm{id}$ was given in Propositions 3.1 and 3.3.

4. Final remarks

We have seen that there might be several representatives of a certain object, in some sense, more or less equivalent. We may think that our object under consideration is something like an *abstract notion*, common to all representatives, and that we deal with particular representations of this abstract object.
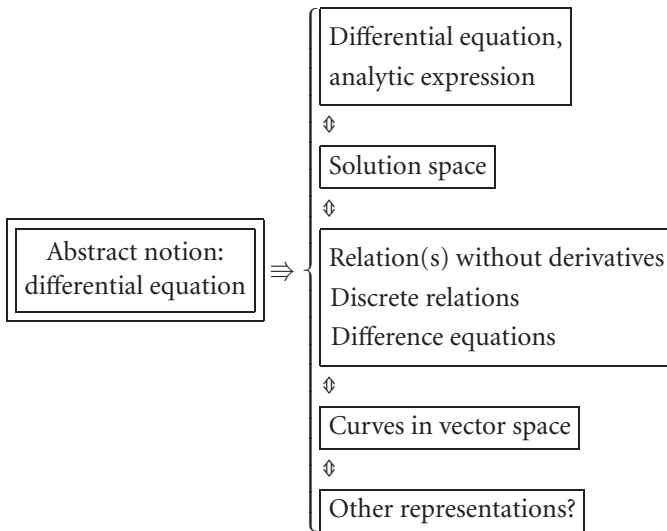
Figure 4.1

For example, linear ordinary linear differential equations can be viewed through Figure 4.1.

Explanation. On the left-hand side, there is an abstract notion, on the right-hand side, its explicit *representations*. The step from an analytic form of a differential equation to its solution space is called *solving* of equation; the backward step is a *construction*, performed by means of *derivatives*. However, an elimination of parameters and arbitrary constants (or functions) from an explicit expression of a solution space may be achieved by using appropriate *algebraic means*. Then we come to relations without derivatives, especially useful when given data are not sufficiently smooth. Qualitative behaviour of solution space and hints for useful constructions can be suggested if a well-visible *geometrical representation* of the studied object is at our disposal. Open problem always remains concerning further representations.

As demonstrated here on the case of linear ordinary differential equations and partial differential equations for decomposable functions, and mentioned also for other cases in different areas of mathematics, the choice of a good representation of a considered object plays an important role. In some sense, "all representations are equal, but some of them are more equal than others" (George Orwell, *Animal Farm* (paraphrased)), meaning that some representations are more suitable than others for expressing particular properties of studied objects.

Acknowledgment

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František Neuman: Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic

E-mail address: neuman@ipm.cz