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Abstract fractional integro-differential equations involving nonlocal initial conditions in α-norm

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Abstract

In the present paper, we deal with the Cauchy problems of abstract fractional integro-differential equations involving nonlocal initial conditions in α-norm, where the operator A in the linear part is the generator of a compact analytic semigroup. New criterions, ensuring the existence of mild solutions, are established. The results are obtained by using the theory of operator families associated with the function of Wright type and the semigroup generated by A, Krasnoselkii's fixed point theorem and Schauder's fixed point theorem. An application to a fractional partial integro-differential equation with nonlocal initial condition is also considered.

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1 Introduction

Let $(A, D(A))$ be the infinitesimal generator of a compact analytic semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$ on a real Banach space $(X, ||\cdot||)$ and $0 \in \rho(A)$. Denote by X_α, the Banach space $D(A^\alpha)$ endowed with the graph norm $||u||_\alpha = ||A^\alpha u||$ for $u \in X_\alpha$. The present paper concerns the study of the Cauchy problem for abstract fractional integro-differential equation involving nonlocal initial condition

$$\begin{cases} {}^cD_t^\beta u(t) = Au(t) + F(t, u(t), u(\kappa_1(t))) \\ \qquad\qquad + \int_0^t K(t-s)G(s, u(s), u(\kappa_2(s)))\mathrm{d}s, \quad t \in [0, T], \\ u(0) + H(u) = u_0 \end{cases} \quad (1.1)$$

in X_α, where ${}^cD_t^\beta$, $0 < \beta < 1$, stands for the Caputo fractional derivative of order β, and $K : [0, T] \to \mathbb{R}^+$, $\kappa_1, \kappa_2 : [0, T] \to [0, T]$, $F, G : [0, T] \times X_\alpha \times X_\alpha \to X$, $H : C([0, T]; X_\alpha) \to X_\alpha$ are given functions to be specified later. As can be seen, H constitutes a nonlocal condition.

The fractional calculus that allows us to consider integration and differentiation of any order, not necessarily integer, has been the object of extensive study for analyzing not only anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [1-3] and references therein), but also fractional phenomena in optimal control (see, e.g., [4-6]). As indicated in [2,5,7] and the related references given there, the advantages of fractional

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derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields. One of the emerging branches of the study is the Cauchy problems of abstract differential equations involving fractional derivatives in time. In recent decades, there has been a lot of interest in this type of problems, its applications and various generalizations (cf. e.g., [8-11] and references therein). It is significant to study this class of problems, because, in this way, one is more realistic to describe the memory and hereditary properties of various materials and processes (cf. [4,5,12,13]).

In particular, much interest has developed regarding the abstract fractional Cauchy problems involving nonlocal initial conditions. For example, by using the fractional power of operators and some fixed point theorems, the authors studied the existence of mild solutions in [14] for fractional differential equations with nonlocal initial conditions and in [15] for fractional neutral differential equations with nonlocal initial conditions and time delays. The existence of mild solutions for fractional differential equations with nonlocal initial conditions in α-norm using the contraction mapping principle and the Schauder's fixed point theorem have been investigated in [16].

We here mention that the abstract problem with nonlocal initial condition was first considered by Byszewski [17], and the importance of nonlocal initial conditions in different fields has been discussed in [18,19] and the references therein. Deng [19], especially, gave the following nonlocal initial values: $H(u) = \sum_{i=1}^{p} C_i u(t_i)$, where C_i ($i = 1$, ..., p) are given constants and $0 < t_1 < \cdots < t_{p-1} < t_p < +\infty$ ($p \in \mathbb{N}$), which is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In the past several years theorems about existence, uniqueness and stability of Cauchy problem for abstract evolution equations with nonlocal initial conditions have been studied by many authors, see for instance [19-28] and references therein.

In this paper, we will study the existence of mild solutions for the fractional Cauchy problem (1.1). New criterions are established. Both Krasnoselkii's fixed point theorem and Schauder's fixed point theorem, and the theory of operator families associated with the function of Wright type and the semigroup generated by A, are employed in our approach. The results obtained are generalizations and continuation of the recent results on this issue.

The paper is organized as follows. In Section 2, some required notations, definitions and lemmas are given. In Section 3, we present our main results and their proofs.

2 Preliminaries

In this section, we introduce some notations, definitions and preliminary facts which are used throughout this work.

We first recall some definitions of fractional calculus (see e.g., [6,13] for more details).

Definition 2.1 *The Riemann-Liouville fractional integral operator of order $\beta > 0$ of function f is defined as*

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, *where* $\Gamma(\cdot)$ *is the gamma function.*

Definition 2.2 *The Caputo fractional derivative of order* $\beta > 0$, $m - 1 < \beta < m$, $m \in \mathbb{N}$, *is defined as*

$$^{c}D^{\beta}f(t) = I^{m-\beta}D_{t}^{m}f(t) = \frac{1}{\Gamma(m-\beta)} \int_{0}^{t} (t-s)^{m-\beta-1}D_{s}^{m}f(s)\mathrm{d}s,$$

where $D_{t}^{m} := \frac{d^{m}}{dt^{m}}$ *and* f *is an abstract function with value in X. If* $0 < \beta < 1$, *then*

$$^{c}D^{\beta}f(t) = \frac{1}{\Gamma(1-\beta)} \int_{0}^{t} \frac{f'(s)}{(t-s)^{\beta}} \mathrm{d}s.$$

Throughout this paper, we let $A : D(A) \to X$ be the infinitesimal generator of a compact analytic semigroup of bounded linear operators $\{T(t)\}_{t \geq 0}$ on X and $0 \in \rho(A)$, which allows us to define the fractional power A^{α} for $0 \leq \alpha < 1$, as a closed linear operator on its domain $D(A^{\alpha})$ with inverse $A^{-\alpha}$.

Let X_{α} denote the Banach space $D(A^{\alpha})$ endowed with the graph norm $||u||_{\alpha} = ||A^{\alpha}u||$ for $u \in X_{\alpha}$ and let $C([0, T];X_{\alpha})$ be the Banach space of all continuous functions from $[0, T]$ into X_{α} with the uniform norm topology

$$|u|_{\alpha} = \sup\{\| u(t)\|_{\alpha}, t \in [0, T]\}.$$

$\mathcal{L}(X)$ stands for the Banach space of all linear and bounded operators on X. Let M be a constant such that

$$M = \sup\{\| T(t)\|_{\mathcal{L}(X)}, \ t \in [0, \infty)\}.$$

For $k > 0$, write

$$\Theta_{k} = \{u \in C([0, T]; X_{\alpha}); |u|_{\alpha} \leq k\}.$$

The following are basic properties of A^{α}.

Theorem 2.1 ([29], pp. 69-75)).

