ON THE SOLVABILITY OF INITIAL-VALUE PROBLEMS FOR NONLINEAR IMPLICIT DIFFERENCE EQUATIONS

PHAM KY ANH AND HA THI NGOC YEN

Received 18 February 2004

Our aim is twofold. First, we propose a natural definition of index for linear nonautonomous implicit difference equations, which is similar to that of linear differential-algebraic equations. Then we extend this index notion to a class of nonlinear implicit difference equations and prove some existence theorems for their initial-value problems.

1. Introduction

Implicit difference equations (IDEs) arise in various applications, such as the Leontief dynamic model of a multisector economy, the Leslie population growth model, and so forth. On the other hand, IDEs may be regarded as discrete analogues of differential-algebraic equations (DAEs) which have already attracted much attention of researchers.

Recently [1, 3], a notion of index 1 linear implicit difference equations (LIDEs) has been introduced and the solvability of initial-value problems (IVPs), as well as multipoint boundary-value problems (MBVPs) for index 1 LIDEs, has been studied. In this paper, we propose a natural definition of index for LIDEs so that it can be extended to a class of nonlinear IDEs. The paper is organized as follows. Section 2 is concerned with index 1 LIDEs and their reduction to ordinary difference equations. In Section 3, we study the index concept and the solvability of IVPs for nonlinear IDEs. The result of this paper can be considered as a discrete version of the corresponding result of [4].

2. Index 1 linear implicit difference equations

Let Q be an arbitrary projection onto a given subspace N of dimension $m-r$ $(1 \leqslant r \leqslant m-1)$ in $\mathbb{R}^m$. Further, let $\{v_i\}_1^r$ and $\{v_j\}_{r+1}^m$ be any bases of $\operatorname{Ker} Q$ and N, respectively. Denote by $V = (v_1, \ldots, v_m)$ a column matrix and denote $\tilde{Q} = \operatorname{diag}(O_r, I_{m-r})$, where O_r and I_{m-r} stand for $r \times r$ zero matrix and $(m-r) \times (m-r)$ identity matrix, respectively. Then V is nonsingular, $Q = V\tilde{Q}V^{-1}$, and this decomposition depends on the choice of the bases $\{v_i\}_1^m$, that is, on V.

Now, suppose N_α and N_β are two subspaces of the same dimension $m-r$ $(1 \leqslant r \leqslant m-1)$ in $\mathbb{R}^m$. Then any projections Q_α and Q_β onto N_α and N_β can be decomposed

Advances in Difference Equations 2004:3 (2004) 195–200
2000 Mathematics Subject Classification: 34A09, 39A10
URL: http://dx.doi.org/10.1155/S1687183904402015

as $Q_\alpha = V_\alpha \tilde{Q} V_\alpha^{-1}$ and $Q_\beta = V_\beta \tilde{Q} V_\beta^{-1}$, respectively. Define an operator connecting two subspaces N_α and N_β (connecting operator, for short) $Q_{\alpha\beta} := V_\alpha \tilde{Q} V_\beta^{-1}$. Clearly,

$$\begin{gathered} Q_{\alpha\beta} = Q_\alpha Q_{\alpha\beta} = Q_{\alpha\beta} Q_\beta = Q_\alpha V_\alpha V_\beta^{-1} = V_\alpha V_\beta^{-1} Q_\beta, \\ Q_{\alpha\beta} Q_{\beta\alpha} = Q_\alpha, \qquad Q_{\beta\alpha} Q_{\alpha\beta} = Q_\beta. \end{gathered} \tag{2.1}$$

We consider a system of LIDEs

$$A_n x_{n+1} + B_n x_n = q_n \quad (n \geqslant 0), \tag{2.2}$$

where $A_n, B_n \in \mathbb{R}^{m\times m}$, $q_n \in \mathbb{R}^m$ are given and $\operatorname{rank} A_n \equiv r$ $(1 \leqslant r \leqslant m-1)$ for all $n \geqslant 0$. Let Q_n be any projection onto $\operatorname{Ker} A_n, P_n = I - Q_n$ and consider decompositions $Q_n = V_n \tilde{Q} V_n^{-1}$ $(n \geqslant 0)$. For definiteness, we put $A_{-1} := A_0$, $Q_{-1} := Q_0$, $P_{-1} := P_0$, and $V_{-1} := V_0$. Thus, the connecting operators $Q_{n-1,n} := V_{n-1} \tilde{Q} V_n^{-1}$ are determined for all $n \geqslant 0$.

