Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A functional equation related to inner product spaces in non-Archimedean normed spaces

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Abstract

In this paper, we prove the Hyers-Ulam stability of a functional equation related to inner product spaces in non-Archimedean normed spaces.
2010 Mathematics Subject Classification: Primary 46S10; 39B52; 47S10; 26E30; 12J25.

Keywords: non-Archimedean spaces, additive and quadratic functional equation, Hyers-Ulam stability

1. Introduction and preliminaries

One of the most interesting questions in the theory of functional analysis concerning the Ulam stability problem of functional equations is as follows: When is it true that a mapping satisfying a functional equation approximately must be close to an exact solution of the given functional equation?

The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940 and affirmatively solved by Hyers [2]. The result of Hyers was generalized by Aoki [3] for approximate additive mappings and by Rassias [4] for approximate linear mappings by allowing the difference Cauchy equation $|| f(x_1 + x_2) - f(x_1) - f(x_2) ||$ to be controlled by $\varepsilon (|| x_1 ||^p + || x_2 ||^p)$. Taking into consideration a lot of influence of Ulam, Hyers and Rassias on the development of stability problems of functional equations, the stability phenomenon that was proved by Rassias is called *Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. In 1994, a generalization of the Rassias' theorem was obtained by Gǎvruta [5], who replaced $\varepsilon (|| x_1 ||^p + || x_2 ||^p)$ by a general control function $\phi(x_1, x_2)$.

Quadratic functional equations were used to characterize inner product spaces [6]. A square norm on an inner product space satisfies the parallelogram equality $||x_1 + x_2||^2 + ||x_1 - x_2||^2 = 2(||x_1||^2 + ||x_1||^2)$. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive mapping [7,8]. It is natural that this equation is called a *quadratic functional equation*, and every solution of the quadratic equation (1.1) is said to be a *quadratic mapping*.

It was shown by Rassias [9] that the norm defined over a real vector space X is induced by an inner product if and only if for a fixed integer $n \geq 2$

$$\sum_{i=1}^{n}\left\|x_i - \frac{1}{n}\sum_{j=1}^{n}x_j\right\|^2 = \sum_{i=1}^{n}\| x_i\|^2 - n\left\|\frac{1}{n}\sum_{i=1}^{n}x_i\right\|^2$$

for all $x_1,..., x_n \in X$.

Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $| \, | : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$, we have

(i)$|a| \geq 0$ and equality holds if and only if $a = 0$,
(ii) $|ab| = |a||b|$,
(iii) $|a + b| \leq \max\{|a|, |b|\}$.

The condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n. We always assume in addition that $| \, |$ is non-trivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $|| \cdot || : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(NA1) $||x|| = 0$ if and only if $x = 0$;
(NA2) $||rx|| = |r|||x||$ for all $r \in \mathbb{K}$ $\mathbb{R}$ and $x \in X$;
(NA3) the strong triangle inequality (ultrametric); namely,

$$\| x + y \| \leq \max\{\| x \|, \| y \|\} \quad (x, y \in X).$$

Then $(X, || \cdot ||)$ is called a non-Archimedean space.

Thanks to the inequality

$$\| x_m - x_l \| \leq \max\{\| x_{j+1} - x_j \| : l \leq j \leq m-1\} \quad (m > l)$$

a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space, we mean a non-Archimedean space in which every Cauchy sequence is convergent.

In 1897, Hensel [10] introduced a normed space which does not have the Archimedean property.

During the last three decades, the theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings [11]. Although many results in the classical normed space theory have a non-Archimedean counterpart, their proofs are essentially different and require an entirely new kind of intuition [12-16].

The main objective of this paper is to prove the Hyers-Ulam stability of the following functional equation related to inner product spaces

$$\sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n}x_j\right) = \sum_{i=1}^{n} f(x_i) - nf\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \tag{1.2}$$

$(n \in \mathbb{N}, n \geq 2)$ in non-Archimedean normed spaces. Interesting new results concerning functional equations related to inner product spaces have recently been obtained by Najati and Rassias [17] as well as for the fuzzy stability of a functional equation related to inner product spaces by Park [18] and Eshaghi Gordji and Khodaei [19]. During the last decades, several stability problems for various functional equations have been investigated by many mathematicians (see [20-49]).