(a) $T(t) : X \to X_{\alpha}$ *for each* $t > 0$, *and* $A^{\alpha}T(t)x = T(t)A^{\alpha}x$ *for each* $x \in X_{\alpha}$ *and* $t \geq 0$.
(b) $A^{\alpha}T(t)$ *is bounded on X for every* $t > 0$ *and there exist* $M_{\alpha} > 0$ *and* $\delta > 0$ *such that*

$$||A^{\alpha}T(t)||_{\mathcal{L}(X)} \leq \frac{M_{\alpha}}{t^{\alpha}}e^{-\delta t}.$$

(c) $A^{-\alpha}$ *is a bounded linear operator in X with* $D(A^{\alpha}) = Im(A^{-\alpha})$.
(d) *If* $0 < \alpha_{1} \leq \alpha_{2}$, *then* $X_{\alpha_{2}} \hookrightarrow X_{\alpha_{1}}$

Lemma 2.1. [27] *The restriction of* $T(t)$ *to* X_{α} *is exactly the part of* $T(t)$ *in* X_{α} *and is an immediately compact semigroup in* X_{α}, *and hence it is immediately norm-continuous.*

Define two families $\{\mathcal{S}_\beta(t)\}_{t\geq 0}$ and $\{\mathcal{P}_\beta(t)\}_{t\geq 0}$ of linear operators by

$$\mathcal{S}_\beta(t)x = \int_0^\infty \Psi_\beta(s)T(t^\beta s)x\,ds, \quad \mathcal{P}_\beta(t)x = \int_0^\infty \beta s \Psi_\beta(s)T(t^\beta s)x\,ds$$

for $x \in X$, $t \geq 0$, where

$$\Psi_\beta(s) = \frac{1}{\pi\beta}\sum_{n=1}^\infty (-s)^{n-1}\frac{\Gamma(1+\beta n)}{n!}\sin(n\pi\beta), \quad s \in (0,\infty)$$

is the function of Wright type defined on $(0, \infty)$ which satisfies

$$\Psi_\beta(s) \geq 0, \quad s \in (0,\infty), \qquad \int_0^\infty \Psi_\beta(s)ds = 1, \;\; and$$
$$\int_0^\infty s^\zeta \Psi_\beta(s)ds = \frac{\Gamma(1+\zeta)}{\Gamma(1+\beta\zeta)}, \quad \zeta \in (-1,\infty). \tag{2.1}$$

The following lemma follows from the results in [15].

Lemma 2.2. *The following properties hold:*

(1) *For every $t \geq 0$, $\mathcal{S}_\beta(t)$ and $\mathcal{P}_\beta(t)$ are linear and bounded operators on X, i.e.,*

$$\parallel \mathcal{S}_\beta(t)x \parallel \leq M \parallel x \parallel, \quad \parallel \mathcal{P}_\beta(t)x \parallel \leq \frac{\beta M}{\Gamma(1+\beta)}\parallel x \parallel$$

for all $x \in X$ and $0 \leq t < \infty$.
(2) *For every $x \in X$, $t \to \mathcal{S}_\beta(t)x$, $t \to \mathcal{P}_\beta(t)x$ are continuous functions from $[0, \infty)$ into X.*
(3) *$\mathcal{S}_\beta(t)$ and $\mathcal{P}_\beta(t)$ are compact operators on X for $t > 0$.*
(4) *For all $x \in X$ and $t \in (0, \infty)$, $\parallel A^\alpha \mathcal{P}_\beta(t)x \parallel \leq C_\alpha t^{-\alpha\beta}\parallel x \parallel$, where $C_\alpha = \frac{M_\alpha \beta \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}$.*

We can also prove the following criterion.

Lemma 2.3. *The functions $t \to A^\alpha \mathcal{P}_\beta(t)$ and $t \to A^\alpha \mathcal{S}_\beta(t)$ are continuous in the uniform operator topology on $(0, +\infty)$.*

Proof. Let $\varepsilon > 0$ be given. For every $r > 0$, from (2.1), we may choose $\delta_1, \delta_2 > 0$ such that

$$\frac{M_\alpha}{r^{\alpha\beta}}\int_0^{\delta_1}\Psi_\beta(s)s^{-\alpha}ds \leq \frac{\varepsilon}{6}, \quad \frac{M_\alpha}{r^{\alpha\beta}}\int_{\delta_2}^\infty \Psi_\beta(s)s^{-\alpha}ds \leq \frac{\varepsilon}{6}. \tag{2.2}$$

Then, we deduce, in view of the fact $t \to A^\alpha T(t)$ that is continuous in the uniform operator topology on $(0, \infty)$ (see [[30], Lemma 2.1]), that there exists a constant $\delta >$ such that

$$\int_{\delta_1}^{\delta_2}\Psi_\beta(s)\left\|A^\alpha T\left(t_1^\beta s\right) - A^\alpha T\left(t_2^\beta s\right)\right\|_{\mathcal{L}(X)}ds \leq \frac{\varepsilon}{3}, \tag{2.3}$$

for $t_1, t_2 \geq r$ and $|t_1 - t_2| < \delta$.

On the other hand, for any $x \in X$, we write

$$
\begin{aligned}
\mathcal{S}_\beta(t_1)x - \mathcal{S}_\beta(t_2)x = \quad & \int_0^{\delta_1} \Psi_\beta(s) \left(T\left(t_1^\beta s\right)x - T\left(t_2^\beta s\right)x \right) ds \\
& + \int_{\delta_1}^{\delta_2} \Psi_\beta(s) \left(T\left(t_1^\beta s\right)x - T\left(t_2^\beta s\right)x \right) ds \\
& + \int_{\delta_2}^{\infty} \Psi_\beta(s)(T(t_1^\beta s)x - T(t_2^\beta s)x)ds.
\end{aligned}
$$

Therefore, using (2.2, 2.3) and Lemma 2.2, we get

$$
\begin{aligned}
& \left\| A^\alpha \mathcal{S}_\beta(t_1)x - A^\alpha \mathcal{S}_\beta(t_2)x \right\| \\
& \leq \int_0^{\delta_1} \Psi_\beta(s) \left(\left\| A^\alpha T\left(t_1^\beta s\right) \right\|_{\mathcal{L}(X)} + \left\| A^\alpha T\left(t_2^\beta s\right) \right\|_{\mathcal{L}(X)} \right) \| x \| \, ds \\
& \quad + \int_{\delta_1}^{\delta_2} \Psi_\beta(s) \| A^\alpha T\left(t_1^\beta s\right) - A^\alpha T\left(t_2^\beta s\right) \|_{\mathcal{L}(X)} \| x \| \, ds \\
& \quad + \int_{\delta_2}^{\infty} \Psi_\beta(s) \left(\left\| A^\alpha T\left(t_1^\beta s\right) \right\|_{\mathcal{L}(X)} + \left\| A^\alpha T\left(t_2^\beta s\right) \right\|_{\mathcal{L}(X)} \right) \| x \| \, ds \\
& \leq \frac{2M_\alpha}{r^{\alpha\beta}} \int_0^{\delta_1} \Psi_\beta(s)s^{-\alpha} \| x \| \, ds \\
& \quad + \int_{\delta_1}^{\delta_2} \Psi_\beta(s) \left\| T\left(t_1^\beta s\right) - T\left(t_2^\beta s\right) \right\|_{\mathcal{L}(X)} \| x \| \, ds \\
& \quad + \frac{2M_\alpha}{r^{\alpha\beta}} \int_{\delta_2}^{\infty} \Psi_\beta(s)s^{-\alpha} \| x \| \, ds \\
& \leq \varepsilon \| x \|,
\end{aligned}
$$

that is,

$$
\| A^\alpha \mathcal{S}_\beta(t_1) - A^\alpha \mathcal{S}_\beta(t_2) \| \leq \varepsilon, \quad \text{for } t_1, t_2 \geq r \text{ and } |t_1 - t_2| < \delta
$$

which together with the arbitrariness of $r > 0$ implies that $A^\alpha \mathcal{P}_\beta(t)$ is continuous in the uniform operator topology for $t > 0$. A similar argument enable us to give the characterization of continuity on $A^\alpha \mathcal{P}_\beta(t)$. This completes the proof. ∎

Lemma 2.4. *For every $t > 0$, the restriction of $\mathcal{S}_\beta(t)$ to X_α and the restriction of $\mathcal{P}_\beta(t)$ to X_α are compact operators in X_α.*

Proof. First consider the restriction of $\mathcal{S}_\beta(t)$ to X_α. For any $r > 0$ and $t > 0$, it is sufficient to show that the set $\{\mathcal{S}_\beta(t)u; u \in B_r\}$ is relatively compact in X_α, where $B_r := \{u \in X_\alpha; \|u\|_\alpha \leq r\}$.

