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Perturbation formula for the two-phase membrane problem

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Abstract

A perturbation formula for the two-phase membrane problem is considered. We perturb the data in the right-hand side of the two-phase equation. The stability of the solution and the free boundary with respect to perturbation in the coefficients and boundary value is shown. Furthermore, continuity and differentiability of the solution with respect to the coefficients are proved.

Keywords: Free boundary problems, Two-phase membrane, Perturbation

Introduction

Let $\lambda^{\pm} : \Omega \to \mathbb{R}$ be non-negative Lipschitz continuous functions, where Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Assume further that $g \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ and g changes sign on $\partial\Omega$. Let $K = \{v \in W^{1,2}(\Omega) : v - g \in W_0^{1,2}(\Omega)\}$. Consider the functional

$$I(v) = \int_{\Omega} \left(\frac{1}{2}|\nabla v|^2 + \lambda^+ \max(v,0) - \lambda^- \min(v,0) \right) dx, \tag{1.1}$$

which is convex, weakly lower semi-continuous and hence attains its infimum at some point $u \in K$. The Euler-Lagrange equation corresponding to the minimizer u is given by Weiss [1] and is called the *two-phase membrane* problem:

$$\begin{cases} \Delta u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u>0\}} & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where χ_A denotes the characteristic function of the set A, and

$$\Gamma(u) = \partial\{x \in \Omega : u(x) > 0\} \cup \partial\{x \in \Omega : u(x) < 0\} \cap \Omega$$

is called the *free boundary*. The free boundary consists of two parts:

$$\Gamma'(u) = \Gamma(u) \cap \{x \in \Omega : \nabla u(x) = 0\}$$

and

$$\Gamma''(u) = \Gamma(u) \cap \{\nabla u(x) \neq 0\}.$$

By $\Omega^+(u)$ and $\Omega^-(u)$ we denote the sets $\{x \in \Omega: u(x) > 0\}$ and $\{x \in \Omega: u(x) < 0\}$, respectively. Also, $\Lambda(u)$ denotes the set $\{x \in \Omega: u(x) = 0\}$.

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The regularity of the solution, the Hausdorff dimension and the regularity of the free boundary are discussed in [2-5]. In [5], on the basis of the monotonicity formula due to Alt, Caffarelli, and Friedman, the boundedness of the second-order derivatives D^2u of solutions to the two-phase membrane problem is proved. Moreover, in [3], a complete characterization of the global two-phase solution satisfying a quadratic growth at a two-phase free boundary point and at infinity is given. In [4] it has been shown that if λ^+ and λ^- are Lipschitz, then, in two dimensions, the free boundary in a neighborhood of each branch point is the union of two C^1-graphs. Also, in higher dimensions, the free boundary has finite $(n - 1)$-dimensional Hausdorff measure. Numerical approximation for the two-phase problem is discussed in [6].

In this article, by *perturbation* we mean the perturbation of the coefficients λ^+ and λ^- and the perturbation of the boundary values g. The case of the one phase obstacle problem has been studied in [7].

For given $(\lambda^+,\lambda^-) \in C^{0,1}(\Omega) \times C^{0,1}(\Omega)$, Equation 1.2 has a unique solution $u \in W^{2,p}_{\text{loc}}(\Omega)$ for

$1 < p < \infty$ (see [8]). Define the map

$$T : (\lambda^+, \lambda^-) \mapsto u, \tag{1.3}$$

where u is the solution of (1.2) corresponding to the coefficients λ^+ and λ^-. The main results in this paper are the following:

1. The stability of solution with respect to boundary value and coefficients is shown.
2. Let $\bar{\lambda} = (\lambda^+, \lambda^-)$, $\bar{h} = (h_1, h_2)$. By $u^{\bar{\lambda}+\varepsilon\bar{h}}$, we mean the solution of problem (1.2) with coefficients $(\lambda^+ + \varepsilon h_1)$ and $(\lambda^- + \varepsilon h_2)$. If we Consider the map $T : (\lambda^+, \lambda^-) \mapsto u$, for given parameters λ^+ and λ^+ and a fixed Dirichlet condition, then the Gateaux derivative of this map is characterized in H^1_0. More precisely, it is shown in Theorem 3.4 that

$$\frac{u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}}}{\varepsilon} \rightharpoonup w^{\bar{\lambda},\bar{h}} \text{ in } H^1_0(\Omega) \text{ as } \varepsilon \to 0,$$

where

$$\Delta w^{\bar{\lambda},\bar{h}} = h_1 \chi_{\{u^{\bar{\lambda}}>0\}} - h_2 \chi_{\{u^{\bar{\lambda}}<0\}} + \frac{(\lambda^+ + \lambda^-)}{|\nabla u^{\bar{\lambda}}|} w^{\bar{\lambda},\bar{h}} \mathcal{H}^{n-1} \lfloor_{\Gamma''(u^{\bar{\lambda}})}.$$

3. (Theorem 3.5) Assuming that all free boundary points are one-phase points (points such that $\nabla u = 0$), a stability result for the free boundary in the flavor of [7] is proved which says that

$$\frac{\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - \chi_{\{u^{\bar{\lambda}}>0\}}}{\varepsilon} \rightharpoonup -\frac{1}{\lambda+}\frac{\partial\delta}{\partial v_1} d\Gamma^+, \text{ in } H^{-1}(\Omega) \text{ as } \varepsilon \to 0,$$
$$\frac{\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}<0\}} - \chi_{\{u^{\bar{\lambda}}<0\}}}{\varepsilon} \rightharpoonup \frac{1}{\lambda^-}\frac{\partial\delta}{\partial v_2} d\Gamma^-, \text{ in } H^{-1}(\Omega) \text{ as } \varepsilon \to 0.$$

Were $\Gamma^{\pm} = \partial \{\pm u(x) > 0\} \cap \Omega$. The function δ is constructed as a solution of certain Dirichlet problem in $\{u^{\bar{\lambda}} > 0\}$. The vector ν_1 stands for the exterior unit normal vector to $\partial\{u^{\bar{\lambda}} > 0\}$.

The structure of article is organized as follows. In the next section, stability of solution with respect to boundary value and coefficients is studied. In Section 3, we prove that the map T is Lipschitz continuous (Theorem 3.1) and differentiable (Theorem 3.4).

Preliminary analysis and stability results

In this section, we state some lemmas which have been proved in the case of one-phase obstacle problem (see [9]). The following proposition shows the stability in L^∞-norm. In what follows, we will denote by $B_r(x_0)$ the ball of radius r centered at x_0 and, for simplicity, we use the notation $B_r = B_r(0)$.

