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Nonlocal conditions for differential inclusions in the space of functions of bounded variations

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Abstract

We discuss the existence of solutions of an abstract differential inclusion, with a right-hand side of bounded variation and subject to a nonlocal initial condition of integral type.

AMS Subject Classification

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1 Introduction

Solutions of differential equations with smooth enough coefficients cannot have jump discontinuities, see for instance [1,2]. The situation is quite different for systems described by differential equations with discontinuous right-hand sides [3]. Examples of such systems are mechanical systems subjected to dry or Coulomb frictions [4], optimal control problems where the control parameters are discontinuous functions of the state [5], impulsive differential equations [6], measure differential equations, pulse frequency modulation systems or models for biological neural nets [7]. For these systems the state variables undergo sudden changes at their points of discontinuity. The mathematical models of many of these systems are described by multivalued differential equations or differential inclusions [8].

Let X be a Banach space with norm $|\cdot|_X$. Then X is a metric space with the distance d_X defined by

$$d_X(x, y) = |x - y|_X, \text{ for any } x, y \in X.$$

Let $I = [0, T]$ be a compact real interval. We are interested in the study of the following multivalued nonlocal initial value problem

$$\begin{cases} \dot{x}(t) \in F(t, x(t)), \ t \in I \\ x(0+) = \int_0^T g(x(t))dt, \end{cases} \tag{1}$$

where $F : I \times X \to X$ is a multivalued map and $g : X \to X$ is continuous.

The investigation of systems subjected to nonlocal conditions started with [9] for partial differential equations and [10] for Sturm-Liouville problems. For more recent work we refer the interested reader to [11] and the references therein.

It is clear that solutions of (1) are solutions of the integral inclusion

$$x(t) \in \int_0^T g(x(t))dt + \int_0^t F(s, x(s))ds. \tag{2}$$

2 Preliminaries

Definition 1 *We say that $f : I \to X$ is of bounded variation, and we write $f \in BV(I, X)$, if*

$$V_{d_X}(f, I) = \sup_{\Pi} \sum_{i=1}^{m} d_X(f(\tau_i), f(\tau_{i-1})) < +\infty,$$

where Π: $\tau_0 = 0 < \tau_1 < \ldots < \tau_m = T$ is any partition of I. The quantity $V_{d_X}(f, I)$ is called the total variation of f.

We shall denote by BV(I, X) the space of all functions of bounded variations on I and with values in X. It is a Banach space with the norm $|\cdot|_b$ given by