Recall that a linear DAE $A(t)x' + B(t)x = q(t)$, $t \in J := [t_0, T]$, where $A, B \in C(J, \mathbb{R}^{m\times m})$, $q \in \mathbb{R}^m$, is said to be of index 1 or transferable (see [4]) if there exists a smooth projection $Q \in C^1(J, \mathbb{R}^{m\times m})$ onto $\operatorname{Ker} A(t)$ such that the matrix $G(t) = A(t) + B(t)Q(t)$ is nonsingular for all $t \in J$. It is proved that the index 1 property (transferability) of linear DAEs does not depend on the choice of smooth projections and is equivalent to the condition $S(t) \cap \operatorname{Ker} A(t) = \{0\}$, where $S(t) := \{\xi \in \mathbb{R}^m : B(t)\xi \in \operatorname{Im} A(t)\}$.

A similar result can be established for LIDEs, namely, the following lemma.

Lemma 2.1. *The matrix $G_n := A_n + B_n Q_{n-1,n}$ is nonsingular if and only if*

$$S_n \cap \operatorname{Ker} A_{n-1} = \{0\}, \tag{2.3}$$

where, as in the DAE case, $S_n := \{\xi \in \mathbb{R}^m : B_n \xi \in \operatorname{Im} A_n\}$.

The proof of Lemma 2.1 repeats that of [3, Lemma 1] with some obvious changes, and uses the fact that condition (2.3) holds if and only if $V_n V_{n-1}^{-1} S_n \cap \operatorname{Ker} A_n = \{0\}$.

Since condition (2.3) does not depend on the representation of connecting operators, we get the following corollary.

Corollary 2.2. *The nonsingularity of G_n does not depend on the choice of connecting operator, that is, if $Q_{n-1,n} := V_{n-1} \tilde{Q} V_n^{-1}$ and $\bar{Q}_{n-1,n} := \bar{V}_{n-1} \tilde{Q} \bar{V}_n^{-1}$, then both matrices $G_n := A_n + B_n Q_{n-1,n}$ and $\bar{G}_n := A_n + B_n \bar{Q}_{n-1,n}$ are singular or nonsingular simultaneously.*

Corollary 2.2 confirms that it suffices to restrict our consideration to orthogonal projections onto $\operatorname{Ker} A_n$, as was done in [3]. However, in the mentioned paper, a singular-value decomposition (SVD) of A_n is employed for constructing an orthogonal projection Q_n onto $\operatorname{Ker} A_n$ and it seems not to be convenient for a further extension of the index notion to nonlinear cases. Corollary 2.2 also allows us to introduce the following notion of index 1 LIDEs, which is quite similar to that of index 1 (transferable) linear DAEs.

Definition 2.3. The LIDEs (2.2) are said to be of index 1 if, for all $n \geqslant 0$,

(i) $\operatorname{rank} A_n = r$;
(ii) $G_n := A_n + B_n Q_{n-1,n}$ *is nonsingular.*

The main difference between linear index 1 DAEs and linear index 1 IDEs is the fact that the pencil $\{A(t),B(t)\}$ in the continuous case is always of index 1 for all $t \in J$, while for $n \geqslant 1$, $\{A_n,B_n\}$ is not necessarily of index 1.

Now, we describe shortly the decomposition technique for index 1 LIDEs. Performing $P_nG_n^{-1}$ and $Q_nG_n^{-1}$ on both sides of (2.2), respectively, we get

$$P_nx_{n+1} + P_nG_n^{-1}B_nx_n = P_nG_n^{-1}q_n, \tag{2.4}$$

$$Q_nG_n^{-1}B_nx_n = Q_n^{-1}G_nq_n. \tag{2.5}$$

Further, denoting $u_n = P_{n-1}x_n$, $v_n = Q_{n-1}x_n$ $(n \geqslant 0)$ and observing that $P_nG_n^{-1}B_nQ_{n-1}x_n = P_nG_n^{-1}B_nQ_{n-1,n}Q_{n,n-1}x_n = P_nQ_{n,n-1}x_n = P_nQ_nQ_{n,n-1}x_n = 0$, we find $P_nG_n^{-1}B_nx_n = P_nG_n^{-1}B_nu_n$. Thus, (2.4) becomes an ordinary difference equation

$$u_{n+1} + P_nG_n^{-1}B_nu_n = P_nG_n^{-1}q_n. \tag{2.6}$$

Since $Q_nG_n^{-1}B_nQ_{n-1}x_n = Q_nG_n^{-1}B_nQ_{n-1,n}Q_{n,n-1}x_n = Q_{n,n-1}x_n = V_nV_{n-1}^{-1}Q_{n-1}x_n = V_nV_{n-1}^{-1}v_n$, (2.5) is reduced to

$$v_n = V_{n-1}V_n^{-1}\left(Q_nG_n^{-1}q_n - Q_nG_n^{-1}B_nu_n\right). \tag{2.7}$$