Proposition 2.1. *Let u_i for $i = 1, 2$ be the solution of the following problem*

$$\begin{cases} \Delta u_i = \lambda^+ \chi_{\{u_i<0\}} - \lambda^- \chi_{\{u_i<0\}} & \text{in } \Omega, \\ u_i = g_i & \text{on } \partial\,\Omega. \end{cases} \tag{1.4}$$

If $g_1 \le g_2 \le g_1 + \varepsilon$, then $u_1 \le u_2 \le u_1 + \varepsilon$. In particular,

$$||u_2 - u_1||_{L^\infty} \le \ ||g_1 - g_2||_{L^\infty}.$$

Proof. First, we show that $u_1 \le u_2$. Denote $\widetilde{\Omega} = \{x \in \Omega | u_1(x) > u_2(x)\}$; then, for all $x \in \tilde{\Omega}$ the following inequalities hold.

$$\chi_{\{u_1>0\}} \ge \chi_{\{u_2>0\}},$$

and

$$\chi_{\{u_1<0\}} \le \chi_{\{u_2<0\}}.$$

These inequalities imply that

$$\Delta u_1 = \lambda^+ \chi_{\{u_1>0\}} - \lambda^- \chi_{\{u_1<0\}} \ge \lambda^+ \chi_{\{u_2>0\}} - \lambda^- \chi_{\{u_2<0\}} = \Delta u_2, \quad \text{in } \widetilde{\Omega},$$

which shows that

$$\Delta(u_1 - u_2) \ge 0, \quad \forall x \in \tilde{\Omega}.$$

One can see that on the boundary of $\widetilde{\Omega}$, the following holds:

$$(u_1 - u_2)|_{\partial\widetilde{\Omega}} = \begin{cases} 0 & x \in \partial\widetilde{\Omega} \backslash \partial\Omega, \\ g_1 - g_2 & x \in \partial\Omega \backslash \partial\widetilde{\Omega}. \end{cases}$$

Note that by assumptions on g_1 and g_2, the inequality $u_1 - u_2 \le 0$ will hold on the $\partial\widetilde{\Omega}$. Thus, we have,

$$\begin{cases} \Delta(u_1 - u_2) \ge 0 \text{ in } \widetilde{\Omega}, \\ (u_1 - u_2) \le 0 \quad \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{1.5}$$

By maximum principle, we obtain that

$$u_1 - u_2 \le 0 \quad \forall x \in \widetilde{\Omega},$$

which is impossible. Therefore, $\widetilde{\Omega} = \emptyset$.

Let u_3 be the solution to the following problem:

$$\begin{cases} \Delta u_3 = \lambda^+ \chi_{\{u_3>0\}} - \lambda^- \chi_{\{u_3<0\}} & \text{in } \Omega, \\ u_3 = g_1 + \varepsilon & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

An analysis similar to the one above shows that if $v = u_1 + \varepsilon - u_3$, then $v \geq 0$, which implies

$$u_1 \leq u_2 \leq u_3 \leq u_1 + \varepsilon.$$

□

Lemma 2.2. *Assume that* $\inf_{B_1(0)} \lambda^- > \varepsilon > 0$. *Let u be a solution to*

$$\Delta u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} \text{ in } B_1,$$

and let u^ε *solve*

$$\Delta u^\varepsilon = (\lambda^+ + \varepsilon)\chi_{\{u^\varepsilon>0\}} - (\lambda^- - \varepsilon)\chi_{\{u^\varepsilon<0\}} \text{ in } B_1,$$

with $u = u^\varepsilon = g$ *on* ∂B_1. *Then*

$$|u^\varepsilon - u| \leq C\varepsilon.$$

Proof. Let $\varepsilon > 0$,; we will show that $u^\varepsilon \leq u$. Set $D = \{x \in B_1 : u^\varepsilon(x) > u(x)\}$. If $u^\varepsilon \leq 0$, on D, then $u < 0$ on D and $\Delta u = -\lambda^- \leq -(\lambda^- - \varepsilon) \leq \Delta u^\varepsilon$: On the other hand, if $u^\varepsilon > 0$; then $\Delta u^\varepsilon = \lambda^+ + \varepsilon \geq \Delta u$. Therefore, $\Delta u^\varepsilon \geq \Delta u$ and, by maximum principle, $D = \emptyset$.

Now we claim that also $u + \varepsilon v \leq u^\varepsilon$ in B_1, where v is the solution to $\Delta v = 1$ with zero Dirichlet boundary data in B_1. Assume that

$$\tilde{\Omega} = \{x \in B_1 : u + \varepsilon v > u^\varepsilon(x)\}.$$

Note that $v(x) \leq 0$ in B_1, and so we have

$$u^\varepsilon < u + \varepsilon v \leq u \text{ in } \tilde{\Omega}.$$

Then, for all $x \in \tilde{\Omega}$, the following inequalities hold:

$$\chi_{\{u>0\}} \geq \chi_{\{u^\varepsilon>0\}},$$

and

$$\chi_{\{u<0\}} \leq \chi_{\{u^\varepsilon<0\}}.$$

In $\tilde{\Omega}$, we have

$$\begin{aligned} \Delta(u + \varepsilon v) &= \Delta u + \varepsilon = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} + \varepsilon \geq \lambda^+ \chi_{\{u^\varepsilon>0\}} - \lambda^- \chi_{\{u^\varepsilon<0\}} + \varepsilon \\ &\geq (\lambda^+ + \varepsilon)\chi_{\{u^\varepsilon>0\}} - (\lambda^- - \varepsilon)\chi_{\{u^\varepsilon<0\}} = \Delta u^\varepsilon. \end{aligned}$$

Therefore, we have

$$\begin{cases} \Delta(u + \varepsilon v - u^\varepsilon) \geq 0 & \text{in } \tilde{\Omega}, \\ u + \varepsilon v - u^\varepsilon = 0 & \text{on } \partial\tilde{\Omega}. \end{cases}$$

This shows that $u + \varepsilon v \leq u^\varepsilon$ in $\tilde{\Omega}$, which is impossible. Since

$$v(x) = \frac{|x|^2 - 1}{2n},$$

this implies that $u^{\varepsilon} \geq -C\varepsilon + u$. Note that in the case when $\varepsilon < 0$, with the assumption $\inf_{B_1(0)} \lambda^+ > -\varepsilon > 0$ one can prove that

$$u \leq u^{\varepsilon} \leq u + \varepsilon v.$$

□

Remark 1. An analysis similar to Lemma 2.2 shows that if the coefficients $\lambda^{\pm}$ be perturbed by $\pm\varepsilon$, then $|u^{\varepsilon} - u| \leq C\varepsilon$.