$$|f|_b = |f(0+)|_X + V_{d_X}(f, I), \text{ for any } f \in \text{BV}(I, X).$$

In order to discuss the integral inclusion (2) we present some facts from set-valued analysis. Complete details can be found in the books [8,12,13]. Let (X, $|\cdot|_X$) and (Y, $|\cdot|_Y$) be Banach spaces. We shall denote the set of all nonempty subsets of X having property ℓ by $\wp_\ell(X)$. For instance, $A \in \wp_{c\ \ell}(X)$ means A closed in X, when $\ell = b$ we have the bounded subsets of X, ℓ = cv for convex subsets, ℓ = cp for compact subsets and ℓ = cp, cv for compact and convex subsets. The domain of a multivalued map $\Re: X \to Y$ is the set dom$\Re = \{z \in X; \Re(z) \neq \emptyset\}$. $\Re$ is convex (closed) valued if $\Re(z)$ is convex (closed) for each $z \in X$: $\Re$ has compact values if $\Re(z) \in \wp_{\text{cv}}(Y)$ for every $z \in X$; $\Re$ is bounded on bounded sets if $\Re(A) = \cup_{z\in A}\Re(z)$ is bounded in Y for all $A \in \wp_b(X)$ (i.e. $\sup_{z\in A}\{\sup\{|y|_Y; y \in \Re(z)\}\} < \infty$): $\Re$ is called upper semicontinuous (u.s.c.) on X if for each $z \in X$ the set $\Re(z) \in \wp_{cl}(Y)$ is nonempty, and for each open subset Λ of Y containing $\Re(z)$, there exists an open neighborhood Π of z such that $\Re(\Pi) \subset \Lambda$. In terms of sequences, $\Re$ is u.s.c. if for each sequence $(z_n) \subset X$, $z_n \to z_0$, and B a closed subset of Y such that $\Re(z_n) \cap B \neq \emptyset$, then $\Re(z_0) \cap B \neq \emptyset$. The set-valued map $\Re$ is called completely continuous if $\Re(A)$ is relatively compact in Y for every $A \in \wp(X)$. If $\Re$ is completely continuous with nonempty compact values, then $\Re$ is u.s.c. if and only if $\Re$ has a closed graph (i.e. $z_n \to z$, $w_n \to w$, $w_n \in \Re(z_n) \Rightarrow w \in \Re(z)$). When $X \subset Y$ then $\Re$ has a fixed point if there exists $z \in X$ such $z \in \Re(z)$. A multivalued map $\Re: J \to \wp_{cl}(X)$ is called measurable if for every $x \in X$, the function $\theta : J \to \mathbb{R}$ defined by $\theta(t) = dist(x, \Re(t)) = \inf\{|x - z|_X ; z \in \Re(t)\}$ is measurable. $|\Re(z)|_Y$ denotes $\sup\{|y|_Y; y \in \Re(z)\}$.

If A and B are two subsets of X, equipped with the metric d_X, such that $d_X(x, y) = |x - y|_X$, the Hausdorff distance between A and B is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\},$$

Where

$$\rho\,(A,B) = \sup_{a\in A} d_X(a,B),\ \text{ and } d_X(a,B) = \inf_{b\in B} d_X(a,b).$$

It is well known that ($\wp_{b,cl}(X)$, d_H) is a metric space and so is ($\wp_{cp}(X)$, d_H).

Definition 2 *(See* [14,15]*)* Θ: $I \to X$ *is of bounded variation (with respect to* d_H*) on* I *if*

$$V(\Theta, I) = V_{d_H}\,(\Theta, I) = \sup_{\Pi}\left[\sum_{i=1}^{m} d_H(\Theta(t_i), \Theta(t_{i-1})\right] < \infty,$$

where the supremum is taken over all partitions $\Pi = \{t_i;\ i = 1, 2, ..., m\}$ *of the interval* I.

Definition 3 *Let* X^I *denote the set of all functions from* I *into* X. *The Nemitskii (or superposition) operator corresponding to* $F: I \times X \to X$ *is the operator*

$$N_F : X^I \to X,$$

defined by

$$N_F(x)(t) = F(t, x(t))\ \textit{for every}\ t \in I.$$

Definition 4 *The multifunction* $F: I\ X \to X$ *is of bounded variation if for any function* $\times \in$ BV(I, X) *the multivalued map* $N_F(x)$: $I \to X$ *is of bounded variation on* I *(in the sense of Definition 2) and*

$$V_{d_H}(F(\cdot, x(\cdot)), I) = V_{d_H}(N_F(x), I).$$

Definition 5 *Let* Δ *be a subset of* $I \times X$. *We say that* Δ *is* $\mathcal{L}\otimes\mathcal{B}$*measurable if* Δ *belongs to the* σ*- algebra generated by all sets of the form* $J \times D$ *where* J *is Lebesgue measurable in* I *and* D *is Borel measurable in* X.

Theorem 6 *(Generalized Helly selection principle) [*[14], *Theorem 5.1 p. 812] Let* K *be a compact subset of the Banach space* $\times$ *and let* $\mathcal{F}$*be a family of maps of uniformly bounded variation from* I *into* K. *Then there exists a sequence of maps* $(f_n)_{n\geq 1} \subset \mathcal{F}$*convergent pointwise on* I *to a map* $f: I \to K$ *of bounded variation such that* $V(f, I) \leq \sup_{\varphi\in\mathcal{F}} V(\varphi, I)$.

