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Positive solutions of the three-point boundary value problem for fractional-order differential equations with an advanced argument

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Abstract

In this article, we consider the existence of at least one positive solution to the three-point boundary value problem for nonlinear fractional-order differential equation with an advanced argument

$$\begin{cases} {}^{C}D^{\alpha}u(t) + a(t)f(u(\theta(t))) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \quad \beta u(\eta) = u(1), \end{cases}$$

where $2 < \alpha \leq 3$, $0 < \eta < 1$, $0 < \beta < \dfrac{1}{\eta}$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative. Using the well-known Guo-Krasnoselskii fixed point theorem, sufficient conditions for the existence of at least one positive solution are established.

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1 Introduction

The study of three-point BVPs for nonlinear integer-order ordinary differential equations was initiated by Gupta [1]. Many authors since then considered the existence and multiplicity of solutions (or positive solutions) of three-point BVPs for nonlinear integer-order ordinary differential equations. To identify a few, we refer the reader to [2-13] and the references therein.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc. [14-17]. In fact, fractional-order models have proved to be more accurate than integer-order models, i.e., there are more degrees of freedom in the fractional-order models. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [18-36] and the references therein.

Differential equations with deviated arguments are found to be important mathematical tools for the better understanding of several real world problems in physics, mechanics, engineering, economics, etc. [37,38]. In fact, the theory of integer order differential equations with deviated arguments has found its extensive applications in realistic mathematical modelling of a wide variety of practical situations and has emerged as an important area of investigation. For the general theory and applications of integer order differential equations with deviated arguments, we refer the reader to the references [39-45].

As far as we know, fractional order differential equations with deviated arguments have not been much studied and many aspects of these equations are yet to be explored. For some recent work on equations of fractional order with deviated arguments, see [46-48] and the references therein. In this article, we consider the following three-point BVPs for nonlinear fractional-order differential equation with an advanced argument

$$\begin{cases} {}^{C}D^{\alpha}u(t) + a(t)f(u(\theta(t))) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \quad \beta u(\eta) = u(1), \end{cases} \tag{1.1}$$

where $2 < \alpha \le 3$, $0 < \eta < 1$, $0 < \beta < \frac{1}{\eta}$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative and $f: [0, \infty) \to [0, \infty)$ is a continuous function.

By a positive solution of (1.1), one means a function $u(t)$ that is positive on $0 < t < 1$ and satisfies (1.1).

Our purpose here is to give the existence of at least one positive solution to problem (1.1), assuming that

(H_1): $a \in C([0, 1], [0, \infty))$ and a does not vanish identically on any subinterval.

(H_2): The advanced argument $\theta \in C((0, 1), (0, 1))$ and $t \le \theta(t) \le 1$, $\forall t \in (0, 1)$.

Let $E = C[0, 1]$ be the Banach space endowed with the sup-norm. Set

$$f_0 = \lim_{u \to 0+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

The main results of this paper are as follows.

Theorem 1.1 *Assume that* (H_1) *and* (H_2) *hold. If* $f_0 = \infty$ *and* $f_\infty = 0$, *then problem* (1.1) *has at least one positive solution.*

Theorem 1.2 *Assume that* (H_1) *and* (H_2) *hold. If* $f_0 = \infty$ *and* $f_\infty = \infty$, *then problem* (1.1) *has at least one positive solution.*

Remark 1.1 *It is worth mentioning that the conditions of our theorems are easily to verify, so they are applicable to a variety of problems, see Examples 4.1 and 4.2.*

The proof of our main results is based upon the following well-known Guo-Krasnoselskii fixed point theorem:

Theorem 1.3 [49] *Let E be a Banach space, and let* $P \subset E$ *be a cone. Assume that* Ω_1, Ω_2 *are open subsets of E with* $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, *and let* $T: P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ *be a completely continuous operator such that*

(i) $||Tu|| \ge ||u||$, $u \in P \cap \partial\Omega_1$, *and* $||Tu|| \le ||u||$, $u \in P \cap \partial\Omega_2$; *or*

(ii) $||Tu|| \le ||u||$, $u \in P \cap \partial\Omega_1$, *and* $||Tu|| \ge ||u||$, $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

2 Preliminaries

For the reader's convenience, we present some necessary definitions from fractional calculus theory and Lemmas.