2. Hyers-Ulam stability in non-Archimedean spaces

In the rest of this paper, unless otherwise explicitly stated, we will assume that G is an additive group and that X is a complete non-Archimedean space. For convenience, we use the following abbreviation for a given mapping $f : G \to X$:

$$\Delta f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) - \sum_{i=1}^{n} f(x_i) + nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

for all $x_1,..., x_n \in G$, where $n \geq 2$ is a fixed integer.

Lemma 2.1. [17]. *Let V_1 and V_2 be real vector spaces. If an odd mapping $f : V_1 \to V_2$ satisfies the functional equation* (1.2), *then f is additive.*

In the following theorem, we prove the Hyers-Ulam stability of the functional equation (1.2) in non-Archimedean spaces for an odd case.

Theorem 2.2. *Let $\phi : G^n \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} \frac{\varphi(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{|2|^m} = 0 = \lim_{m\to\infty} \frac{1}{|2|^m} \Phi(2^{m-1}x) \tag{2.1}$$

for all $x, x_1, x_2,..., x_n \in G$, and

$$\tilde{\varphi}_a(x) = \lim_{m\to\infty} \max\left\{\frac{1}{|2|^k}\Phi(2^k x) : 0 \leq k < m\right\} \tag{2.2}$$

exists for all $x \in G$, where

$$\begin{aligned}\Phi(x) := \max\Big\{&\varphi(2x, 0, \ldots, 0), \frac{1}{|2|}\max\{n\varphi(x, x, 0, \ldots, 0),\\ &\varphi(x, -x, \ldots, -x), \varphi(-x, x, \ldots, x)\}\Big\}\end{aligned} \tag{2.3}$$

for all $x \in G$. Suppose that an odd mapping $f : G \to X$ satisfies the inequality

$$\| \Delta f(x_1, \ldots, x_n) \| \leq \varphi(x_1, x_2, \ldots, x_n) \tag{2.4}$$

for all $x_1, x_2,..., x_n \in G$. Then there exists an additive mapping $A : G \to X$ such that

$$\| f(x) - A(x) \| \leq \frac{1}{|2|}\tilde{\varphi}_a(x) \tag{2.5}$$

for all $x \in G$, and if

$$\lim_{\ell\to\infty}\lim_{m\to\infty} \max\left\{\frac{1}{|2|^k}\Phi(2^k x) : \ell \leq k < m + \ell\right\} = 0 \tag{2.6}$$

then A is a unique additive mapping satisfying (2.5).

Proof. Letting $x_1 = nx_1$, $x_i = nx_1' \quad (i = 2, \ldots, n)$ in (2.4) and using the oddness of f, we obtain that

$$\begin{aligned}&\| nf(x_1+(n-1)x'_1)+f((n-1)(x_1-x'_1))-(n-1)f(x_1-x'_1)\\&\quad -f(nx_1)-(n-1)f(nx'_1)\| \le \varphi(nx_1,nx'_1,\ldots,nx'_1)\end{aligned} \tag{2.7}$$

for all $x_1, x'_1 \in G$. Interchanging x_1 with x'_1 in (2.7) and using the oddness of f, we get

$$\begin{aligned}&\| nf((n-1)x_1+x'_1)-f((n-1)(x_1-x'_1))+(n-1)f(x_1-x'_1)\\&\quad -(n-1)f(nx_1)-f(nx'_1)\| \le \varphi(nx'_1,nx_1,\ldots,nx_1)\end{aligned} \tag{2.8}$$

for all $x_1, x'_1 \in G$. It follows from (2.7) and (2.8) that

$$\begin{aligned}&\| nf(x_1+(n-1)x'_1)-nf((n-1)x_1+x'_1)+2f((n-1)(x_1-x'_1))\\&\qquad -2(n-1)f(x_1-x'_1)+(n-2)f(nx_1)-(n-2)f(nx'_1)\|\\&\quad \le \max\{\varphi(nx_1,nx'_1,\ldots,nx'_1),\varphi(nx'_1,nx_1,\ldots,nx_1)\}\end{aligned} \tag{2.9}$$