Remark 2. The proofs of Proposition 2.1 and Lemma 2.2 show that if u and v solve the following problems, respectively:

$$\begin{cases} \Delta u = \lambda_1^+ \chi_{\{u>0\}} - \lambda_1^- \chi_{\{u<0\}} & \text{in} \quad B_1, \\ u = g_1 & \text{on} \quad \partial B_1, \end{cases}$$

and

$$\begin{cases} \Delta v = \lambda_2^+ \chi_{\{v>0\}} - \lambda_2^- \chi_{\{v<0\}} & \text{in} \quad B_1, \\ v = g_2 & \text{on} \quad \partial B_1, \end{cases}$$

with $\lambda_2^+ \geq \lambda_1^+,\ \lambda_2^- \leq \lambda_1^-,\ g_2 \leq g_1$, then $u \geq v$. In particular,

$$\Lambda(u) \subseteq \Lambda(v),\ \Omega^+(v) \subseteq \Omega^+(u) \text{ and } \Omega^-(v) \subseteq \Omega^-(u).$$

Theorem 2.3. *Let u_k be a sequence of minimizer to (1.1), respectively with data g_k and $\lambda_k^{\pm}$, such that*

$$g_k \to g \quad in \quad H^{\frac{1}{2}}(\partial\Omega),$$

and

$$\lambda_k^{\pm} \to \lambda^{\pm} \quad in \quad C^0(\Omega).$$

Then,

$$u_k \to u \quad in \quad H^1(\Omega),$$

where u is the minimizer of (1.1) with data g and potential $\lambda^{\pm}$.

Proof. First, one can see that g is an admissible boundary data, i.e., g changes sign on the boundary by the strong convergence of g_k in $H^{\frac{1}{2}}(\partial\Omega)$. We denote by u^* the solution to minimization problem (1.1) with data g and $\lambda^{\pm}$. Consider the minimum levels $c_k = I_k(u_k)$ and $c^* = I(u^*)$. Also the convergence of the boundary traces g_k and of the $\lambda_k^{\pm}$, ensures a bound on the sequence c_k. Since the sequence of functionals $\{I_k\}$ is uniformly coercive, from the fact that $I_k(u_k) \leq C$, we infer a bound on the sequence $||u_k||_{H^1(\Omega)}$; therefore, we can assume, up to a subsequence, that

$$c_k \to c_0 \text{ and } u_k \rightharpoonup u \text{ weakly in } H^1(\Omega).$$

Furthermore, by the weak continuity of the trace operator, we obtain

$$u|_{\partial\Omega} = g.$$

The weak lower semi-continuity of the norm implies

$$\int_\Omega \frac{1}{2}|\nabla u|^2 dx \leq \text{Lim inf} \int_\Omega \frac{1}{2}|\nabla u_k|^2 dx,$$

and we also have

$$\int_\Omega (\lambda^+ \max(u,0) - \lambda^- \min(u,0))dx \leq \text{Lim inf} \int_\Omega (\lambda_k^+ \max(u_k,0) - \lambda_k^- \min(u_k,0))\, dx.$$

Note that the level

$$c = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \lambda^+ \max(u,0) - \lambda^- \min(u,0)\right) dx,$$

is not necessarily a minimum, but, by the previous discussion it satisfies the inequalities

$$c_0 \geq c \geq c^*.$$

We shall prove that $c_0 = c^*$. Suppose, by contradiction, that $c^* < c_0$. Consider the harmonic extensions (denoted with the same notations) on Ω of g_i's and of g and introduce

$$h_k = g_k - g.$$

Then, by construction

$$\begin{cases} h_k \to 0 \quad\quad \text{in } H^1(\Omega), \\ h_k\,|_{\partial\Omega} \to 0 \text{ in } H^{\frac{1}{2}}(\partial\Omega). \end{cases} \tag{1.7}$$

We define $w_k = u^* + h_k$, and observe that $w_k|_{\partial\Omega} = g_k$. Moreover, by (1.7),

$$w_k \to u^* \text{ in } H^1(\Omega). \tag{1.8}$$

Hence, it follows from the definition of c_k that

$$\int_\Omega \left(\frac{1}{2}|\nabla w_k|^2 + \lambda_k^+ \max(w_k,0) - \lambda_k^- \min(w_k,0)\right) dx \geq c_k.$$

On the other hand, (1.8) gives

$$\int_\Omega \left(\frac{1}{2}|\nabla w_k|^2 + \lambda_k^+ \max(w_k,0) - \lambda_k^- \min(w_k,0)\right) dx \to c^*,$$

which implies that $c^* \geq c_0$. Finally, from the equality of the minima $c_0 = c = c^*$, we also deduce the strong convergence of u_k in $H^1(\Omega)$. □

Perturbation formula for the free boundary

In this section, we prove the continuity and differentiability of the map T. The case of one-phase obstacle problem was studied by Stojanovic [7].

Theorem 3.1. *Assume $\lambda^+, \lambda^- \in L^p(\Omega)$ for $p > \frac{n}{2}$. The map $(\lambda^+, \lambda^-) \mapsto u$ is Lipschitz continuous in the following sense. If u_i for $i = 1, 2$ solves*

$$\begin{cases} \Delta u_i = \lambda_i^+ \chi_{\{u_i>0\}} - \lambda_i^- \chi_{\{u_i<0\}} & \text{in } \Omega, \\ u_i = g & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

then

$$||u_2 - u_1||_{H^1(\Omega)} \leq C(||\lambda_1^+ - \lambda_2^+||_{H^{-1}(\Omega)} + ||\lambda_1^- - \lambda_2^-||_{H^{-1}(\Omega)}),$$

and for $p > \frac{n}{2}$

$$||u_2 - u_1||_{L^\infty(\Omega)} \leq C(||\lambda_1^+ - \lambda_2^+||_{L^p(\Omega)} + ||\lambda_1^- - \lambda_2^-||_{L^p(\Omega)}).$$

We first prove the following lemma:

Lemma 3.2. *If*

$$(\lambda_1^+ - \lambda_2^+) \leq \varepsilon \in L^p, \quad p > n/2, \quad \varepsilon \geq 0, \quad \text{and } \lambda_1^- = \lambda_2^-, \tag{1.10}$$

then

$$u_2 - u_1 \leq \delta \in L^\infty(\Omega),$$

where $\delta > 0$, $\delta \in W^{2,p} \cap H_0^1$ *solves*

$$\Delta\delta = -\varepsilon. \tag{1.11}$$

Moreover, the same argument can be applied with

$$(\lambda_2^- - \lambda_1^-) \leq \varepsilon \text{ and } \lambda_2^+ = \lambda_1^+. \tag{1.12}$$