Since by Lemma 2.1, the restriction of $T(t)$ to X_α is compact for $t > 0$ in X_α, for each $t > 0$ and $\varepsilon \in (0, t)$,

$$
\begin{aligned}
& \left\{ \int_\varepsilon^{\infty} \Psi_\beta(s)T\left(t^\beta s\right) u \, ds; u \in B_r \right\} \\
& = \left\{ T\left(t^\beta \varepsilon\right) \int_\varepsilon^{\infty} \Psi_\beta(s)T\left(t^\beta s - t^\beta \varepsilon\right) u \, ds; u \in B_r \right\}
\end{aligned}
$$

is relatively compact in X_α. Also, for every $u \in B_r$, as

$$\int_\varepsilon^\infty \Psi_\beta(s) T\left(t^\beta s\right) u \, ds \to \mathcal{S}_\beta(t)u, \quad \varepsilon \to 0$$

in X_α, we conclude, using the total boundedness, that the set $\{\mathcal{S}_\beta(t)u; u \in B_r\}$ is relatively compact, which implies that the restriction of $\mathcal{S}_\beta(t)$ to X_α is compact. The same idea can be used to prove that the restriction of $\mathcal{P}_\beta(t)$ to X_α is also compact. ∎

The following fixed point theorems play a key role in the proofs of our main results, which can be found in many books.

Lemma 2.5 (Krasnoselskii's Fixed Point Theorem). *Let E be a Banach space and B be a bounded closed and convex subset of E, and let F_1, F_2 be maps of B into E such that $F_1 x + F_2 y \in B$ for every pair x, $y \in B$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1 x + F_2 x = x$ has a solution on B.*

Lemma 2.6 (Schauder Fixed Point Theorem). *If B is a closed bounded and convex subset of a Banach space E and $F : B \to B$ is completely continuous, then F has a fixed point in B.*

3 Main results

Based on the work in [[15], Lemma 3.1 and Definition 3.1], in this paper, we adopt the following definition of mild solution of Cauchy problem (1.1).

Definition 3.1. *By a mild solution of Cauchy problem* (1.1), *we mean a function* $u \in C([0, T]; X_\alpha)$ *satisfying*

$$u(t) = \mathcal{S}_\beta(t)(u_0 - H(u)) + \int_0^t (t-s)^{\beta-1} \mathcal{P}_\beta(t-s)(F(s, u(s), u(\kappa_1(s)))$$
$$+ \int_0^s K(s-\tau)G(\tau, u(\tau), u(\kappa_2(\tau)))d\tau)ds$$

for $t \in [0, T]$.

Let us first introduce our basic assumptions.

(H_0) $\kappa_1, \kappa_2 \in C([0, T]; [0, T])$ and $K \in C([0, T]; \mathbb{R}^+)$.

(H_1) $F, G : [0, T] \times X_\alpha \times X_\alpha \to X$ are continuous and for each positive number $k \in \mathbb{N}$, there exist a constant $\gamma \in [0, \beta(1 - \alpha))$ and functions $\phi_k(\cdot) \in L^{1/\gamma}(0, T; \mathbb{R}^+)$, $\varphi_k(\cdot) \in L^\infty(0, T; \mathbb{R}^+)$ such that

$$\sup_{\|u\|_\alpha, \|v\|_\alpha \leq k} \| F(t, u, v) \| \leq \varphi_k(t) \quad and \quad \liminf_{k \to +\infty} \frac{\| \varphi_k \|_{L^{1/\gamma}(0,T)}}{k} = \sigma_1 < \infty,$$

$$\sup_{\|u\|_\alpha, \|v\|_\alpha \leq k} \| G(t, u, v) \| \leq \phi_k(t) \quad and \quad \liminf_{k \to +\infty} \frac{\| \phi_k \|_{L^\infty(0,T)}}{k} = \sigma_2 < \infty.$$

(H_2) $F, G : [0, T] \times X_\alpha \times X_\alpha \to X$ are continuous and there exist constants L_F, L_K such that

$$\| F(t, u_1, v_1) - F(t, u_2, v_2) \| \leq L_F(\| u_1 - u_2 \|_\alpha + \| v_1 - v_2 \|_\alpha),$$
$$\| G(t, u_1, v_1) - G(t, u_2, v_2) \| \leq L_G(\| u_1 - u_2 \|_\alpha + \| v_1 - v_2 \|_\alpha)$$

for all $(t, u_1, v_1), (t, u_2, v_2) \in [0, T] \times X_\alpha \times X_\alpha$.

(H_3) $H : C([0, T]; X_\alpha) \to X_\alpha$ is Lipschitz continuous with Lipschitz constant L_H.

(H_4) $H : C([0, T]; X_\alpha) \to X_\alpha$ is continuous and there is a $\eta \in (0, T)$ such that for any $u, w \in C([0, T]; X_\alpha)$ satisfying $u(t) = w(t)(t \in [\eta, T])$, $H(u) = H(w)$.

(H_5) There exists a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $u \in \Theta_k$,

$$\| H(u) \|_\alpha \leq \Phi(k), \quad and \quad \liminf_{k \to +\infty} \frac{\Phi(k)}{k} = \mu < \infty.$$

Remark 3.1. *Let us note that (H_4) is the case when the values of the solution $u(t)$ for t near zero do not affect $H(u)$. We refer to* [19]*for a case in point.*

In the sequel, we set $\widetilde{k} := \int_0^T K(t) dt$. We are now ready to state our main results in this section.