for all $x_1, x'_1 \in G$. Setting $x_1 = nx_1$, $x_2 = -nx'_1$, $x_i = 0$ ($i = 3,\ldots, n$) in (2.4) and using the oddness of f, we get

$$\begin{aligned}&\| f((n-1)x_1+x'_1)-f(x_1+(n-1)x'_1)+2f(x_1-x'_1)\\&\quad -f(nx_1)+f(nx'_1)\| \le \varphi(nx_1,-nx'_1,0,\ldots,0)\end{aligned} \tag{2.10}$$

for all $x_1, x'_1 \in G$. It follows from (2.9) and (2.10) that

$$\begin{aligned}&\| f((n-1)(x_1-x'_1))+f(x_1-x'_1)-f(nx_1)+f(nx'_1)\|\\&\quad \le \frac{1}{|2|}\max\{n\varphi(nx_1,-nx'_1,0,\ldots,0),\\&\quad \varphi(nx_1,nx'_1,\ldots,nx'_1),\varphi(nx'_1,nx_1,\ldots,nx_1)\}\end{aligned} \tag{2.11}$$

for all $x_1, x'_1 \in G$. Putting $x_1 = n(x_1 - x'_1)$, $x_i = 0$ ($i = 2,\ldots, n$) in (2.4), we obtain

$$\| f(n(x_1-x'_1))-f((n-1)(x_1-x'_1))-f((x_1-x'_1))\| \le \varphi(n(x_1-x'_1),0,\ldots,0) \tag{2.12}$$

for all $x_1, x'_1 \in G$. It follows from (2.11) and (2.12) that

$$\begin{aligned}&\| f(n(x_1-x'_1))-f(nx_1)+f(nx'_1)\|\\&\quad \le \max\Big\{\varphi(n(x_1-x'_1),0,\ldots,0),\frac{n}{|2|}\varphi(nx_1,-nx'_1,0,\ldots,0),\\&\quad \frac{1}{|2|}\max\{\varphi(nx_1,nx'_1,\ldots,nx'_1),\varphi(nx'_1,nx_1,\ldots,nx_1)\}\Big\}\end{aligned} \tag{2.13}$$

for all $x_1, x'_1 \in G$. Replacing x_1 and x'_1 by $\frac{x}{n}$ and $\frac{-x}{n}$ in (2.13), respectively, we obtain

$$\begin{aligned}&\| f(2x)-2f(x)\| \le \max\{\varphi(2x,0,\ldots,0),\\&\frac{1}{|2|}\max\{n\varphi(x,x,0,\ldots,0),\varphi(x,-x,\ldots,-x),\varphi(-x,x,\ldots,x)\}\}\end{aligned}$$

for all $x \in G$. Hence,

$$\left\|\frac{f(2x)}{2}-f(x)\right\| \le \frac{1}{|2|}\Phi(x) \tag{2.14}$$

for all $x \in G$. Replacing x by $2^{m-1}x$ in (2.14), we have

$$\left\|\frac{f(2^{m-1}x)}{2^{m-1}}-\frac{f(2^m x)}{2^m}\right\| \le \frac{1}{|2|^m}\Phi(2^{m-1}x) \tag{2.15}$$

for all $x \in G$. It follows from (2.1) and (2.15) that the sequence $\{\frac{f(2^m x)}{2^m}\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{f(2^m x)}{2^m}\}$ is convergent. So one can define the mapping $A : G \to X$ by $A(x) := \lim_{m\to\infty} \frac{f(2^m x)}{2^m}$ for all $x \in G$. It follows from (2.14) and (2.15) that

$$\left\| f(x) - \frac{f(2^m x)}{2^m} \right\| \leq \frac{1}{|2|} \max\left\{ \frac{1}{|2|^k} \Phi(2^k x) : 0 \leq k < m \right\} \tag{2.16}$$

for all $m \in \mathbb{N}$ and all $x \in G$. By taking m to approach infinity in (2.16) and using (2.2), one gets (2.5). By (2.1) and (2.4), we obtain

$$\begin{aligned} \| \Delta A(x_1, x_2, \ldots, x_n) \| &= \lim_{m\to\infty} \frac{1}{|2|^m} \| \Delta f(2^m x_1, 2^m x_2, \ldots, 2^m x_n) \| \\ &\leq \lim_{m\to\infty} \frac{1}{|2|^m} \varphi(2^m x_1, 2^m x_2, \ldots, 2^m x_n) = 0 \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in G$. Thus, the mapping A satisfies (1.2). By Lemma 2.1, A is additive.