Proof. Let

$$\lambda_3^+ = \lambda_1^+ \chi_{\{u_1>0\}}, \qquad \lambda_3^- = \lambda_1^- \chi_{\{u_1<0\}}, \tag{1.13}$$

$$\lambda_4^+ = \min\{\lambda_2^+, \lambda_3^+\}, \qquad \lambda_4^- = \lambda_3^-. \tag{1.14}$$

Then, by the same proof as in the first part of Lemma 2.2, one gets

$$u_3 = u_1, \quad u_4 \geq u_2,$$

where u_3 and u_4 solve Equation 1.2 with coefficients $\lambda_3^\pm$, $\lambda_4^\pm$, respectively. Relation (1.10) gives

$$(\lambda_3^+ - \lambda_4^+)\chi_{\{u_1>0\}} \leq \varepsilon. \tag{1.15}$$

Also, by the choice of λ_4^+, we have

$$\lambda_4^+\chi_{\{u_1\leq 0\}} = 0, \quad \lambda_4^-\chi_{\{u_1\geq 0\}} = 0. \tag{1.16}$$

We will show that

$$(u_4 - (u_3 + \delta))^+ = 0.$$

First, note that

$$\Delta u_4 = \lambda_4^+\chi_{\{u_4>0\}} - \lambda_4^-\chi_{\{u_4<0\}},$$
$$\Delta(u_3 + \delta) = \lambda_3^+\chi_{\{u_3>0\}} - \lambda_3^-\chi_{\{u_3<0\}} - \varepsilon.$$

Therefore,

$$\Delta(u_4 - (u_3 + \delta)) = \lambda_4^+\chi_{\{u_4>0\}} - \lambda_4^-\chi_{\{u_4<0\}} - \lambda_3^+\chi_{\{u_3>0\}} + \lambda_3^-\chi_{\{u_3<0\}} + \varepsilon.$$

Rearranging the above terms gives

$$\Delta(u_4 - (u_3 + \delta)) - \lambda_4^+ \chi_{\{u_4>0\}} + \lambda_4^+ \chi_{\{u_3>0\}} + \lambda_4^- \chi_{\{u_4<0\}} - \lambda_3^- \chi_{\{u_3<0\}}$$
$$= (\lambda_4^+ - \lambda_3^+)\chi_{\{u_3>0\}} + \varepsilon \geq 0.$$

Multiplying by $(u_4 -(u_3 + \delta))^+$ and integrating by parts gives

$$0 \leq \int_\Omega [(u_4 - (u_3 + \delta))^+ \Delta(u_4 - (u_3 + \delta))]\, dx$$
$$- \int_\Omega [\lambda_4^+(\chi_{\{u_4>0\}} - \chi_{\{u_3>0\}}) - \lambda_4^- \chi_{\{u_4<0\}} + \lambda_3^- \chi_{\{u_3<0\}}]\,(u_4 - (u_3 + \delta))^+ dx. \tag{1.17}$$

Then,

$$- \int_\Omega |\nabla((u_4 - (u_3 + \delta))^+|^2 dx$$
$$- \int_\Omega [\lambda_4^+(\chi_{\{u_4>0\}} - \chi_{\{u_3>0\}}) - \lambda_4^- \chi_{\{u_4<0\}} + \lambda_3^- \chi_{\{u_3<0\}}]\,(u_4 - (u_3 + \delta))^+ dx \geq 0.$$

It follows that

$$\int_\Omega |\nabla((u_4 - (u_3 + \delta))^+|^2 dx$$
$$+ \int_{\{u_4-(u_3+\delta)>0\}} [\lambda_4^+ \chi_{\{u_4>0\}} - \chi_{\{u_3>0\}} - \lambda_4^- \chi_{\{u_4<0\}} + \lambda_3^- \chi_{\{u_3<0\}}]\,(u_4 - (u_3 + \delta))\, dx \leq 0.$$

Note that

$$[\lambda_4^+(\chi_{\{u_4>0\}} - \chi_{\{u_3>0\}}) - \lambda_4^- \chi_{\{u_4<0\}} + \lambda_3^- \chi_{\{u_3<0\}}](u_4 - u_3) \geq 0.$$

Then, we have

$$\int_\Omega |\nabla((u_4 - (u_3 + \delta))^+|^2 dx$$
$$- \int_{\{u_4-(u_3+\delta)>0\}} [\lambda_4^+ \chi_{\{u_4>0\}} - \chi_{\{u_3>0\}} - \lambda_3^-(\chi_{\{u_4<0\}} - \chi_{\{u_3<0\}}]\,\delta\, dx \leq 0.$$

However,

$$\int_{\{u_4-(u_3+\delta)>0\}} [\lambda_4^+(\chi_{\{u_4>0\}} - \chi_{\{u_3>0\}}) - \lambda_4^-(\chi_{\{u_4<0\}} - \chi_{\{u_3<0\}})]\delta\, dx$$
$$= \int_{\{u_4-(u_3+\delta)>0\}} \lambda_4^+(\chi_{u_4>0}\chi_{u_3\leq 0})\delta\, dx - \int_{\{u_4-(u_3+\delta)>0\}} \lambda_4^-(\chi_{\{u_4<0\}}\chi_{\{u_3\geq 0\}})\delta\, dx = 0.$$

In the last equation, we have used (1.16).

□

Thus we completed the proof of Theorem 3.1.

Proof of Theorem 3.1. By elliptic regularity and Lemma 3.2, we have

$$\delta \in W^{2,p}_{1oc}(\Omega) \cap H^1_0(\Omega),$$

and, consequently, the Sobolev embedding $W^{2,p}_{loc} \hookrightarrow C^{0,\frac{n}{2p}}_{loc}$ for $p > \frac{n}{2}$, implies

$$\delta \in C^{0,\alpha}(\Omega), \text{ with } 0 < \alpha < 1.$$

Therefore,

$$||\delta||_{L^\infty} \leq C||\varepsilon||_{L^p}.$$

Now if we assume $|\lambda_1^+ - \lambda_2^+| \leq \varepsilon$, then it will follows that $|u_2 - u_1| < \delta$. To complete the proof, assume that

$$(\lambda_1^+ - \lambda_2^+) \leq \varepsilon \text{ and } (\lambda_2^- - \lambda_1^-) \leq \varepsilon.$$

Set $\Delta u' = \lambda_2^+ \chi_{\{u'>0\}} - \lambda_1^- \chi_{\{u'<0\}}$. Then, we have

$$u_2 - u_1 = u_2 - u' + u' - u_1 \leq 2\delta,$$

and

$$||u_2 - u_1||_{L^\infty} \leq ||u_2 - u'||_{L^\infty} + ||u' - u_1||_{L^\infty}.$$

By Equation 1.11, we obtain

$$||u_2 - u_1||_{L^\infty} \leq C(||\lambda_1^+ - \lambda_2^+||_{L^p} + ||\lambda_1^- - \lambda_2^-||_{L^p}).$$

□

The proof of Theorem 3.4 uses the following theorem, proved by I. Blank in [9].