In the next theorem we shall denote by $\bar{U}$ and ∂U the closure and the boundary of a set U.

Theorem 7 *(*[16], *Theorem 3.4, p. 34]) Let* U *be an open subset of a Banach space* Z *with* $0 \in U$. *Let* $A: \bar{U} \to Z$*be a single-valued operator and* $B: \bar{U} \to \wp_{\text{cp,cv}}(Z)$*be a multivalued operator such that*

(i) $A(\bar{U}) + B(\bar{U})$ is bounded,
(ii) A is a contraction with constant $k \in (0, 1/2)$,
(iii) B is u.s.c and compact.

Then either

(a) the operator inclusion $\lambda x \in Ax + Bx$ has a solution for $\lambda = 1$, or
(b) there is an element $u \in \partial U$ such that $\lambda u \in Au + Bu$ for some $\lambda > 1$.

3 Main results

In this section we state and prove our main result. We should point out that no semi-continuity property is assumed on the multifunction F, which is usually the case in the literature. We refer the interested reader to the nice collection of papers in [17] and the references therein.

Theorem 8 *Assume that the following conditions hold.*

(H1) $g : X \to X$ is continuous, $g(0) = 0$ and there exists $\theta : [0, +\infty) \to [0, +\infty)$ continuous and $\theta(r) \le \beta r$, with $\beta < 1/2$ and $\beta T \ne 1$, such that

$$|g(u) - g(v)|_X < \theta(|u - v|_X),$$

(H2) $F : I \times X \to \wp_{\mathrm{cp,cv}}(X)$ is of bounded variation such that

(i) $(t, x) \mapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,

(ii) there exists an integrable function $q : I \to [0, +\infty)$ with

$$|F(t,x)|_X \le q(t) \text{ for } (t,x) \in I \times X,$$

(iii) $x_k \to x$ as $k \to \infty$ pointwise implies $d_H(F(t, x_k), F(t, x)) \to 0, k \to \infty$.

Then problem (1) has at least one solution in BV(I, X).

Proof. Let $Q = \sup_{t \in I} \int_0^t q(s)ds$. We show that there exists $M > 0$ such that all possible solutions of (2) in BV(I, X), satisfy

$$|x|_b \le M.$$

Recall that solutions of (1) satisfy

$$x(t) \in \int_0^T g(x(t))dt + \int_0^t F(s, x(s))ds = \int_0^T g(x(t))dt + \int_0^t N_F(x)(s)ds. \tag{3}$$

Since the multivalued map $N_F(x)$: $I \to X$ is of bounded variation it admits a selector $f : I \to X$ of bounded variation such that

$$V_{d_X}(f, I) \le V_{d_H}(N_F(x), I),$$

see [[18], Theorem A, p. 250].

It follows from (3) that

$$x(t) = \int_0^T g(x(t))dt + \int_0^t f(s)ds, \ t \in I. \tag{4}$$

This implies

$$\begin{aligned} |x(t)|_X &\le \left| \int_0^T g(x(t))dt \right|_X + \left| \int_0^t f(s)ds \right|_X \\ &\le \int_0^T |g(x(t))|_X dt + \int_0^t |f(s)|_X ds. \end{aligned}$$

The condition on g and (H2) (ii) imply

$$|x(t)|_X \leq \beta \int_0^T |x(t)|_X dt + \int_0^t q(s)ds.$$

Hence

$$\int_0^T |x(t)|_X dt \leq \beta T \int_0^T |x(t)|_X dt + \int_0^T \int_0^t q(s)dsdt.$$