Theorem 3.1. *Let the assumptions (H_0), (H_1) and (H_3) be satisfied. Then, for $u_0 \in X_\alpha$, the fractional Cauchy problem* (1.1) *has at least one mild solution provided that*

$$ML_H + C_\alpha \sigma_1 T^{(1-\alpha)\beta - \gamma} \left(\frac{1 - \gamma}{(1-\alpha)\beta - \gamma} \right)^{1-\gamma} + \frac{C_\alpha \sigma_2 \widetilde{k} T^{(1-\alpha)\beta}}{(1-\alpha)\beta} < 1. \tag{3.1}$$

Proof. Let $v \in C([0, T]; X_\alpha)$ be fixed with $|v|_\alpha \equiv 0$. From (3.1) and (H_1), it is easy to see that there exists a $k_0 > 0$ such that

$$M(\| u_0 \|_\alpha + L_H k_0 + \| H(v) \|_\alpha) + C_\alpha \left(\frac{1 - \gamma}{(1-\alpha)\beta - \gamma} \right)^{1-\gamma} T^{(1-\alpha)\beta - \gamma} \| \varphi_{k_0} \|_{L^{1/\gamma}(0,T)}$$
$$+ \frac{C_\alpha \widetilde{k} T^{(1-\alpha)\beta}}{(1-\alpha)\beta} \| \phi_{k_0} \|_{L^\infty(0,T)} \leq k_0.$$

Consider a mapping Γ defined on Θ_{k_0} by

$$(\Gamma u)(t) = \mathcal{S}_\beta(t)(u_0 - H(u)) + \int_0^t (t-s)^{\beta-1} \mathcal{P}_\beta(t-s) \left(F(s, u(s), u(\kappa_1(s))) \right.$$
$$\left. + \int_0^s K(s - \tau) G(\tau, u(\tau), u(\kappa_2(\tau))) d\tau \right) ds$$
$$:= (\Gamma_1 u)(t) + (\Gamma_2 u)(t), \quad t \in [0, T].$$

It is easy to verify that $(\Gamma u)(\cdot) \in C([0, T]; X_\alpha)$ for every $u \in \Theta_{k_0}$. Moreover, for every pair $v, u \in \Theta_{k_0}$ and $t \in [0, T]$, by (H_1) a direct calculation yields

$$\| (\Gamma_1 v)(t) + (\Gamma_2 u)(t) \|_\alpha$$
$$\leq \| \mathcal{S}_\beta(t)(u_0 - H(v)) \|_\alpha + \int_0^t (t-s)^{\beta-1} \| A^\alpha \mathcal{P}_\beta(t-s) \|_{L(X)} \Big\| F(s, u(s), u(\kappa_1(s)))$$
$$+ \int_0^s K(s - \tau) G(\tau, u(\tau), u(\kappa_2(\tau))) d\tau \Big\| ds$$
$$\leq M(\| u_0 \|_\alpha + L_H k_0 + \| H(v) \|_\alpha)$$
$$+ C_\alpha \int_0^t (t-s)^{\beta(1-\alpha)-1} (\varphi_{k_0}(s) + \widetilde{k} \| \phi_{k_0} \|_{L^\infty(0,T)}) ds$$
$$\leq M(\| u_0 \|_\alpha + L_H k_0 + \| H(v) \|_\alpha)$$
$$+ C_\alpha \left(\frac{1 - \gamma}{(1-\alpha)\beta - \gamma} \right)^{1-\gamma} T^{(1-\alpha)\beta - \gamma} \| \varphi_{k_0} \|_{L^{1/\gamma}(0,T)} \frac{C_\alpha \widetilde{k} T^{(1-\alpha)\beta}}{(1-\alpha)\beta} \| \phi_{k_0} \|_{L^\infty(0,T)}$$
$$\leq k_0.$$

That is, $\Gamma_1 v + \Gamma_2 u \in \Theta_{k_0}$ for every pair $v, u \in \Theta_{k_0}$. Therefore, the fractional Cauchy problem (1.1) has a mild solution if and only if the operator equation $\Gamma_1 u + \Gamma_2 u = u$ has a solution in Θ_{k_0}.

In what follows, we will show that Γ_1 and Γ_2 satisfy the conditions of Lemma 2.5. From (H_3) and (3.1), we infer that Γ_1 is a contraction. Next, we show that Γ_2 is completely continuous on Ω_{k_0}.

We first prove that Γ_2 is continuous on Θ_{k_0}. Let $\{u_n\}_{n=1}^{\infty} \subset \Theta_{k_0}$ be a sequence such that $u_n \to u$ as $n \to \infty$ in $C([0, T]; X_\alpha)$. Therefore, it follows from the continuity of F, G, κ_1 and κ_2 that for each $t \in [0, T]$,

$$F(t, u_n(t), u_n(\kappa_1(t))) \to F(t, u(t), u(\kappa_1(t))) \quad \text{as } n \to \infty,$$
$$G(t, u_n(t), u_n(\kappa_1(t))) \to G(t, u(t), u(\kappa_2(t))) \quad \text{as } n \to \infty.$$

Also, by (H_1), we see

$$\int_0^t (t - s)^{\beta - 1 - \alpha\beta} \parallel F(s, u_n(s), u_n(\kappa_1(s))) - F(s, u(s), u(\kappa_1(s))) \parallel \mathrm{d}s$$
$$\leq 2 \int_0^t (t - s)^{\beta - 1 - \alpha\beta} \varphi_{k_0}(s) \mathrm{d}s$$
$$\leq 2 \left(\frac{1 - \gamma}{(1 - \alpha)\beta - \gamma} \right)^{1-\gamma} T^{(1-\alpha)\beta - \gamma} \parallel \varphi_{k_0} \parallel_{L^{1/\gamma}(0,T)},$$

and

$$\int_0^t (t - s)^{\beta - 1 - \alpha\beta} \int_0^s K(s - \tau) \parallel G(\tau, u_n(\tau), u_n(\kappa_2(\tau)))$$
$$- G(\tau, u(\tau), u(\kappa_2(\tau))) \parallel \mathrm{d}\tau \mathrm{d}s$$
$$\leq 2 \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)} \int_0^t (t - s)^{\beta - 1 - \alpha\beta} \mathrm{d}s$$
$$\leq \frac{2 \widetilde{k} T^{(1-\alpha)\beta}}{(1 - \alpha)\beta} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}.$$

Hence, as

$$\parallel (\Gamma_2 u_n)(t) - (\Gamma_2 u)(t) \parallel_\alpha$$
$$\leq C_\alpha \int_0^t (t - s)^{\beta - 1 - \alpha\beta} \parallel F(s, u_n(s), u_n(\kappa_1(s))) - F(s, u(s), u(\kappa_1(s))) \parallel \mathrm{d}s$$
$$+ C_\alpha \int_0^t (t - s)^{\beta - 1 - \alpha\beta} \int_0^s K(s - \tau) \parallel G(\tau, u_n(\tau), u_n(\kappa_2(\tau)))$$
$$- G(\tau, u(\tau), u(\kappa_2(\tau))) \parallel \mathrm{d}\tau \mathrm{d}s,$$

we conclude, using the Lebesgue dominated convergence theorem, that for all $t \in [0, T]$,

$$\parallel (\Gamma_2 u_n)(t) - (\Gamma_2 u)(t) \parallel_\alpha \to 0, \quad \text{as } n \to \infty,$$

which implies that

$$|\Gamma_2 u_n - \Gamma_2 u|_\alpha \to 0, \quad \text{as } n \to \infty.$$

This proves that Γ_2 is continuous on Θ_{k_0}.