If A' is another additive mapping satisfying (2.5), then

$$\begin{aligned} \| A(x) - A'(x) \| &= \lim_{\ell\to\infty} |2|^{-\ell} \| A(2^\ell x) - A'(2^\ell x) \| \\ &\leq \lim_{\ell\to\infty} |2|^{-\ell} \max\{\| A(2^\ell x) - f(2^\ell x) \|, \| f(2^\ell x) - Q'(2^\ell x) \|\} \\ &\leq \frac{1}{|2|} \lim_{\ell\to\infty} \lim_{m\to\infty} \max\left\{ \frac{1}{|2|^k} \tilde{\varphi}(2^k x) : \ell \leq k < m + \ell \right\} = 0 \end{aligned}$$

for all $x \in G$, Thus $A = A'$. □

Corollary 2.3. *Let* $\rho : [0, \infty) \to [0, \infty)$ *be a function satisfying*

(*i*) $\rho(|2|t) \leq \rho(|2|)\rho(t)$ *for all* $t \geq 0$,
(*ii*) $\rho(|2|) < |2|$.

Let $\varepsilon > 0$ *and let G be a normed space. Suppose that an odd mapping* $f : G \to X$ *satisfies the inequality*

$$\| \Delta f(x_1, \ldots, x_n) \| \leq \varepsilon \sum_{i=1}^{n} \rho(\| x_i \|)$$

for all $x_1, \ldots, x_n \in G$. *Then there exists a unique additive mapping* $A : G \to X$ *such that*

$$\| f(x) - A(x) \| \leq \frac{2n}{|2|^2} \varepsilon \rho(\| x \|)$$

for all $x \in G$.

Proof. Defining $\phi : G^n \to [0, \infty)$ by $\varphi(x_1, \ldots, x_n) := \varepsilon \sum_{i=1}^{n} \rho(\| x_i \|)$, we have

$$\lim_{m\to\infty} \frac{1}{|2|^m} \varphi(2^m x_1, \ldots, 2^m x_n) \leq \lim_{m\to\infty} \left(\frac{\rho(|2|)}{|2|} \right)^m \varphi(x_1, \ldots, x_n) = 0$$

for all $x_1,..., x_n \in G$. So we have

$$\tilde{\varphi}_a(x) := \lim_{m\to\infty} \max\left\{\frac{1}{|2|^k}\Phi(2^k x) : 0 \le k < m\right\} = \Phi(x)$$

and

$$\lim_{\ell\to\infty}\lim_{m\to\infty} \max\left\{\frac{1}{|2|^k}\Phi(2^k x) : \ell \le k < m+\ell\right\} = \lim_{\ell\to\infty}\frac{1}{|2|^\ell}\Phi(2^\ell x) = 0$$

for all $x \in G$. It follows from (2.3) that

$$\begin{aligned}\Phi(x) &= \max\left\{\varepsilon\rho(\| 2x \|), \frac{1}{|2|}\max\{2n\varepsilon\rho(\| x \|), n\varepsilon\rho(\| x \|), n\varepsilon\rho(\| x \|)\}\right\}\\ &= \max\left\{\varepsilon\rho(\| 2x \|), \frac{1}{|2|}\max\{2n\varepsilon\rho(\| x \|), n\varepsilon\rho(\| x \|)\}\right\}\\ &= \max\left\{\varepsilon\rho(\| 2x \|), \frac{1}{|2|}2n\varepsilon\rho(\| x \|)\right\} = \frac{2n}{|2|}\varepsilon\rho(\| x \|).\end{aligned}$$

Applying Theorem 2.2, we conclude that

$$\| f(x) - A(x) \| \le \frac{1}{|2|}\tilde{\varphi}_a(x) = \frac{1}{|2|}\Phi(x) = \frac{2n}{|2|^2}\varepsilon\rho(\| x \|)$$

for all $x \in G$. □

Lemma 2.4. [17]. *Let V_1 and V_2 be real vector spaces. If an even mapping $f : V_1 \to V_2$ satisfies the functional equation* (1.2), *then f is quadratic.*

In the following theorem, we prove the Hyers-Ulam stability of the functional equation (1.2) in non-Archimedean spaces for an even case.