Finally,

$$x_n = u_n + v_n = \left(I - Q_{n-1,n}G_n^{-1}B_n\right)u_n + Q_{n-1,n}G_n^{-1}q_n. \tag{2.8}$$

Thus, if (2.2) is of index 1, then, for given $u_0 = P_{-1}x_0 = P_0x_0$, we can compute u_{n+1}, v_n, and x_n $(n \geqslant 0)$ by (2.6), (2.7), and (2.8), respectively. As in the DAEs case, we only need to initialize the P_0-component of x_0. Further, putting $n = 0$ in (2.8) and noting that $V_{-1} = V_0$, $u_0 = P_{-1}x_0 = P_0x_0$, we find that a consistent initial value x_0 must satisfy a "hidden" constraint, namely, $Q_0(I + G_0^{-1}B_0P_0)x_0 = Q_0G_0^{-1}q_0$.

3. Nonlinear implicit difference equations

We begin this section by recalling the following version of the Hadamard theorem on homeomorphism.

Theorem 3.1 [2, page 222]. *Suppose $F \in C^1(X,Y)$ is a local homeomorphism between two Banach spaces X, Y and $\zeta(R) := \inf_{\|x\|\leqslant R}(\|[F'(x)]^{-1}\|)^{-1}$. Then if $\int_0^\infty \zeta(R)dR = +\infty$, F is a (global) homeomorphism of X into Y.*

In particular, if $\|[F'(x)]^{-1}\| \leqslant \alpha\|x\| + \beta$ for all $x \in X$, where $\alpha \geqslant 0$, $\beta > 0$, then F is a homeomorphism of X into Y. Further, suppose $F = T + H$, where $T \in C^1(X,Y)$, $\|[T'(x)]^{-1}\| \leqslant \gamma$, for all $x \in X$, and $\|H(x) - H(y)\| \leqslant L\|x - y\|$, for all $x, y \in X$, then if $L\gamma < 1$, F is a homeomorphism of X into Y.

Consider a system of nonlinear IDEs

$$f_n(x_{n+1},x_n) = 0 \quad (n \geqslant 0), \tag{3.1}$$

where $f_n : \mathbb{R}^m \to \mathbb{R}^m$ are given vector functions.

Definition 3.2. Equation (3.1) is said to be of index 1 if

(i) the function f_n is continuously differentiable, moreover, $\operatorname{Ker}(\partial f_n/\partial y)(y,x) = N_n$, $\dim N_n = m - r$, for all $n \geqslant 0$, $y,x \in \mathbb{R}^m$, where $1 \leqslant r \leqslant m-1$;
(ii) the matrix $G_n = (\partial f_n/\partial y)(y,x) + (\partial f_n/\partial x)(y,x)Q_{n-1,n}$ $(n \geqslant 0)$ is nonsingular.

Here, we put $N_{-1} = N_0$, $V_{-1} = V_0$, $Q_{-1} = Q_0$, and denote by $Q_{n-1,n}$ an operator connecting two subspaces N_{n-1}, N_n.

In the remainder of this paper, for the sake of simplicity, the norm of $\mathbb{R}^m$ is assumed to be Euclidean.

Theorem 3.3. *Let (3.1) be of index 1. Moreover, suppose that*

$$||G_n^{-1}(y,x)|| \leqslant \alpha_n \|y\| + \beta_n \|x\| + \gamma_n \quad \forall y,x \in \mathbb{R}^m,\ \forall n \geqslant 0, \tag{3.2}$$

where $\alpha_n, \beta_n \geqslant 0$, $\gamma_n > 0$ are constants. Then the problem of finding x_n from (3.1) and the initial condition

$$P_0 x_0 = p_0 \tag{3.3}$$

has a unique solution.