It suffice to prove that Γ_2 is compact on Ω_{k_0}. For the sake of brevity, we write

$$\mathcal{N}(t, u(t)) = F(t, u(t), u(\kappa_1(t))) + \int_0^t K(t - \tau) G(\tau, u(\tau), u(\kappa_2(\tau))) \mathrm{d}\tau.$$

Let $t \in [0, T]$ be fixed and $\varepsilon, \varepsilon_1 > 0$ be small enough. For $u \in \Omega_{k_0}$, we define the map $\Gamma^{\varepsilon, \varepsilon_1}$ by

$$(\Gamma^{\varepsilon, \varepsilon_1} u)(t) = \int_0^{t-\varepsilon} \int_{\varepsilon_1}^{\infty} \beta \tau \Psi_\beta(\tau) T((t-s)^\beta \tau) \mathcal{N}(s, u(s)) d\tau ds$$

$$= T(\varepsilon^\beta \varepsilon_1) \int_0^{t-\varepsilon} \int_{\varepsilon_1}^{\infty} \beta \tau \Psi_\beta(\tau) T((t-s)^\beta \tau - \varepsilon^\beta \varepsilon_1) \mathcal{N}(s, u(s)) d\tau ds.$$

Therefore, from Lemma 2.1 we see that for each $t \in (0, T]$, the set $\{\Gamma^{\varepsilon, \varepsilon_1} u)(t); u \in \Omega_{k_0}\}$ is relatively compact in X_α. Then, as

$$\| (\Gamma_2 u)(t) - (\Gamma^{\varepsilon, \varepsilon_1} u)(t) \|_\alpha$$

$$\leq \left\| \int_0^t \int_0^{\varepsilon_1} \beta \tau (t-s)^{\beta-1} \Psi_\beta(\tau) T((t-s)^\beta \tau) \mathcal{N}(s, u(s)) d\tau ds \right\|_\alpha$$

$$+ \left\| \int_{t-\varepsilon}^t \int_{\varepsilon_1}^{\infty} \beta \tau (t-s)^{\beta-1} \Psi_\beta(\tau) T((t-s)^\beta \tau) \mathcal{N}(s, u(s)) d\tau ds \right\|_\alpha$$

$$\leq \beta M_\alpha \left[\int_0^t (t-s)^{\beta(1-\alpha)-1} (\varphi_{k_0}(s) + \tilde{k} \| \phi_{k_0} \|_{L^\infty(0,T)}) ds \int_0^{\varepsilon_1} \tau^{1-\alpha} \Psi_\beta(\tau) d\tau \right.$$

$$\left. + \int_{t-\varepsilon}^t (t-s)^{\beta(1-\alpha)-1} (\varphi_{k_0}(s) + \tilde{k} \| \phi_{k_0} \|_{L^\infty(0,T)}) ds \int_{\varepsilon_1}^{\infty} \tau^{1-\alpha} \Psi_\beta(\tau) d\tau \right]$$

$$\leq \beta M_\alpha \left[\left(\frac{1-\gamma}{(1-\alpha)\beta - \gamma} \right)^{1-\gamma} T^{(1-\alpha)\beta-\gamma} \| \varphi_{k_0} \|_{L^{1/\gamma}(0,T)} \right.$$

$$+ \frac{\tilde{k} T^{(1-\alpha)\beta}}{(1-\alpha)\beta} \| \phi_{k_0} \|_{L^\infty(0,T)} \left] \int_0^{\varepsilon_1} \tau^{1-\alpha} \Psi_\beta(\tau) d\tau \right.$$

$$+ \frac{\beta M_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} \left[\left(\frac{1-\gamma}{(1-\alpha)\beta - \gamma} \right)^{1-\gamma} \| \varphi_{k_0} \|_{L^{1/\gamma}(0,T)} \varepsilon^{(1-\alpha)\beta-\gamma} \right.$$

$$+ \frac{\tilde{k}}{(1-\alpha)\beta} \| \phi_{k_0} \|_{L^\infty(0,T)} \varepsilon^{(1-\alpha)\beta} \right]$$

$$\to 0 \quad as \; \varepsilon, \varepsilon_1 \to 0^+$$

in view of (2.1), we conclude, using the total boundedness, that for each $t \in [0, T]$, the set $\{\Gamma_2 u)(t); u \in \Omega_{k_0}\}$ is relatively compact in X_α.

On the other hand, for $0 < t_1 < t_2 \leq T$ and $\varepsilon' > 0$ small enough, we have

$$\| (\Gamma_2 u)(t_1) - (\Gamma_2 u)(t_2) \|_\alpha \leq A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = \int_{t_1}^{t_2} (t_2 - s)^{\beta-1-\alpha\beta} \| \mathcal{N}(s, u(s)) \| ds,$$

$$A_2 = \int_0^{t_1-\varepsilon'} (t_1 - s)^{\beta-1} \| A^\alpha \mathcal{P}_\beta(t_2 - s) - A^\alpha \mathcal{P}_\beta(t_1 - s) \|_{\mathcal{L}(X)} \| \mathcal{N}(s, u(s)) \| ds,$$

$$A_3 = \int_{t_1-\varepsilon'}^{t_1} (t_1 - s)^{\beta-1} \left((t_2 - s)^{-\alpha\beta} + (t_1 - s)^{-\alpha\beta} \right) \| \mathcal{N}(s, u(s)) \| ds,$$

$$A_4 = \int_0^{t_1} \left| (t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1} \right| \cdot (t_2 - s)^{-\alpha\beta} \| \mathcal{N}(s, u(s)) \| ds.$$

Therefore, it follows from (H_1), Lemma 2.2, and Lemma 2.3 that

$$
\begin{aligned}
A_1 &\leq C_\alpha \int_{t_1}^{t_2} (t_2 - s)^{\beta - 1 - \alpha\beta} (\varphi_{k_0}(s) + \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}) ds \\
&\leq C_\alpha \left(\frac{1 - \gamma}{(1 - \alpha)\beta - \gamma} \right)^{1-\gamma} \parallel \varphi_{k_0} \parallel_{L^{1/\gamma}(0,T)} (t_2 - t_1)^{(1-\alpha)\beta - \gamma} \\
&\quad + \frac{C_\alpha \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}}{(1 - \alpha)\beta} (t_2 - t_1)^{(1-\alpha)\beta}, \\
A_2 &\leq \sup_{s \in [0, t_1 - \varepsilon']} \parallel A^\alpha \mathcal{P}_\alpha(t_2 - s) - A^\alpha \mathcal{P}_\alpha(t_1 - s) \parallel_{\mathcal{L}(X)} \\
&\quad \times \int_0^{t_1 - \varepsilon'} (t_1 - s)^{\beta - 1} (\varphi_{k_0}(s) + \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}) ds \\
&\leq \left[\left(\frac{1 - \gamma}{\beta - \gamma} \right)^{1-\gamma} \parallel \varphi_{k_0} \parallel_{L^{1/\gamma}(0,T)} \left(t_1^{\frac{\beta - \gamma}{1 - \gamma}} - \varepsilon'^{\frac{\beta - \gamma}{1 - \gamma}} \right)^{1-\gamma} \right. \\
&\quad \left. + \frac{\widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}}{\beta} \left(t_1^\beta - \varepsilon'^\beta \right) \right] \\
&\quad \times \sup_{s \in [0, t_1 - \varepsilon']} \parallel A^\alpha \mathcal{P}_\alpha(t_2 - s) - A^\alpha \mathcal{P}_\alpha(t_1 - s) \parallel_{\mathcal{L}(X)}, \\
A_3 &\leq C_\alpha \int_{t_1 - \varepsilon'}^{t_1} (t_1 - s)^{\beta - 1} \left((t_2 - s)^{-\alpha\beta} + (t_1 - s)^{-\alpha\beta} \right) \\
&\quad \times (\varphi_{k_0}(s) + \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}) ds \\
&\leq 2 C_\alpha \int_{t_1 - \varepsilon'}^{t_1} (t_1 - s)^{\beta - 1 - \alpha\beta} (\varphi_{k_0}(s) + \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}) ds \\
&\leq C_\alpha \left(\frac{1 - \gamma}{(1 - \alpha)\beta - \gamma} \right)^{1-\gamma} \parallel \varphi_{k_0} \parallel_{L^{1/\gamma}(0,T)} \varepsilon'^{(1-\alpha)\beta - \gamma} \\
&\quad + \frac{C_\alpha \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}}{(1 - \alpha)\beta} \varepsilon'^{(1-\alpha)\beta}, \\
A_4 &\leq C_\alpha \int_0^{t_1} ((t_1 - s)^{\beta - 1} - (t_2 - s)^{\beta - 1})(t_2 - s)^{-\alpha\beta} \\
&\quad \times (\varphi_{k_0}(s) + \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}) ds \\
&\leq C_\alpha \int_0^{t_1} ((t_1 - s)^{(1-\alpha)\beta - 1} - (t_2 - s)^{(1-\alpha)\beta - 1})(\varphi_{k_0}(s) + \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}) ds \\
&\leq C_\alpha \left(\frac{1 - \gamma}{(1 - \alpha)\beta - \gamma} \right)^{1-\gamma} \parallel \varphi_{k_0} \parallel_{L^{1/\gamma}(0,T)} \\
&\quad \times \left[t_1^{(1-\alpha)\beta - \gamma} - \left(t_2^{\frac{(1-\alpha)\beta - \gamma}{1 - \gamma}} - (t_2 - t_1)^{\frac{(1-\alpha)\beta - \gamma}{1 - \gamma}} \right)^{1-\gamma} \right] \\
&\quad + \frac{2\widetilde{k}}{(1 - \alpha)\beta} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)} \left(t_1^{(1-\alpha)\beta} - t_2^{(1-\alpha)\beta} + (t_2 - t_1)^{(1-\alpha)\beta} \right),
\end{aligned}
$$