Theorem 3.3. *(Linear Stability of the Free Boundary in the one phase case). Suppose that the free boundary is locally uniformly* $C^{1,\alpha}$ *regular in* B_1*. Let* w, w^ε *be the solutions of the following one-phase problems, respectively,*

$$\begin{cases} \Delta w = \lambda^+ \chi_{\{w>0\}} & in \quad B_1, \\ w = g & on \quad \partial B_1, \end{cases}$$

and

$$\begin{cases} \Delta w^\varepsilon = (\lambda^+ + \varepsilon)\chi_{\{w^\varepsilon>0\}} & in \quad B_1, \\ w^\varepsilon = g & on \quad \partial B_1. \end{cases}$$

Then, for ε *small enough, we have*

$$\operatorname{dist}(\Gamma(w), \Gamma(w^\varepsilon)) \leq C\varepsilon, \quad on \quad B_{\frac{1}{2}}. \tag{1.18}$$

Remark 3. The analogue of Theorem 3.3 can be proved for the two-phase membrane problem in the following cases:

(1) When all the points are regular one-phase points (cf. Theorem 3.3).
(2) When all the points are two-phase points with $|\nabla u| = 0$ (branching points).
(3) When $|\nabla u|$ is uniformly bounded from below (cf. Estimate 1.19).

Although we could not prove this theorem for the two-phase case in general, there are grounds, however, to suggest that it holds true in this case as well.

The proof of part (3) is as follows. Suppose $\varepsilon > 0$, $h_1 > 0$, $h_2 < 0$ and $\inf_\Omega \lambda^- > -\varepsilon h_2$. Then Lemma 2.2 implies that

$$-C\varepsilon + u^{\bar{\lambda}} \leq u^{\bar{\lambda}+\varepsilon\bar{h}} \leq u^{\bar{\lambda}}.$$

Also, $u^{\bar{\lambda}} \geq C' \operatorname{dist}(x, \Gamma''(u^{\bar{\lambda}}))$ for $x \in \Omega^+ \cap B_r$ where r is small enough, which gives

$$u^{\bar{\lambda}+\varepsilon\bar{h}} \geq -C\varepsilon + C' \operatorname{dist}(x, \Gamma''(u^{\bar{\lambda}})).$$

Thus, $u^{\bar{\lambda}+\varepsilon\bar{h}}$ is positive provided that $\operatorname{dist}(x, \Gamma''(u^{\bar{\lambda}})) \geq C''\varepsilon$, which shows

$$\operatorname{dist}(\Gamma''(u^{\bar{\lambda}}), \Gamma''(u^{\bar{\lambda}+\varepsilon\bar{h}})) \leq C_1\varepsilon. \tag{1.19}$$

Now we shall prove that the map $(\lambda^+, \lambda^-) \mapsto u^{\bar{\lambda}}$ is differentiable in the following sense:

Theorem 3.4. *The mapping*

$$T : C^{0,1}(\Omega) \times C^{0,1}(\Omega) \to W^{2,p}(\Omega),$$

defined by $u = T(\lambda^+, \lambda^-)$ *is differentiable. Furthermore, if* $\bar{\lambda}, \bar{h} \in C^{0,1}(\Omega) \times C^{0,1}(\Omega)$. *Then, there exists* $w^{\bar{\lambda},\bar{h}} \in H_0^1$, *such that*

$$\frac{u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}}}{\varepsilon} \rightharpoonup w^{\bar{\lambda},\bar{h}} \text{ in } H_0^1(\Omega) \text{ as } \varepsilon \to 0,$$

where

$$\Delta w^{\bar{\lambda},\bar{h}} = h_1\chi_{\{u^{\bar{\lambda}}>0\}} - h_2\chi_{\{u^{\bar{\lambda}}<0\}} + \frac{(\lambda^+ + \lambda^-)}{|\nabla u^{\bar{\lambda}}|} w^{\bar{\lambda},\bar{h}} \mathcal{H}^{n-1} \lfloor_{\Gamma''(u^{\bar{\lambda}})}. \tag{1.20}$$

In Equation 1.20, $\mathcal{H}^{n-1}$*denotes the* $(n - 1)$*-dimensional Hausdorff measure.*

Proof. We have

$$\Delta u^{\bar{\lambda}} = \lambda^+\chi_{\{u^{\bar{\lambda}}>0\}} - \lambda^-\chi_{\{u^{\bar{\lambda}}<0\}},$$

and

$$\Delta u^{\bar{\lambda}+\varepsilon\bar{h}} = (\lambda^+ + \varepsilon h_1)\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - (\lambda^- + \varepsilon h_2)\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}<0\}}.$$

Therefor,

$$\begin{aligned}\Delta(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}}) = \lambda^+(\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - \chi_{\{u^{\bar{\lambda}}>0\}}) + \lambda^-(\chi_{\{u^{\bar{\lambda}}<0\}} \\ -\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}<0\}}) + \varepsilon h_1\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - \varepsilon h_2\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}<0\}}.\end{aligned} \tag{1.21}$$

We multiply both sides of (1.21) by $(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}})$ and integrate by parts and we obtain

$$\begin{aligned}\int_\Omega |\nabla(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}})|^2 dx = &-\int_\Omega \lambda^+(\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - \chi_{\{u^{\bar{\lambda}}>0\}})(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}})\,dx \\ &+\int_\Omega \lambda^-(\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}<0\}} - \chi_{\{u^{\bar{\lambda}}<0\}})(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}})\,dx \\ &-\int_\Omega \varepsilon(h_1\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - h_2\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}<0\}})(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}})\,dx.\end{aligned}$$

Note that

$$(\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - \chi_{\{u^{\bar{\lambda}}>0\}})(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}}) \geq 0,$$

and

$$(\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}<0\}} - \chi_{\{u^{\bar{\lambda}}<0\}})(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}}) \leq 0.$$