Proof. Since

$$f_n(x_{n+1},x_n) - f_n(P_n x_{n+1},x_n) = \int_0^1 \frac{\partial f_n}{\partial y}(P_n x_{n+1} + tQ_n x_{n+1},x_n)Q_n x_{n+1}\,dt = 0, \tag{3.4}$$

equation (3.1) becomes

$$f_n(P_n x_{n+1}, P_{n-1}x_n + Q_{n-1}x_n) = 0 \quad (n \geqslant 0). \tag{3.5}$$

Suppose $u_n = P_{n-1}x_n$ $(n \geqslant 0)$ is found (for $n = 0$, $u_0 = P_{-1}x_0 = P_0 x_0 = p_0$ is given). We have to find $u = P_n x_{n+1} \in \operatorname{Im} P_n \subset \mathbb{R}^r$ and $v = Q_{n-1}x_n \in \operatorname{Im} Q_{n-1} \subset \mathbb{R}^{m-r}$. Define an operator $F : \mathbb{R}^m \to \mathbb{R}^m$ by $F : z := (u^T, v^T)^T \mapsto f_n(u, u_n + v)$. Let $w = (\Delta u^T, \Delta v^T)^T$, where $\Delta u \in \operatorname{Im} P_n$, $\Delta v \in \operatorname{Im} Q_{n-1}$, then $F'(z)w = (\partial f_n/\partial y)(u,u_n+v)\Delta u + (\partial f_n/\partial x)(u,u_n+v)\Delta v$. Consider the linearized equation

$$F'(z)w = q, \tag{3.6}$$

where $q \in \mathbb{R}^m$ is an arbitrary fixed vector. First, observe that $G_n P_n = (\partial f_n/\partial y)P_n + (\partial f_n/\partial x)Q_{n-1,n}Q_n P_n = (\partial f_n/\partial y)P_n = \partial f_n/\partial y$, hence $G_n^{-1}(\partial f_n/\partial y) = P_n$ and $G_n Q_n = (\partial f_n/\partial x)Q_{n-1,n}Q_n = (\partial f_n/\partial x)Q_{n-1,n}$, therefore $G_n^{-1}(\partial f_n/\partial x)Q_{n-1,n} = Q_n$, where G_n, $\partial f_n/\partial y$, $\partial f_n/\partial x$ are valued at $(u, u_n + v)$. Further, since $P_n\Delta u = \Delta u$, $\Delta v = Q_{n-1}\Delta v = Q_{n-1,n}Q_{n,n-1}\Delta v$, then by the action of G_n^{-1} on both sides of (3.6) and using the last observations, we get

$$\Delta u + Q_{n,n-1}\Delta v = G_n^{-1}(u, u_n + v)q. \tag{3.7}$$

Now, applying P_n and Q_n to both sides of (3.7), respectively, we find $\Delta u = P_nG_n^{-1}q$ and $Q_{n,n-1}\Delta v = Q_nG_n^{-1}q$. The last equality leads to $\Delta v = V_{n-1}V_n^{-1}Q_nG_n^{-1}q$. Thus, (3.6) has a unique solution $w = (\Delta u^T, \Delta v^T)^T$. Moreover, $\|\Delta u\| \leqslant \|P_n\|\|G_n^{-1}\|\|q\|$ and $\|\Delta v\| \leqslant \|V_{n-1}V_n^{-1}Q_n\|\|G_n^{-1}\|\|q\|$, that is, $F'(z)$ has a bounded inverse. A simple calculation shows that $\|[F'(z)]^{-1}\| \leqslant \omega_n\|z\| + \delta_n$, where $\omega_n = \sqrt{2}\rho_n \max\{\alpha_n, \beta_n\}$, $\delta_n = \rho_n(\gamma_n + \beta_n\|u_n\|)$, and $\rho_n = (\|P_n\|^2 + \|V_{n-1}V_n^{-1}Q_n\|^2)^{1/2}$. By the Hadamard theorem on homeomorphism, (3.1) has a unique solution P_nx_{n+1} and $Q_{n-1}x_n$ for all $n \geqslant 0$. This completes the proof of Theorem 3.3. □

In the next theorem, without loss of generality, we will use orthogonal projections onto N_n, that is, $Q_n = V_n\tilde{Q}V_n^T$ and $V_nV_n^T = V_n^TV_n = I$. In this case, $Q_{n-1,n} = V_{n-1}\tilde{Q}V_n^T$ and $\|Q_n\| = \|P_n\| = \|V_n\| = 1$.