from which it is easy to see that A_i $(i = 1, 2, 3, 4)$ tends to zero independently of $u \in \Omega_{k_0}$ as $t_2 - t_1 \to 0$ and $\varepsilon' \to 0$. Hence, we can conclude that

$$
\parallel (\Gamma_2 u)(t_1) - (\Gamma_2 u)(t_2) \parallel_\alpha \to 0, \quad as\ t_2 - t_1 \to 0,
$$

and the limit is independently of $u \in \Omega_{k_0}$.

For the case when $0 = t_1 < t_2 \leq T$, since

$$
\begin{aligned}
&\parallel (\Gamma_2 u)(t_1) - (\Gamma_2 u)(t_2) \parallel_\alpha \\
&= \left\Vert \int_0^{t_2} (t_2 - s)^{\beta - 1} \mathcal{P}_\beta(t_2 - s) \mathcal{N}(s, u(s)) ds \right\Vert_\alpha \\
&\leq C_\alpha \int_0^{t_2} (t_2 - s)^{\beta - 1 - \alpha\beta} (\varphi_{k_0}(s) + \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}) ds \\
&\leq C_\alpha \left(\frac{1 - \gamma}{(1 - \alpha)\beta - \gamma} \right)^{1-\gamma} \parallel \varphi_k \parallel_{L^{1/\gamma}(0,T)} t_2^{(1-\alpha)\beta - \gamma} + \frac{C_\alpha \widetilde{k} \parallel \phi_{k_0} \parallel_{L^\infty(0,T)}}{(1 - \alpha)\beta} t_2^{(1-\alpha)\beta}.
\end{aligned}
$$

$\parallel (\Gamma_2 u)(t_1) - (\Gamma_2 u)(t_2) \parallel_\alpha$ can be made small when t_2 is small independently of $u \in \Omega_{k_0}$. Consequently, the set $\{(\Gamma_2)(\cdot); \cdot \in [0, T], u \in \Omega_{k_0}\}$ is equicontinuous. Now applying the Arzela-Ascoli theorem, it follows that Γ_2 is compact on Ω_{k_0}.

Therefore, applying Lemma 2.5, we conclude that Γ has a fixed point, which gives rise to a mild solution of Cauchy problem (1.1). This completes the proof. ∎

The second result of this paper is the following theorem.

Theorem 3.2. *Let the assumptions* (H_0), (H_2), (H_4) *and* (H_5) *be satisfied. Then, for* u_0 $\in X_\alpha$, *the fractional Cauchy problem* (1.1) *has at least one mild solution provided that*

$$M\mu + \frac{2C_\alpha T^{(1-\alpha)\beta}(L_F + \tilde{k}L_G)}{(1-\alpha)\beta} < 1. \tag{3.2}$$

Proof. The proof is divided into the following two steps.

Step 1. Assume that $w \in C([\eta, T]; X_\alpha)$ is fixed and set

$$\tilde{w}(t) = \begin{cases} w(t), & t \in [\eta, T], \\ w(\eta), & t \in [0, \eta]. \end{cases}$$

It is clear that $w \in C([0, T]; X_\alpha)$. We define a mapping Γ_w on $C([0, T]; X_\alpha)$ by

$$(\Gamma_w u)(t) = \mathcal{S}_\beta(t)(u_0 - H(\tilde{w})) + \int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)(F(s, u(s), u(\kappa_1(s))))$$

$$+ \int_0^s K(s-\tau)G(\tau, u(\tau), u(\kappa_2(\tau)))d\tau)ds, \quad t \in [0, T].$$

Clearly, $(\Gamma_w u)(\cdot) \in C([0, T]; X_\alpha)$ for every $u \in C([0, T]; X_\alpha)$. Moreover, for $u \in \Theta_k$, from (H_2), it follows that

$$\begin{aligned}
&\| (\Gamma_w u)(t)\|_\alpha \\
&\leq \| \mathcal{S}_\beta(t)(u_0 - H(\tilde{w}))\|_\alpha \\
&\quad + \int_0^t (t-s)^{\beta-1} \| \mathcal{P}_\beta(t-s)\left(F(s, u(s), u(\kappa_1(s)))\right) \\
&\qquad + \int_0^s K(s-\tau)G(\tau, u(\tau), u(\kappa_2(\tau)))d\tau)\|_\alpha ds \\
&\leq M(\| u_0\|_\alpha + \| H(\tilde{w})\|_\alpha) \\
&\quad + C_\alpha \int_0^t (t-s)^{\beta(1-\alpha)-1}\left[L_F(\| u(s)\|_\alpha + \| u(\kappa_1(s))\|_\alpha) + \| F(s, v, v)\| \right. \\
&\qquad \left. + L_G \int_0^s K(s-\tau)(\| u(\tau)\|_\alpha + \| u(\kappa_2(\tau))\|_\alpha + \| G(s, v, v)\|)d\tau\right]ds \\
&\leq M(\| u_0\|_\alpha + \| H(\tilde{w})\|_\alpha) + \frac{2kC_\alpha(L_F + \tilde{k}L_G)t^{(1-\alpha)\beta}}{(1-\alpha)\beta} \\
&\quad + \frac{C_\alpha\left(\max_{0\leq s\leq T}\| F(s, v, v)\| + \tilde{k}\max_{0\leq s\leq T}\| G(s, v, v)\|\right)T^{(1-\alpha)\beta}}{(1-\alpha)\beta},
\end{aligned}$$

where $v \in C([0, T]; X_\alpha)$ is fixed with $|v|_\alpha \equiv 0$, which implies that there exists a integer $k_0 > 0$ such that Γ_w maps Θ_{k_0} into itself. In fact, if this is not the case, then for each $k > 0$, there would exist $u_k \in \Theta_k$ and $t_k \in [0, T]$ such that $||(\Gamma_w u_k)(t_k)||_\alpha > k$. Thus, we have

$$\begin{aligned}
k < \| (\Gamma_w u_k)(t_k)\|_\alpha &\leq M(\| u_0\|_\alpha + \| H(\tilde{u})\|_\alpha) + \frac{2kC_\alpha(L_F + \tilde{k}L_G)T^{(1-\alpha)\beta}}{(1-\alpha)\beta} \\
&\quad + \frac{C_\alpha\left(\max_{0\leq s\leq T}\| F(s, v, v)\| + \tilde{k}\max_{0\leq s\leq T}\| G(s, v, v)\|\right)T^{(1-\alpha)\beta}}{(1-\alpha)\beta}
\end{aligned}$$