Definition 2.1 *For a function* $f: [0, \infty) \to \mathbb{R}$, *the Caputo derivative of fractional order* α *is defined as*

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s, \quad n-1 < \alpha < n,\ n = [\alpha] + 1,$$

where $[\alpha]$ *denotes the integer part of real number* α.

Definition 2.2 *The Riemann-Liouville fractional integral of order* α *is defined as*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s, \quad \alpha > 0,$$

provided the integral exists.

Definition 2.3 *The Riemann-Liouville fractional derivative of order* α *for a function* $f(t)$ *is defined by*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) \mathrm{d}s, \quad n = [\alpha] + 1,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 [15] *Let* $\alpha > 0$, *then fractional differential equation*

$$^{C}D^{\alpha} u(t) = 0$$

has solution

$$u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1},\ C_i \in \mathbb{R}, \quad i = 0, 1, 2,\ \ldots,\ n-1,$$

where n is the smallest integer greater than or equal to α.

Lemma 2.2 [15] *Let* $\alpha > 0$, *then*

$$I^{\alpha C} D^{\alpha} u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}$$

for some $C_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, N-1$, *where* N *is the smallest integer greater than or equal to* α.

Lemma 2.3 *Let* $2 < \alpha \leq 3$, $1 \neq \beta\eta$. *Assume* $y(t) \in C[0, 1]$, *then the following problem*

$$^{C}D^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u''(0) = 0, \quad \beta u(\eta) = u(1), \tag{2.2}$$

has a unique solution

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \mathrm{d}s + \frac{1}{1-\beta\eta} \int_0^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, \mathrm{d}s - \frac{\beta}{1-\beta\eta} \int_0^{\eta} \frac{t(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, \mathrm{d}s.$$

Proof. We may apply Lemma 2.2 to reduce Equation (2.1) to an equivalent integral equation

$$u(t) = -I^{\alpha} y(t) - b_1 - b_2 t - b_3 t^2 = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, \mathrm{d}s - b_1 - b_2 t - b_3 t^2, \tag{2.3}$$

for some $b_1, b_2, b_3 \in \mathbb{R}$.

In view of the relation ${}^{C}D^{\alpha}\, I^{\alpha}u(t) = u(t)$ and $I^{\alpha}I^{\beta}u(t) = I^{\alpha+\beta}u(t)$ for α, β >0, we can get that

$$u'(t) = -\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)\mathrm{d}s - b_2 - 2b_3t.$$

$$u''(t) = -\int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)\mathrm{d}s - 2b_3.$$

By $u(0) = u''(0) = 0$, it follows $b_1 = b_3 = 0$. Then by the condition $\beta u(\eta) = u(1)$, we have

$$-b_2 = \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \frac{\beta}{1-\beta\eta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,\mathrm{d}s.$$

Combine with (2.3), we get

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + \frac{1}{1-\beta\eta}\int_0^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \frac{\beta}{1-\beta\eta}\int_0^\eta \frac{t(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,\mathrm{d}s.$$

This complete the proof.

Lemma 2.4 *Let* $2 < \alpha \le 3$, $0 < \beta < \frac{1}{\eta}$. *Assume* $y \in C([0,1],[0,\infty))$, *then the unique solution u of* (2.1) *and* (2.2) *satisfies* $u(t) \ge 0$, $\forall t \in [0,1]$.

Proof. By Lemma 2.3, we know that $u''(t) = -\int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}y(s)\mathrm{d}s \le 0$. It means that the graph of $u(t)$ is concave down on (0, 1).

In addition,

$$\begin{aligned}
u(1) &= -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s + \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \frac{\beta}{1-\beta\eta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s \\
&= \frac{\beta\eta}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \frac{\beta}{1-\beta\eta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s \\
&\ge \frac{\beta}{1-\beta\eta}\left[\int_0^\eta \frac{\eta(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s - \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\mathrm{d}s\right] \\
&= \frac{\beta}{(1-\beta\eta)\Gamma(\alpha)}\int_0^\eta \left[\eta(1-s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right]y(s)\mathrm{d}s \\
&\ge \frac{\beta}{(1-\beta\eta)\Gamma(\alpha)}\int_0^\eta \left[(\eta-\eta s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right]y(s)\mathrm{d}s \ge 0
\end{aligned}$$

Combine with $u(0) = 0$, it follows $u(t) \ge 0$, $\forall t \in [0,1]$.