Therefore,

$$\int_\Omega |\nabla(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}})|^2 dx \leq \int_\Omega \varepsilon(h_1\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - h_2\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}<0\}})(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}})\, dx.$$

The Hölder inequality implies

$$\begin{aligned}||\nabla(u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}})||^2_{L^2(\Omega)} &\leq \varepsilon||h_1\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - h_2\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}<0\}}||_{L^2(\Omega)}||u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}}||_{L^2(\Omega)} \\ &\leq \varepsilon(||h_1||_{L^2(\Omega)} + ||h_2||_{L^2(\Omega)})||u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}}||_{L^2(\Omega)}.\end{aligned}$$

Moreover, by the Poincaré inequality, we have

$$||\nabla(\frac{u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}}}{\varepsilon})||_{L^2(\Omega)} \leq C(||h_1||_{L^2(\Omega)} + ||h_2||_{L^2(\Omega)}). \tag{1.22}$$

From (1.22), the weak convergence to a limit, denoted by $w^{\bar{\lambda},\bar{h}}$, follows (for a subsequence). Here, we show that $w^{\bar{\lambda},\bar{h}}$ satisfies (1.20). Multiply (1.21) by a test function φ, where φ has compact support in $\{u^{\bar{\lambda}} > 0\}$, and then divide by ε,

$$-\int_\Omega \nabla(\frac{u^{\bar{\lambda}+\varepsilon\bar{h}} - u^{\bar{\lambda}}}{\varepsilon}) \cdot \nabla\phi dx = \int_\Omega \frac{\lambda^+}{\varepsilon}(\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - \chi_{\{u^{\bar{\lambda}}>0\}})\phi dx + \int_\Omega h_1\chi_{\{u^{\bar{\lambda}}>0\}}\phi\, dx. \tag{1.23}$$

Assume that d is the distance between supp(φ) and $\Gamma^+(u^{\bar{\lambda}})$. If $u^{\bar{\lambda}}(x) \geq cd^2$, then, (since $u^{\bar{\lambda}+\varepsilon\bar{h}} \to u^{\bar{\lambda}}$) for ε small enough, we have

$$|u^{\bar{\lambda}+\varepsilon\bar{h}}(x) - u^{\bar{\lambda}}(x)| \ \leq \frac{cd^2}{2},$$

and so $u^{\bar{\lambda}+\varepsilon\bar{h}}(x) \geq \frac{cd^2}{2} > 0$. This means that, for each φ, one can chose ε small enough such that

$$(\chi_{\{u^{\bar{\lambda}+\varepsilon\bar{h}}>0\}} - \chi_{\{u^{\bar{\lambda}}>0\}}) = 0 \text{ in supp } \phi.$$

In particular, passing to the limit in (1.23), we obtain that in the set $\{u^{\bar{\lambda}} > 0\}$, equation

$$\Delta w^{\bar{\lambda},\bar{h}} = h_1,$$

holds. Similarly, in the set $\{u^{\bar{\lambda}} > 0\}$, one has

$$\Delta w^{\bar{\lambda},\bar{h}} = -h_2.$$

Now let x_0 be a one-phase regular point for $u^{\bar{\lambda}}$ and $x_\varepsilon \in \Gamma(u^{\bar{\lambda}+\varepsilon\bar{h}})$ where x_ε has minimal distance to x_0.

Assumption In what follows, we assume that the estimate (1.18) in Theorem 3.3 also holds for one-phase points in our case. A straightforward calculation gives

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{u^{\bar\lambda+\varepsilon\bar h}(x_0)-u^{\bar\lambda}(x_0)}{\varepsilon}\\
&=\lim_{\varepsilon\to 0}\frac{u^{\bar\lambda+\varepsilon\bar h}(x_\varepsilon)+(x_\varepsilon-x_0)^T\cdot\nabla u^{\bar\lambda+\varepsilon\bar h}(x_\varepsilon)+O((x_\varepsilon-x_0)^2)-u^{\bar\lambda}(x_0)}{\varepsilon}\\
&=\lim_{\varepsilon\to 0}\frac{(x_\varepsilon-x_0)^T\cdot\nabla u^{\bar\lambda+\varepsilon\bar h}(x_\varepsilon)}{\varepsilon}=0,
\end{aligned}$$

which shows that $w^{\bar\lambda,\bar h}=0$ at one-phase regular points.

To complete the proof, let us assume that $x_0\in\Gamma''(u^{\bar\lambda})$. Let ν denote the normal to the free boundary $\Gamma''(u^{\bar\lambda})$ at x_0, that is $\nu=\frac{\nabla u(x_0)}{|\nabla u(x_0)|}$. Assume that $B_r(x_0)$ is a ball centered at x_0 where r is small enough. Since $\nabla u(x_0)\neq 0$, then $\Gamma''(u^{\bar\lambda})$ can be represented as $(x', f(x'))$ where f is a $C^{1,\alpha}$ graph. We have

$$\nu=e_n+O(r^\alpha). \tag{1.24}$$

Let Ω_ε be the region between $\Gamma''(u^{\bar\lambda})$ and $\Gamma''(u^{\bar\lambda+\varepsilon\bar h})$. From (1.21) we obtain

$$\Delta\Big(\frac{u^{\bar\lambda+\varepsilon\bar h}-u^{\bar\lambda}}{\varepsilon}\Big)=\frac{\lambda^++\lambda^-}{\varepsilon}\chi\,\Omega_\varepsilon+h_1\chi_{\{u^{\bar\lambda+\varepsilon\bar h}>0\}}-h_2\chi_{\{u^{\bar\lambda+\varepsilon\bar h}<0\}}.$$

The term $\frac{1}{\varepsilon}\chi_{\Omega_\varepsilon}$ converges weakly as $\varepsilon\to 0$, to a measure μ with support on $\Gamma''(u)$. For any ball $B_r(x_0)$ with $x_0\in\Gamma''(u)$, set

$$\mu(B_r)=\lim_{\varepsilon\to 0}\int_{B_r}\frac{1}{\varepsilon}\chi_{\Omega_\varepsilon}\,dx.$$

Estimate (1.19) shows that μ is a finite measure, since

$$\mu(B_r)=\lim_{\varepsilon\to 0}\int_{B_r\cap\Omega_\varepsilon}\frac{1}{\varepsilon}\,dx=\lim_{\varepsilon\to 0}\frac{|B_r\cap\Omega_\varepsilon|}{\varepsilon}\le C.$$