Dividing on both sides by k and taking the lower limit as $k \to +\infty$, we get

$$1 \le \frac{2C_\alpha(L_F + \widetilde{k}L_G)T^{(1-\alpha)\beta}}{(1-\alpha)\beta},$$

this contradicts (3.2). Also, for $u, v \in \Theta_{k_0}$, a direct calculation yields

$$\| (\Gamma_w u)(t) - (\Gamma_w v)(t) \|_\alpha$$

$$= \left\| \int_0^t (t-s)^{\beta-1} \mathcal{P}_\beta(t-s) \left[F(s, u(s), u(\kappa_1(s))) - F(s, v(s), v(\kappa_1(s))) \right. \right.$$

$$\left. + \int_0^s K(s-\tau)(G(\tau, u(\tau), u(\kappa_2(\tau))) - G(\tau, v(\tau), v(\kappa_2(\tau))))d\tau \right] ds \Bigg\|_\alpha$$

$$\le C_\alpha \int_0^t (t-s)^{\beta-1-\alpha\beta} \left[\| F(s, u(s), u(\kappa_1(s))) - F(s, v(s), v(\kappa_1(s))) \| \, ds \right.$$

$$\left. + \int_0^s K(s-\tau) \| G(\tau, u(\tau), u(\kappa_2(\tau))) - G(\tau, v(\tau), v(\kappa_2(\tau))) \| \, d\tau \right] ds$$

$$\le C_\alpha \int_0^t (t-s)^{\beta-1-\alpha\beta} \left[L_F(\| u(s) - v(s) \|_\alpha + \| u(\kappa_1(s)) - v(\kappa_1(s)) \|_\alpha) \right.$$

$$\left. + L_G \int_0^s K(s-\tau)(\| u(\tau) - v(\tau) \|_\alpha + \| u(\kappa_2(\tau)) - v(\kappa_2(\tau)) \|_\alpha)d\tau \right] ds$$

$$\le \frac{2C_\alpha T^{(1-\alpha)\beta}(L_F + \widetilde{k}L_G)}{(1-\alpha)\beta} |u - v|_\alpha,$$

which together with (3.2) implies that Γ_w is a contraction mapping on Θ_{k_0}. Thus, by the Banach contraction mapping principle, Γ_w has a unique fixed point $u_w \in \Theta_{k_0}$, i.e.,

$$u_w = \mathcal{S}_\beta(t)(u_0 - H(\widetilde{w})) + \int_0^t (t-s)^{\beta-1} \mathcal{P}_\beta(t-s)(F(s, u_w(s), u_w(\kappa_1(s)))$$

$$+ \int_0^s K(s-\tau)G(\tau, u_w(\tau), u_w(\kappa_2(\tau)))d\tau)ds$$

for $t \in [0, T]$.
Step 2. Write

$$\Theta_{k_0}^\eta = \{u \in C([\eta, T]; X_\alpha); \ \| u(t) \|_\alpha \le k_0 \quad for \ all \ t \in [\eta, T]\}.$$

It is clear that $\Theta_{k_0}^\eta$ is a bounded closed convex subset of $C([\eta, T]; X_\alpha)$.

Based on the argument in Step 1, we consider a mapping $\mathcal{F}$ on $\Theta_{k_0}^\eta$ defined by

$$(\mathcal{F}w)(t) = u_w, \quad t \in [\eta, T].$$

It follows from (H_5) and (3.2) that $\mathcal{F}$ maps $\Theta_{k_0}^\eta$ into itself. Moreover, for $w_1, w_2 \in \Theta_{k_0}^\eta$, from Step 1, we have

$$\left(1 - \frac{2C_\alpha T^{(1-\alpha)\beta}(L_F + \widetilde{k}L_G)}{(1-\alpha)\beta} \right) |u_{w_1} - u_{w_2}|_\alpha \le M \| H(\overline{w}_1) - H(\overline{w}_2) \|_\alpha,$$

that is,

$$\sup_{t \in [\eta, T]} \| \mathcal{F}w_1(t) - \mathcal{F}w_2(t) \|_\alpha \to 0 \quad as \ w_1 \to w_2 \ in \ \Theta_{k_0}^\eta,$$

which yields that $\mathcal{F}$ is continuous. Next, we prove that $\mathcal{F}$ has a fixed point in $\Theta_{k_0}^\eta$. It will suffice to prove that $\mathcal{F}$ is a compact operator. Then, the result follows from Lemma 2.6.

Let's decompose the mapping $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ as

$$(\mathcal{F}_1 w)(t) = \mathcal{S}_\beta(t)(u_0 - H(\widetilde{w})),$$
$$(\mathcal{F}_2 w)(t) = \int_0^t (t-s)^{\beta-1} \mathcal{P}_\beta(t-s)(F(s, u_w(s), u_w(\kappa_1(s)))$$
$$+ \int_0^s K(s-\tau)G(\tau, u_w(\tau), u_w(\kappa_2(\tau)))d\tau)ds.$$

Since assumption (H_5) implies that the set $\left\{H(\widetilde{w}); w \in \Theta_{k_0}^\eta\right\}$ is bounded in X_α, it follows from Lemma 2.4 that for each $t \in [\eta, T]$, $\left\{(\mathcal{F}_1 w)(t); w \in \Theta_{k_0}^\eta\right\}$ is relatively compact in X_α. Also, for $\eta \le t_1 \le t_2 \le T$,

$$\left\|(\mathcal{S}_\beta(t_2) - \mathcal{S}_\beta(t_1))(u_0 - H(\widetilde{w}))\right\|_\alpha \to 0 \quad as\ t_2 - t_1 \to 0$$

independently of $w \in \Theta_{k_0}^\eta$. This proves that the set $\left\{(\mathcal{F}_1 w)(\cdot); w \in \Theta_{k_0}^\eta\right\}$ is equicontinuous. Thus, an application of Arzela-Ascoli's theorem yields that $\mathcal{F}_1$ is compact.