This last inequality yields

$$\int_0^T |x(t)|_X dt \leq \frac{1}{1-\beta T} \int_0^T \int_0^t q(s)dsdt.$$

Since

$$\int_0^T \int_0^t q(s)dsdt = \int_0^T (T-s)q(s)ds,$$

we obtain

$$\int_0^T |x(t)|_X dt \leq \frac{1}{1-\beta T} \int_0^T (T-s)q(s)ds,$$

so that

$$\int_0^T |x(t)|_X dt \leq \frac{2T}{1-\beta T} Q. \tag{5}$$

Inequality (5) and the condition on g imply that

$$\int_0^T |g(x(t))|_X dt \leq \frac{2\beta T}{1-\beta T} Q.$$

Hence any possible solution x of (2) in BV(I, X), satisfies

$$|x(0+)|_X \leq \frac{2\beta T}{1-\beta T} Q.$$

Let $\Pi = \{t_i;\ i = 1, 2, ..., m\}$ be any partition of the interval I, and let $x \in$ BV(I, X) be any possible solution of (2). It follows from (4) that

$$x(t_i) - x(t_{i-1}) = \int_{t_{i-1}}^{t_i} f(s)ds,\ i = 1, \ldots, m.$$

It is easily shown that

$$V_{d_X}(x, I) \leq V_{d_X}(f, I) \leq \sup_{\Pi} \left[\sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} q(s)ds \right] \leq Q.$$

Therefore

$$|x|_b \leq \frac{2\beta T}{1-\beta T}Q + Q.$$

Letting $M := \dfrac{1+\beta T}{1-\beta T}Q$,, we see that

$$|x|_b \leq M.$$

Let

$$\Omega := \{x \in \mathrm{BV}(I, X);\ |x|_b < M + 1\}.$$

Define two operators

$$A : \Omega \to X,\ B : \Omega \to X$$

by

$$Ax(t) = \int_0^T g(x(t))dt,$$

and

$$Bx(t) = \int_0^t F(s, x(s))ds = \int_0^t N_F(x)(s)\,ds.$$

First, we show that $A(\bar{\Omega}) + B(\bar{\Omega})$ is bounded, i.e. $\sup_{x\in\bar{\Omega}}\{\sup\{|y|_b; y \in A(x) + B(x)\}\} < \infty$.

Let $y \in A(\bar{\Omega}) + B(\bar{\Omega})$. Then there exists $x \in \bar{\Omega}$ such that

$$y \in A(x) + B(x).$$

It follows from (3) that $|y|_b \leq M$.

(H1) implies that the single-valued operator A is a contraction with constant $k \in (0, 1/2)$.

Claim 1. The multivalued operator B has compact and convex values. For, since $F : I \times X \to \wp_{\mathrm{cp,cv}}(X)$ it follows that $NF : X^I \to \wp_{\mathrm{cp,cv}}(X)$, i.e. has compact and convex values. This implies that the Aumann integral

$$\int_0^t N_F(x)(s)\,ds$$

has compact and convex values. See for instance [5].

Claim 2. B is completely continuous, i.e. $B(\Omega)$ is a relatively compact subset of BV(I, X). Let $q \in \Omega$ be arbitrary. Then for every $f \in N_F(q)$ the function $u : I \to X$ defined by

$$u(t) = \int_0^t f(s)ds,$$

satisfies

$$\dot{u}(t) = f(t), \quad \mathrm{u}(0+) = 0.$$

If we write

$$u = \Upsilon f,$$

then the operator Υ: $X \to X$ is continuous and

$$B = \Upsilon \circ N_F.$$

Let $(Bx_k)_{k\geq 1}$ be a sequence in $B\ (\Omega)$. Then the sequence $(x_k)_{k\geq 1}$ is uniformly bounded and is of bounded variation. Theorem 4 shows that there exists a subsequence, which we label the same, and which converges pointwise to $y \in \Omega$. We have

$$\left|Bx_k - By\right|_b \leq \sup_{\Pi} \left[\sum_{i=1}^{m} \int_{\tau_{i-1}}^{\tau_i} \left|F(s, x_k(s)) - F(s, y(s))\right|_X ds\right].$$

Assumption (H2) (iii) implies that

$$\left|Bx_k - By\right|_b \to 0 \text{ as } k \to 0.$$

This proves the claim.