We want to prove that

$$\lim_{r\to 0}\frac{\mu(B_r(x_0))}{\mathcal{H}^{n-1}\lfloor_{\Gamma''(u)}(B_r(x_0))}=\frac{w^{\bar\lambda,\bar h}(x_0)}{|\nabla u^{\bar\lambda}(x_0)|}. \tag{1.25}$$

Then, μ can be written as (see [10], Chapter I)

$$\mu=\lim_{r\to 0}\frac{\mu(B_r)}{\mathcal{H}^{n-1}\lfloor_{\Gamma''(u^{\bar\lambda})}(B_r)}\cdot\mathcal{H}^{n-1}.$$

Let d be the distance of x_0 to $\Gamma''(u^{\bar\lambda+\varepsilon\bar h})$ in direction of ν, using Taylor expansion, we get

$$d=\frac{u^{\bar\lambda+\varepsilon\bar h}(x_0)}{|\nabla u^{\bar\lambda+\varepsilon\bar h}(x_\varepsilon)|}+O(\varepsilon). \tag{1.26}$$

In order to show (1.25), we have

$$\mu(B_r) = \lim_{\varepsilon\to 0}\int_{B_r}\frac{1}{\varepsilon}\chi_{\Omega_\varepsilon}dx = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}d\,|B_r^{'}| + O(r^{n-1}) = \qquad \text{by } (1.26)$$
$$= \lim_{\varepsilon\to 0}\frac{u^{\bar\lambda+\varepsilon\bar h}(x_0) - u^{\bar\lambda}(x_0)}{\varepsilon}\frac{1}{|\nabla u^{\bar\lambda+\varepsilon\bar h}(x_\varepsilon)|}\,|B_r^{'}| + O(r^{n-1})$$
$$= \frac{w^{\bar\lambda,\bar h}(x_0)}{|\nabla u^{\bar\lambda}(x_0)|}\,|B_r^{'}| + O(r^{n-1}),$$

where $|B_r'|$ is the measure of $B_r' = B_r \cap \{x_n = 0\}$. In addition, we have

$$\int_{B_r} d\mathcal{H}^{n-1}\lfloor_{\Gamma''(u^{\bar\lambda})} = \int_{B_r^{'}}\sqrt{1+|\nabla f|^2} = |B_r^{'}| + r^{n-1}O(r^{2\alpha}).$$

Therefore,

$$\lim_{r\to 0}\frac{\mu(B_r)}{\int_{B_r} d\mathcal{H}^{n-1}\lfloor_{\Gamma''(u^{\bar\lambda})}} = \lim_{r\to 0}\frac{\frac{w^{\bar\lambda,\bar h}(x_0)}{|\nabla u^{\bar\lambda}(x_0)|}|B_r^{'}|}{|B_r'|} = \frac{w^{\bar\lambda,\bar h}(x_0)}{|\nabla u^{\bar\lambda}(x_0)|}.$$

We deduce that, $w^{\bar\lambda,\bar h} \in H_0^1(\Omega)$ satisfies (1.20).

□

Remark 4. If for all free boundary points $\nabla u = 0$, which means that $\Gamma(u) = \Gamma'(u)$, then

$$w^{\bar\lambda,\bar h} = \begin{cases} \delta^{\bar\lambda,\bar h} \text{ in } \{u^{\bar\lambda} > 0\}, \\ 0 \quad\text{ in } \{u^{\bar\lambda} = 0\}, \\ \delta^{\bar\lambda,\bar h} \text{ in } \{u^{\bar\lambda} < 0\}, \end{cases}$$

where $\delta^{\bar\lambda,\bar h}$ is the unique solution of the elliptic equation

$$\begin{cases} \Delta\delta = h_1 & \text{in } \{u^{\bar\lambda} > 0\}, \\ \delta = 0 & \text{on } \partial\{u^{\bar\lambda} > 0\}, \\ \Delta\delta = -h_2 & \text{in } \{u^{\bar\lambda} < 0\}, \\ \delta = 0 & \text{on } \partial\{u^{\bar\lambda} < 0\}. \end{cases}$$

Remark 5. Consider the following two-phase problem in dimension one ($n = 1$), where λ_1, λ_2 are constants.

$$\begin{cases} u'' = \lambda_1\chi_{\{u>0\}} - \lambda_2\chi_{\{u<0\}} & \text{in}(-1,+1), \\ u(-1) = a < 0, \quad u(+1) = b > 0. \end{cases}$$

Straightforward calculations show that if $\sqrt{\frac{b-a}{\lambda_1}} + \sqrt{\frac{b-a}{\lambda_2}} \le 2$, then the set $\{x \in \Omega: u(x) = 0\}$ has a positive measure. In this setting, an interesting question is which conditions in higher dimensions will imply that the zero set has positive measure in B_1.

Example 1 Let $\bar\lambda = (4, 2)$, $\bar h = (1, 1)$. Consider the equation

$$\begin{cases} u'' = 4\chi_{\{u>0\}} - 2\chi_{\{u<0\}}, \\ u(+1) = +1, u(-1) = -1. \end{cases}$$

One can obtain

$$u^{\lambda+\varepsilon h}=\begin{cases}(2+\frac{\varepsilon}{2})x^2-(4+\varepsilon)(1-\sqrt{\frac{2}{4+\varepsilon}})x & 1-\sqrt{\frac{2}{4+\varepsilon}}\le x\le 1,\\ -(1+\frac{\varepsilon}{2})+(4+\varepsilon)(1-\sqrt{\frac{2}{4+\varepsilon}}) & \\ 0 & -1+\sqrt{\frac{2}{2+\varepsilon}}\le x\le 1-\sqrt{\frac{2}{4+\varepsilon}},\\ (-1-\frac{\varepsilon}{2})x^2-(2+\varepsilon)(-1+\sqrt{\frac{2}{2+\varepsilon}})x & -1\le x\le -1+\sqrt{\frac{2}{2+\varepsilon}}.\\ +(\frac{\varepsilon}{2})+\sqrt{\frac{2}{2+\varepsilon}}+(2+\varepsilon)(-1+\sqrt{\frac{2}{2+\varepsilon}}) & \end{cases}$$

Consequently, one computes

$$\lim_{\varepsilon\to 0}\frac{u^{\bar\lambda+\varepsilon\bar h}-u^{\bar\lambda}}{\varepsilon}=\begin{cases}\frac{x^2}{2}-\frac{1}{2} & 1-\frac{\sqrt{2}}{2}\le x\le 1,\\ 0 & 0\le x\le 1-\frac{\sqrt{2}}{2},\\ -(\frac{x^2}{2}+\frac{x}{2}) & -1\le x\le 0.\end{cases}$$

By Weiss [1], we know that the Hausdorff dimension of $\Gamma=\partial\{u>0\}\cup\partial\{u<0\}$ is less than or equal to n - 1 and by Edquist et al. [2] the regularity of the free boundary is C^1. Let $d\Gamma$ denote the measure $d\Gamma=\mathcal{H}^{n-1}\llcorner\Gamma$; the restriction of the (n - 1)-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on the set Γ. Moreover, let ν_1 be the unit normal exterior to $\partial\{u>0\}$ and ν_2 be the unit normal to $\partial\{u<0\}$ exterior to $\{u<0\}$.