Observe that the set

$$\left\{F(t, u(t), u(\kappa_1(t))) + \int_0^t K(t-\tau)G(\tau, u(\tau), u(\kappa_2(\tau)))d\tau;\ t \in [0, T],\ w \in \Theta_{k_0}^\eta\right\}$$

is bounded in X. Therefore, using Lemma 2.1, Lemma 2.2 and Lemma 2.3, it is not difficult to prove, similar to the argument with Γ_2 in Theorem 3.1, that $\mathcal{F}_2$ is compact. Hence, making use of Lemma 2.6 we conclude that $\mathcal{F}$ has a fixed point $w^* \in \Theta_{k_0}^\eta$. Put $q = u_{w^*}$. Then,

$$q(t) = \mathcal{S}_\beta(t)\left(u_0 - H(\widetilde{w^*})\right) + \int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)\left(F(s, q(s), q(\kappa_1(s)))\right.$$

$$\left. + \int_0^s K(s-\tau)G(\tau, q(\tau), q(\kappa_2(\tau)))d\tau\right)ds, \quad t \in [0, T].$$

Since $u_{w^*} = \mathcal{F}w^* = w^*$ $(t \in [\eta, T])$, $H(w^*) = H(q)$ and hence q is a mild solution of the fractional Cauchy problem (1.1). This completes the proof. ∎

4 Example

In this section, we present an example to our abstract results, which do not aim at generality but indicate how our theorem can be applied to concrete problem.

We consider the partial differential equation with Dirichlet boundary condition and nonlocal initial condition in the form

$$\begin{cases} {}^c\partial_t^{\frac{1}{2}} u(t,x) = \dfrac{\partial^2 u(t,x)}{\partial x^2} + \dfrac{q_1(t)|u(\sin(t),x)|}{1+|u(\sin(t),x)|} \\ \qquad\qquad + \int\limits_0^t \dfrac{q_2(s)}{1+(t-s)^2} \dfrac{|u(s,x)|}{1+|u(s,x)|} ds, \quad 0 \le t \le 1,\ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le 1, \\ u(0,x) = \int\limits_\eta^1 \ln[e^{u(s,x)}(|u(s,x)|+1)]ds + u_0(x), \quad 0 \le x \le \pi, \end{cases} \quad (4.1)$$

where the functions q_1, q_2 are continuous on [0, 1] and $0 < \eta < 1$.

Let $X = L^2[0, \pi]$ and the operators $A = \frac{\partial^2}{\partial x^2} : D(A) \subset X \mapsto X$ be defined by

$$D(A) = \{u \in X;\ u, u' \text{ are absolutely continuous, } u'' \in X, \text{ and}$$
$$u(0) = u(\pi) = 0\}.$$

Then, A has a discrete spectrum and the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with the corresponding normalized eigenvectors $\gamma_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Moreover, A generates a compact, analytic semigroup $\{T(t)\}_{t\ge 0}$. The following results are well also known (see [29] for more details):

(1) $T(t)u = \sum_{n=1}^\infty e^{-n^2 t}(u, \gamma_n)\gamma_n$, $\| T(t) \|_{\mathcal{L}(X)} \le e^{-t}$ for all $t \ge 0$.

(2) $A^{-\frac{1}{2}}u = \sum_{n=1}^\infty \frac{1}{n}(u, \gamma_n)\gamma_n$ for each $u \in X$. In particular, $\left\| A^{-\frac{1}{2}} \right\|_{\mathcal{L}(X)} = 1$.

(3) $A^{\frac{1}{2}}u = \sum_{n=1}^\infty n(u, \gamma_n)\gamma_n$ with the domain

$$D\left(A^{\frac{1}{2}}\right) = \left\{ u \in X; \sum_{n=1}^\infty n(u, \gamma_n)\gamma_n \in X \right\}.$$

Denote by $E_{\zeta,\beta}$, the generalized Mittag-Leffler special function (cf., e.g., [4]) defined by

$$E_{\zeta,\beta}(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(\zeta k + \beta)} \quad \zeta, \beta > 0, \quad t \in \mathbb{R}.$$

Therefore, we have

$$\mathcal{S}_\beta(t)u = \sum_{n=1}^\infty E_\beta(-n^2 t^\beta)(u, \gamma_n)\gamma_n, \quad u \in X; \quad \| \mathcal{S}_\beta(t) \|_{\mathcal{L}(X)} \le 1,$$
$$\mathcal{P}_\beta(t)u = \sum_{n=1}^\infty e_\beta(-n^2 t^\beta)(u, \gamma_n)\gamma_n, \quad u \in X; \quad \| \mathcal{P}_\beta(t) \|_{\mathcal{L}(X)} \le \frac{\beta}{\Gamma(1+\beta)}$$

for all $t \ge 0$, where $E_\beta(t) := E_{\beta,1}(t)$ and $e_\beta(t) := E_{\beta,\beta}(t)$.

The consideration of this section also needs the following result.

Lemma 4.1. [31]*If $w \in D(A^{\frac{1}{2}})$, then w is absolutely continuous, $w' \in X$, and*

$$\| w' \| = \left\| A^{\frac{1}{2}} w \right\|.$$

Define

$$F(t, u(t), u(\kappa_1(t)))(x) = \frac{q_1(t)|u(\sin(t), x)|}{1 + |u(\sin(t), x)|},$$

$$K(t) = \frac{1}{t^2 + 1}, \quad \kappa_1(t) = \sin(t), \quad \kappa_2(t) = t,$$

$$G(t, u(t), u(\kappa_2(t)))(x) = \frac{q_2(t)|u(t, x)|}{1 + |u(t, x)|},$$

$$H(u)(x) = \int_\eta^1 \ln[e^{u(s, x)}(|u(s, x)| + 1)]ds.$$

Therefore, it is not difficult to verify that $F, G : [0, 1] \times X_{\frac{1}{2}} \times X_{\frac{1}{2}} \to X$ and $H : C([0, 1]; X_{\frac{1}{2}}) \to X_{\frac{1}{2}}$ are continuous,

$$\| F(t, u(t), u(\kappa_1(t))) - F(t, v(t), v(\kappa_1(t))) \|$$
$$\leq \mu_1 \left\| A^{-\frac{1}{2}} \right\|_{\mathcal{L}(X)} \| u(\kappa_1(t)) - v(\kappa_1(t)) \|_{\frac{1}{2}},$$
$$\| G(t, u(t), u(\kappa_1(t))) - G(t, v(t), v(\kappa_1(t))) \|$$
$$\leq \mu_2 \left\| A^{-\frac{1}{2}} \right\|_{\mathcal{L}(X)} \| u(t) - v(t) \|_{\frac{1}{2}},$$

where $\mu_i := \sup_{t \in [0, 1]} |q_i(t)|$, and for any u satisfying $|u|_{\frac{1}{2}} \leq k$,

$$\| H(u) \|_{\frac{1}{2}} = \left\| A^{\frac{1}{2}} H(u)(\cdot) \right\| = \| H(u)'(\cdot) \| \leq 2(1 - \eta)k$$

in view of Lemma 4.1.

Now, we note that the problem (4.1) can be reformulated as the abstract problem (1.1) and the assumptions (H_0), (H_2), (H_4) and (H_5) hold with

$$\alpha = \frac{1}{2}, \quad T = 1, \quad L_F = \mu_1, \quad L_G = \mu_2, \quad \Phi(k) = 2(1 - \eta)k, \quad \mu = 2(1 - \eta).$$

Thus, when $1 - \eta + 4M_{\frac{1}{2}}(\mu_1 + \mu_2) < \frac{1}{2}$ such that condition (3.2) is verified, (4.1) has at least one mild solution due to Theorem 3.2.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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