Lemma 2.5 *Let* $2 < \alpha \le 3$, $0 < \beta < \frac{1}{\eta}$. *Assume* $y \in C([0,1],[0,\infty))$, *then the unique solution u of* (2.1) *and* (2.2) *satisfies*

$$\inf_{t\in[\eta,1]} u(t) \geq \gamma||u||,$$

where $\gamma = \min\{\beta\eta, \dfrac{\beta(1-\eta)}{1-\beta\eta}, \eta\}$.

Proof. Note that $u''(t) \leq 0$, by applying the concavity of u, the proof is easy. So we omit it.

3 Proofs of main theorems

Define the operator $T : C[0, 1] \to C[0, 1]$ as follows,

$$Tu(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s + \frac{1}{1-\beta\eta}\int_0^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\ - \frac{\beta}{1-\beta\eta}\int_0^\eta \frac{t(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))ds. \tag{3.1}$$

Then the problem (1.1) has a solution if and only if the operator T has a fixed point.

Define the cone $P = \{u | u \in C[0,1],\ u \geq 0,\ \inf_{t\in[\eta,1]} u(\theta(t)) \geq \gamma||u||\}$, where $\gamma = \min\{\beta\eta, \dfrac{\beta(1-\eta)}{1-\beta\eta}, \eta\}$.

Proof of Theorem 1.1. The operator T is completely continuous. Obviously, T is continuous.

Let $\Omega \subset C[0, 1]$ be bounded, then there exists a constant $K > 0$ such that $||a(t) f(u(\theta(t))|| \leq K$, $\forall u \in \Omega$. Thus, we have

$$\begin{aligned} Tu(t) &\leq \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\ &\leq \frac{K}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s \\ &= \frac{K}{(1-\beta\eta)\Gamma(\alpha+1)}, \end{aligned}$$

which implies $||Tu|| \leq \dfrac{K}{(1-\beta\eta)\Gamma(\alpha+1)}$.

On the other hand, we have

$$\begin{aligned} |(Tu)'(t)| &\leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} a(s)f(u(\theta(s)))\mathrm{d}s + \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\ &\quad + \frac{\beta}{1-\beta\eta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\ &\leq K\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\mathrm{d}s + \frac{K}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s + \frac{K\beta}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s \\ &= \frac{K}{\Gamma(\alpha)} + \frac{(1+\beta)K}{(1-\beta\eta)\Gamma(\alpha+1)} := M. \end{aligned}$$

Hence, for each $u \in \Omega$, let $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, we have

$$|(Tu)(t_2) - (Tu)(t_1)| \le \int_{t_1}^{t_2} \left|(Tu)'(s)\right| \mathrm{d}s \le M(t_2 - t_1).$$

So, T is equicontinuous. The Arzela-Ascoli Theorem implies that $T : C[0, 1] \to C[0, 1]$ is completely continuous.

Since $t \le \theta(t) \le 1$, $t \in (0, 1)$, then

$$\inf_{t\in[\eta,1]} u(\theta(t)) \ge \inf_{t\in[\eta,1]} u(t) \ge \gamma ||u||. \tag{3.2}$$

Thus, Lemmas 2.5 and 3.2 show that $TP \subset P$. Then, $T : P \to P$ is completely continuous.

In view of $f_0 = \infty$, there exists a constant $\rho_1 > 0$ such that $f(u) \ge \delta_1 u$ for $0 < u < \rho_1$, where $\delta_1 > 0$ satisfies

$$\frac{\eta\delta_1\gamma}{(1-\beta\eta)} \int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)\mathrm{d}s \ge 1. \tag{3.3}$$