Claim 3. B is u.s.c. Since B is completely continuous it is enough to show that its graph is closed. Let $\{(x_n, y_n)\}_{n\geq 1}$ be a sequence in *graph*(B) and let $(x, y) = \lim_{n\to\infty} (x_n, y_n)$. Then $y_n \in B(x_n)$, i.e $y_n\,(t) \in \int_0^t F(s, x_n(s))ds$, $t \in I$. This implies that

$$y_n\,(t) \in \int_0^t F(s, x(s))ds + \int_0^t \left[F(s, x_n(s)) - F(s, x(s))\right] ds.$$

Since $x_n \to x$ in X it follows from (H2)(ii) that

$$\lim_{n\to\infty} y_n\,(t) \in \int_0^t F(s, x(s))ds,$$

which shows that

$$y \in B(x).$$

Hence $(x, y) \in$ *graph*(B), and B has a closed graph.

Finally, alternative (b) in Theorem 5 cannot hold due to (3) and the choice of Ω.

By Theorem 5 the inclusion

$$x \in Ax + Bx,$$

has at least one solution in BV(I, X). This completes the proof of the theorem.

For our second result we consider the case when $\int_0^T g(x(t))dt = \int_0^T \psi\,(t)\,x(t)dt$, where $\psi : I \to \mathbb{R}$ is continuous. Let

$$\psi_0 = \int_0^T \psi(t)dt \text{ and } \lambda\,(\mathrm{s}) = \int_{\mathrm{s}}^{\mathrm{T}} \frac{\psi(\mathrm{t})\mathrm{dt}}{1 - \psi_0}.$$

From the definition of the function λ we infer that, if $\psi^* = \max_{t\in I} |\psi(t)|$,

$$\left|\lambda(s)\right| \leq \frac{2T}{1 - \psi_0}\psi^* \text{ for any } s \in I.$$

Theorem 9 *Assume that the following conditions hold*

(H3) $\psi : I \to \mathbb{R}$ is continuous and $\psi_0 \neq 1$,

(H4) $F : I \times X \to \wp_{\mathrm{cp,cv}}(X)$ is of bounded variation such that

(i) $(t, x) \mapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,

(ii) there exists $\omega : I \times [0, \infty) \to (0, \infty)$ continuous, nondecreasing with respect to its second argument and

$$\limsup_{\rho\to\infty} \frac{1}{\rho}\left(\frac{1-\psi_0+2\psi^*T}{1-\psi_0}\right)\int_0^T \omega(s,\rho)\,ds < 1, \tag{6}$$

such that $|F(t, x)\,|_X \leq \omega \to (t, |x|_b)$.

(iii) $x_k \to x$ pointwise as $k \to \infty$ implies $d_H(F(t, x_k), F(t, x)) \to 0$ as $k \to \infty$.

Then problem (1) has at least one solution in $\mathrm{BV}(I, X)$.

Proof. Since the multivalued map $N_F(x)$: $I \to X$ is of bounded variation it admits a selector $h : I \to X$ of bounded variation such that

$$V_{d_X}(h, I) \leq V_{d_H}(N_F(x), I),$$

see [[18], Theorem A, p. 250].