Theorem 3.4. *Suppose $f_n(y,x) = g_n(y,x) + h_n(y,x)$, where*

(i) *$g_n(y,x)$ is continuously differentiable, moreover*

$$\operatorname{Ker}\frac{\partial g_n}{\partial y}(y,x) = N_n, \quad \dim N_n = m - r, \quad \forall n \geqslant 0,\ \forall x,y \in \mathbb{R}^m; \tag{3.8}$$

(ii) *$G_n(y,x) = (\partial g_n/\partial y)(y,x) + (\partial g_n/\partial x)(y,x)Q_{n-1,n}$ $(n \geqslant 0)$ has uniformly bounded inverses, that is, $\|G_n^{-1}(y,x)\| \leqslant \gamma_n$ for all $n \geqslant 0$, $y,x \in \mathbb{R}^m$;*

(iii) *$h_n(y,x) = h_n(P_ny,x)$ for all $n \geqslant 0$, $y,x \in \mathbb{R}^m$;*

(iv) *$\|h_n(y,x) - h_n(\bar{y},\bar{x})\| \leqslant L_n(\|y-\bar{y}\|^2 + \|x-\bar{x}\|^2)^{1/2}$ for all $n \geqslant 0$, $y,x,\bar{y},\bar{x} \in \mathbb{R}^m$.*

Then, if $\gamma_nL_n < 1/\sqrt{2}$ for all $n \geqslant 0$, the IVP (3.1), (3.3) has a unique solution.

Proof. Using the notations of Theorem 3.3, we define two operators $T(z) = g_n(u, u_n + v)$ and $H(z) = h_n(u, u_n + v)$, where, as before, $z := (u^T, v^T)^T$, $u := P_nx_{n+1}$, $v := Q_{n-1}x_n$, and $u_n := P_{n-1}x_n$. From the proof of Theorem 3.3, it follows that $\|[T'(z)]^{-1}\| \leqslant \sqrt{2}\gamma_n$. On the other hand, $H(z)$ is Lipschitz continuous with a Lipschitz constant L_n and $\sqrt{2}\gamma_nL_n < 1$. Thus, the mapping $F(z) = T(z) + H(z)$ is a homeomorphism of X onto Y, therefore the IVP (3.1), (3.3) has a unique solution. □

Corollary 3.5. *Suppose $f_n(y,x) = A_ny + B_nx + h_n(y,x)$, where $A_n, B_n \in \mathbb{R}^{m\times m}$, and $h_n : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ satisfy the following conditions:*

(i) *$\operatorname{rank} A_n \equiv r$ and the matrix $G_n = A_n + B_nQ_{n-1,n}$ is nonsingular for all $n \geqslant 0$, where $Q_{n-1,n}$ is a connecting operator of $\operatorname{Ker} A_{n-1}$ and $\operatorname{Ker} A_n$, $A_{-1} := A_0$;*

(ii) *$h_n(y,x)$ is continuously differentiable, moreover*

$$\begin{gathered}\operatorname{Ker} A_n \subset \operatorname{Ker}\frac{\partial f_n}{\partial y}(y,x) \quad \forall n \geqslant 0,\ \forall y,x \in \mathbb{R}^m,\\ \|h_n(y,x) - h_n(\bar{y},\bar{x})\| \leqslant L_n(\|y-\bar{y}\|^2 + \|x-\bar{x}\|^2)^{1/2}, \quad \forall n \geqslant 0,\ \forall y,x,\bar{y},\bar{x} \in \mathbb{R}^m.\end{gathered} \tag{3.9}$$

Then, if $L_n\|G_n^{-1}\| < 1/\sqrt{2}$, the IVP (3.1), (3.3) is uniquely solvable.

It can be shown that the explicit Euler method applied to nonlinear transferable DAEs [4] leads to nonlinear index 1 IDEs. This and other problems related to connections between DAEs and IDEs will be discussed in our forthcoming paper.

Acknowledgment

The first author thanks Prof. Shao Xiumin and Prof. Liang Guoping for their hospitality during his visit to the Institute of Mathematics, Chinese Academy of Sciences.

References

[1] P. K. Anh and L. C. Loi, *On multipoint boundary-value problems for linear implicit non-autonomous systems of difference equations*, Vietnam J. Math. **29** (2001), no. 3, 281–286.

[2] M. S. Berger, *Nonlinearity and Functional Analysis*, Lectures on Nonlinear Problems in Mathematical Analysis. Pure and Applied Mathematics, Academic Press, New York, 1977.

[3] L. C. Loi, N. H. Du, and P. K. Anh, *On linear implicit non-autonomous systems of difference equations*, J. Difference Equ. Appl. **8** (2002), no. 12, 1085–1105.

[4] R. März, *On linear differential-algebraic equations and linearizations*, Appl. Numer. Math. **18** (1995), no. 1–3, 267–292.

Pham Ky Anh: Department of Mathematics, Mechanics, and Informatics, College of Science, Vietnam National University, 334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam

E-mail address: anhpk@vnu.edu.vn

Ha Thi Ngoc Yen: Department of Applied Mathematics, Hanoi University of Technology, 1 Dai Co Viet, 10000 Hanoi, Vietnam

E-mail address: hangocyen02@yahoo.com