Theorem 3.5. *Assume that the free boundary points are one-phase points, and let* δ *be the same as defined in Remark 4. Then, we have*

$$\frac{\chi_{\{u^{\bar\lambda+\varepsilon\bar h}>0\}}-\chi_{\{u^{\bar\lambda}>0\}}}{\varepsilon}\rightharpoonup-\frac{1}{\lambda+}\frac{\partial\delta}{\partial\nu_1}d\Gamma^+,$$

weakly in $H^{-1}(\Omega)$ *as* $\varepsilon\to 0$. *In addition*

$$\frac{\chi_{\{u^{\bar\lambda+\varepsilon\bar h}<0\}}-\chi_{\{u^{\bar\lambda}<0\}}}{\varepsilon}\rightharpoonup\frac{1}{\lambda^-}\frac{\partial\delta}{\partial\nu_2}d\Gamma^-.$$

Proof. To begin with, observe that

$$\Delta u^{\bar\lambda}=\lambda^+\chi_{\{u^{\bar\lambda}>0\}}-\lambda^-\chi_{\{u^{\bar\lambda}<0\}},$$

$$\Delta u^{\bar\lambda+\varepsilon\bar h}=(\lambda^++\varepsilon h_1)\chi_{\{u^{\bar\lambda+\varepsilon\bar h}>0\}}-(\lambda^-+\varepsilon h_2)\chi_{\{u^{\bar\lambda+\varepsilon\bar h}<0\}}.$$

Then, for a test function $\phi\in H_0^1(\Omega)$ one obtains

$$\begin{aligned}\int_\Omega\Delta(\frac{u^{\bar\lambda+\varepsilon\bar h}-u^{\bar\lambda}}{\varepsilon})\phi\,dx&=\int_\Omega h_1\chi_{\{u^{\bar\lambda+\varepsilon\bar h}>0\}}\phi\,dx-\int_\Omega h_2\chi_{\{u^{\bar\lambda+\varepsilon\bar h}<0\}}\phi\,dx\\&+\int_\Omega\frac{\lambda^+}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}>0\}}-\chi_{\{u^{\bar\lambda}>0\}})\phi\,dx-\int_\Omega\frac{\lambda^-}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}<0\}}-\chi_{\{u^{\bar\lambda}<0\}})\phi\,dx.\end{aligned}\qquad(1.27)$$

The left-hand side of Equation 1.27 is

$$\int_\Omega\Delta(\frac{u^{\bar\lambda+\varepsilon\bar h}-u^{\bar\lambda}}{\varepsilon})\phi\,dx=-\int_\Omega\nabla(\frac{u^{\bar\lambda+\varepsilon\bar h}-u^{\bar\lambda}}{\varepsilon})\nabla\phi\,dx.$$

Let $\varepsilon \to 0$, in (1.27); then, by the notations introduced in Remark 4, one has

$$-\int_\Omega \nabla\delta\nabla\phi\, dx = \int_\Omega h_1 \chi_{\{u^{\bar\lambda}>0\}}\phi\, dx - \int_\Omega h_2 \chi_{\{u^{\bar\lambda}<0\}}\phi\, dx$$
$$+\lim_{\varepsilon\to 0}\int_\Omega \frac{\lambda^+}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}>0\}} - \chi_{\{u^{\bar\lambda>0}\}})\phi\, dx - \lim_{\varepsilon\to 0}\int_\Omega \frac{\lambda^-}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}<0\}} - \chi_{\{u^{\bar\lambda}<0\}})\phi\, dx.$$

Integrating by parts gives

$$\begin{aligned}-\int_\Omega \nabla\delta\nabla\phi\, dx &= \int_{\{u^{\bar\lambda}>0\}} \Delta\delta\,\phi\, dx - \int_{\partial\{u^{\bar\lambda}>0\}} \frac{\partial\delta}{\partial\nu_1}\phi\, d\sigma\\
&+ \int_{\{u^{\bar\lambda}<0\}} \Delta\delta\,\phi\, dx - \int_{\partial\{u^{\bar\lambda}<0\}} \frac{\partial\delta}{\partial\nu_2}\phi\, d\sigma\\
&= \int_\Omega h_1 \chi_{\{u^{\bar\lambda}>0\}}\phi\, dx - \int_\Omega h_2 \chi_{\{u^{\bar\lambda}<0\}}\varphi\, dx\\
&+ \lim_{\varepsilon\to 0}\int_\Omega \frac{\lambda^+}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}>0\}} - \chi_{\{u^{\bar\lambda}>0\}})\phi\, dx\\
&- \lim_{\varepsilon\to 0}\int_\Omega \frac{\lambda^-}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}<0\}} - \chi_{\{u^{\bar\lambda}<0\}})\phi\, dx.\end{aligned}$$

In the view of Remark 4, we have

$$\lim_{\varepsilon\to 0}[\int_\Omega \frac{\lambda^+}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}>0\}} - \chi_{\{u^{\bar\lambda}>0\}})\phi\, dx - \int_\Omega \frac{\lambda^-}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}<0\}} - \chi_{\{u^{\bar\lambda}<0\}})\phi\, dx]$$
$$= -\int_{\partial\{u>0\}} \phi\frac{\partial\delta}{\partial\nu_1}d\sigma + \int_{\partial\{u>0\}} \phi\frac{\partial\delta}{\partial\nu_2}d\sigma.$$

Finally, we conclude that

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{\lambda^+}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}>0\}} - \chi_{\{u^{\bar\lambda}>0\}})\phi\, dx = -\int_{\partial\{u>0\}} \phi\frac{\partial\delta}{\partial\nu_1}d\sigma,$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{\lambda^-}{\varepsilon}(\chi_{\{u^{\bar\lambda+\varepsilon\bar h}<0\}} - \chi_{\{u^{\bar\lambda}<0\}})\phi\, dx = \int_{\partial\{u<0\}} \phi\frac{\partial\delta}{\partial\nu_2}d\sigma.$$

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