Take $u \in P$, such that $||u|| = \rho_1$. Then, we have

$$\begin{aligned}
||Tu|| &\ge Tu(\eta) \\
&= -\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s + \frac{1}{1-\beta\eta}\int_0^1 \frac{\eta(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\
&\quad - \frac{\beta}{1-\beta\eta}\int_0^\eta \frac{\eta(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\
&= -\frac{1}{1-\beta\eta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s + \frac{\eta}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\
&= -\frac{1}{1-\beta\eta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s + \frac{\eta}{1-\beta\eta}\int_0^\eta \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\
&\quad + \frac{\eta}{1-\beta\eta}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\
&\ge -\frac{1}{1-\beta\eta}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s + \frac{1}{1-\beta\eta}\int_0^\eta \frac{(\eta-\eta s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\
&\quad + \frac{\eta}{1-\beta\eta}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\
&\ge \frac{\eta}{1-\beta\eta}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)f(u(\theta(s)))\mathrm{d}s \\
&\ge \frac{\eta}{1-\beta\eta}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)\delta_1 u(\theta(s))\mathrm{d}s \\
&\ge \frac{\eta}{1-\beta\eta}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)\delta_1\gamma ||u||\mathrm{d}s \\
&= \frac{\eta\delta_1\gamma}{(1-\beta\eta)}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} a(s)\mathrm{d}s||u|| \ \ge \ ||u||.
\end{aligned} \tag{3.4}$$

Let $\Omega_{\rho 1} = \{u \in C\,0[1] \mid ||u|| < \rho_1\}$. Thus, (3.4) shows $||Tu|| \ge ||u||$, $u \in P \cap \partial\Omega_{\rho 1}$.

Next, in view of $f_\infty = 0$, there exists a constant $R > \rho_1$ such that $f(u) \le \delta_2 u$ for $u \ge R$, where $\delta_2 > 0$ satisfies

$$\frac{\delta_2}{(1-\beta\eta)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\mathrm{d}s \le 1. \tag{3.5}$$

We consider the following two cases.

Case one. f is bounded, which implies that there exists a constant $r_1 > 0$ such that $f(u) \le r_1$ for $u \in [0, \infty)$. Now, we may choose $u \in P$ such that $||u|| = \rho_2$, where $\rho_2 \ge \max\{\mu, R\}$.

Then we have

$$\begin{aligned}
Tu(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)f(u(\theta(s)))\mathrm{d}s + \frac{1}{1-\beta\eta}\int_0^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)f(u(\theta(s)))\mathrm{d}s \\
&\quad - \frac{\beta}{1-\beta\eta}\int_0^\eta \frac{t(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)f(u(\theta(s)))\mathrm{d}s \\
&\le \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)f(u(\theta(s)))\mathrm{d}s \\
&\le \frac{r_1}{(1-\beta\eta)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\mathrm{d}s \\
&\triangleq \mu \le \rho_2 = ||u||.
\end{aligned}$$

Case two. f is unbounded, which implies then there exists a constant $\rho_2 > \frac{R}{\gamma} > R$ such that $f(u) \le f(\rho_2)$ for $0 < u \le \rho_2$ (note that $f \in C([0, \infty), [0, \infty))$. Let $u \in P$ such that $||u|| = \rho_2$, we have

$$\begin{aligned}
Tu(t) &\le \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)f(u(\theta(s)))\mathrm{d}s \\
&\le \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)f(\rho_2)\mathrm{d}s \\
&\le \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\delta_2\rho_2\mathrm{d}s \\
&= \frac{\delta_2}{(1-\beta\eta)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\mathrm{d}s||u|| \\
&\le ||u||.
\end{aligned}$$

Hence, in either case, we may always let $\Omega_{\rho 2} = \{u \in C[0, 1] \mid ||u|| < \rho_2\}$ such that $||Tu|| \le ||u||$ for $u \in P \cap \partial\Omega_{\rho 2}$.

Thus, by the first part of Guo-Krasnoselskii fixed point theorem, we can conclude that (1.1) has at least one positive solution.

Proof of Theorem 1.2. Now, in view of $f_0 = 0$, there exists a constant $r_1 > 0$ such that $f(u) \le \tau_1 u$ for $0 < u < r_1$, where $\tau_1 > 0$ satisfies

$$\frac{\tau_1}{(1-\beta\eta)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\mathrm{d}s \le 1. \tag{3.6}$$

Take $u \in P$, such that $||u|| = r_1$. Then, we have

$$\begin{aligned} Tu\,(t) &\le \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)f(u(\theta(s)))\mathrm{d}s \\ &\le \frac{1}{1-\beta\eta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\tau_1 u(\theta(s))\mathrm{d}s \\ &\le \frac{\tau_1}{(1-\beta\eta)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\mathrm{d}s||u|| \\ &\le ||u||. \end{aligned} \tag{3.7}$$

Let $\Omega_1 = \{u \in C[0, 1] \mid ||u|| < r_1\}$. Thus, (3.7) shows $||Tu|| \le ||u||$, $u \in P \cap \partial\Omega_1$.