Solutions of (2) satisfy

$$x(t) = x(0+) + \int_0^t h(s)ds, \quad h \in N_F(x). \tag{7}$$

Substituting the initial condition in (7) we obtain

$$x(t) = \int_0^T \psi(t)\,x(t)dt + \int_0^t h(s)ds, \quad h \in N_F(x)$$

Since $\psi_0 \neq 1$ it follows that

$$x(t) = \int_0^T \frac{\psi(t)}{1-\psi_0}\int_0^t h(s)dsdt + \int_0^t h(s)ds, \quad h \in N_F(x).$$

Thus, solutions of (2) are solutions of

$$x(t) = \int_0^T \lambda(s)h(s)ds + \int_0^t h(s)ds, h \in N_F(x), \tag{8}$$

and vice versa. It follows from (8)

$$|x(t)|_X \leq \int_0^T |\lambda(s)|\,\omega(s, |x|_b)\,ds + \int_0^t \omega(s, |x|_b)\,ds.$$

The upper bound on $|\lambda(s)|$ implies

$$|x(t)|_X \leq \frac{2T}{1-\psi_0}\psi^*\int_0^T \omega(s, |x|_b)\,ds + \int_0^t \omega(s, |x|_b)\,ds, \tag{9}$$

which gives

$$\left|x(0+)\right|_X \le \frac{2T}{1-\psi_0}\psi^* \int_0^T \omega\,(s,|x|_b)\,ds.$$

Let $\Pi = \{t_i;\ i = 1, 2, ..., m\}$ be any partition of the interval I, and let $x \in \mathrm{BV}(I, X)$ be any possible solution of (2). Then, it follows from (7) that

$$x(t_i) - x(t_{i-1}) = \int_{t_{i-1}}^{t_i} h(s)ds,\ \ i = 1, \dots, m,$$

which leads to

$$V_{d_X}(x, I) \le V_{d_X}(h, I) \le \int_0^T \omega\,(s,|x|_b)\,ds.$$

Since $|x|_b = \left|x(0+)\right|_X + V_{d_X}(x, I)$, we have

$$|x|_b \le \frac{2T}{1-\psi_0}\psi^* \int_0^T \omega\,(s,|x|_b)\,ds + \int_0^T \omega\,(s,|x|_b)\,ds.$$

Finally, we see that

$$|x|_b \le \frac{1-\psi_0+2\psi^* T}{1-\psi_0} \int_0^T \omega\,(s,|x|_b)\,ds. \tag{10}$$

Let

$$\rho_0 = |x|_b.$$

Then (10) yields

$$1 \le \frac{1}{\rho_0}\left(\frac{1-\psi_0+2\psi^* T}{1-\psi_0}\right) \int_0^T \omega\,(s,\rho_0)\,ds. \tag{11}$$

The condition on the function ω implies that there exists $\rho^* > 0$ such that for all $\rho > \rho^*$

$$\frac{1}{\rho}\left(\frac{1-\psi_0+2\psi^* T}{1-\psi_0}\right) \int_0^T \omega\,(s,\rho)\,ds < 1. \tag{12}$$

Comparing inequalities (11) and (12) we see that

$$\rho_0 = |x|_b \le \rho^*.$$

Let

$$\Sigma = \{x \in \mathrm{BV}(I, X); \quad |x|_b \le \rho^*\}.$$

Then Σ is nonempty, closed, bounded and convex.
Define a multivalued operator

$$\mathcal{F} : \mathrm{BV}(I, X) \to \wp_{\mathrm{cp,\,cv}}(X),$$

by

$$\mathcal{F}x(t) = \int_0^T \lambda(s) N_F(x)(s)ds + \int_0^t N_F(x)(s)ds. \tag{13}$$

Then solutions of (2) are fixed point of the multivalued operator $\mathcal{F} : \Sigma \to \wp_{\text{cp, cv}}(X)$.

It is clear that $\mathcal{F}(\Sigma) \subset \Sigma$. Proceeding as in the above claims we can show that $\mathcal{F}$ is u.s.c. and $\overline{\mathcal{F}(\Sigma)}$ is compact. By the Theorem of Bohnenblust and Karlin (see Corollary 11.3 in [8]) $\mathcal{F}$ has a fixed point in Σ, which is a solution of the inclusion (2), and therefore a solution of (1).

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Authors' contributions
Both authors have read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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