Next, in view of $f_\infty = \infty$, there exists a constant $r_2 > r_1$ such that $f(u) \ge \tau_2 u$ for $u \ge r_2$, where $\tau_2 > 0$ satisfies

$$\frac{\tau_2\eta\gamma}{(1-\beta\eta)}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\mathrm{d}s \ge 1. \tag{3.8}$$

Let $\Omega_2 = \{u \in C[0, 1] \mid ||u|| < \rho_2\}$,where $\rho_2 > \dfrac{r_2}{\gamma} > r_2$, then, $u \in P$ and $||u|| = \rho_2$ implies

$$\inf_{t\in[\eta,1]} u(\theta(t)) \ge \gamma||u|| > r_2,$$

and so

$$\begin{aligned} ||Tu|| &\ge Tu(\eta) \\ &\ge \frac{\eta}{1-\beta\eta}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)f(u(\theta(s)))\mathrm{d}s \\ &\ge \frac{\eta}{1-\beta\eta}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\tau_2 u(\theta(s))\mathrm{d}s \\ &\ge \frac{\eta}{1-\beta\eta}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\tau_2\gamma||u||\mathrm{d}s \\ &= \frac{\tau_2\eta\gamma}{(1-\beta\eta)}\int_\eta^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}a(s)\mathrm{d}s||u|| \ \ge \ ||u||. \end{aligned}$$

This shows that $||Tu|| \ge ||u||$ for $u \in P \cap \partial\Omega_2$.

Therefore, by the second part of Guo-Krasnoselskii fixed point theorem, we can conclude that (1.1) has at least one positive solution $u \in P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

4 Examples

Example 4.1 *Consider the fractional differential equation*

$$\begin{cases} {}^{C}D^{\alpha}u(t) + e^{-t}f(u(\theta(t))) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \quad \beta u(\eta) = u(1), \end{cases} \tag{4.1}$$

where $2 < \alpha \leq 3$, $0 < \eta < 1$, $0 < \beta < \frac{1}{\eta}$, $\theta(t) = t^{\nu}$, $0 < \nu < 1$ *and*

$$f(u) = \begin{cases} \dfrac{\sin u}{u^2}, & 0 \leq u \leq \dfrac{\pi}{2}, \\ \dfrac{4\sqrt{2u} + \cos u}{\pi^{\frac{5}{2}}}, & u > \dfrac{\pi}{2}. \end{cases}$$

Note that conditions (H_1) and (H_2) of Theorem 1.1 hold. Through a simple calculation we can get $f_0 = \infty$ and $f_\infty = 0$. Thus, by Theorem 1.1, we can get that the problem (4.1) has at least one positive solution.

Example 4.2 *Consider the fractional differential equation*

$$\begin{cases} {}^{C}D^{\alpha}u(t) + a(t)f(u(\theta(t))) = 0, \; 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad \beta u(\eta) = u(1), \end{cases} \tag{4.2}$$

where $2 < \alpha \leq 3$, $0 < \eta < 1$, $0 < \beta < \frac{1}{\eta}$, $\theta(t) = \sqrt{t}$, $a(t) = e^{\tan t}$ *and*

$$f(u) = u^{\frac{3}{2}} \ln(1 + u) + u^{3+\sin u}.$$

Obviously, it is not difficult to verify conditions (H_1) and (H_2) of Theorem 1.2 hold. Through a simple calculation we can get $f_0 = 0$ and $f_\infty = \infty$. Thus, by Theorem 1.2, we can get that the problem (4.2) has at least one positive solution.

Remark 4.1 *In the above two examples,* α, β, η *could be any constants which satisfy* $2 < \alpha \leq 3$, $0 < \eta < 1$, $0 < \beta < \frac{1}{\eta}$. *For example, we can take* $\alpha = 2.5$, $\eta = 0.5$, $\beta = 1.5$.

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Authors' contributions
GW completed the main part of this paper, SKN and LZ corrected the main theorems and gave two examples. All authors read and approved the final manuscript.

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The authors declare that they have no competing interests.

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