Theorem 2.5. *Let $\phi : G^n \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty}\frac{\varphi(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{|2|^{2m}} = 0 = \lim_{m\to\infty}\frac{1}{|2|^{2m}}\tilde{\varphi}(2^{m-1}x) \tag{2.17}$$

for all $x, x_1, x_2,..., x_n \in G$, and

$$\tilde{\varphi}_q(x) = \lim_{m\to\infty}\max\left\{\frac{1}{|2|^{2k}}\tilde{\varphi}(2^k x) : 0 \le k < m\right\} \tag{2.18}$$

exists for all $x \in G$, where

$$\begin{aligned}\tilde{\varphi}(x) :=& \frac{1}{|n-1|}\max\left\{\frac{1}{|2|}\varphi(nx, nx, 0, \ldots, 0),\right.\\ &\left.\varphi(nx, 0, \ldots, 0), \varphi(x, (n-1)x, 0, \ldots, 0), \Psi(x)\right\}\end{aligned} \tag{2.19}$$

and

$$\Psi(x) := \frac{1}{|2|}\max\{n\varphi(nx, 0, \ldots, 0), \varphi(nx, 0, \ldots, 0), \varphi(0, nx, \ldots, nx)\} \tag{2.20}$$

for all $x \in G$. Suppose that an even mapping $f : G \to X$ with $f(0) = 0$ satisfies the inequality (2.4) *for all $x_1, x_2,..., x_n \in G$. Then there exists a quadratic mapping $Q : G \to X$ such that*

$$\| f(x) - Q(x) \| \le \frac{1}{|2|^2}\tilde{\varphi}_q(x) \tag{2.21}$$

for all $x \in G$, *and if*

$$\lim_{\ell\to\infty}\lim_{m\to\infty}\max\left\{\frac{1}{|2|^{2k}}\tilde{\varphi}(2^k x) : \ell \le k < m + \ell\right\} = 0 \tag{2.22}$$

then Q is a unique quadratic mapping satisfying (2.21).

Proof. Letting $x_1 = nx_1$, $x_i = nx_2$ (i = 2,..., n) in (2.4) and using the evenness of f, we obtain

$$\begin{aligned}&\| nf(x_1 + (n-1)x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2)\\ &\quad -f(nx_1) - (n-1)f(nx_2) \| \le \varphi(nx_1, nx_2, \ldots, nx_2)\end{aligned} \tag{2.23}$$

for all $x_1, x_2 \in G$. Interchanging x_1 with x_2 in (2.23) and using the evenness of f, we obtain

$$\begin{aligned}&\| nf((n-1)x_1 + x_2) + f((n-1)(x_1 - x_2)) + (n-1)f(x_1 - x_2)\\ &\quad -(n-1)f(nx_1) - f(nx_2) \| \le \varphi(nx_2, nx_1, \ldots, nx_1)\end{aligned} \tag{2.24}$$

for all $x_1, x_2 \in G$. It follows from (2.23) and (2.24) that

$$\begin{aligned}&\| nf((n-1)x_1 + x_2) + nf(x_1 + (n-1)x_2) + 2f((n-1)(x_1 - x_2))\\ &\quad +2(n-1)f(x_1 - x_2) - nf(nx_1) - nf(nx_2) \|\\ &\le \max\{\varphi(nx_1, nx_2, \ldots, nx_2), \varphi(nx_2, nx_1, \ldots, nx_1)\}\end{aligned} \tag{2.25}$$

for all $x_1, x_2 \in G$. Setting $x_1 = nx_1$, $x_2 = -nx_2$, $x_i = 0$ (i = 3,..., n) in (2.4) and using the evenness of f, we obtain

$$\begin{aligned}&\| f((n-1)x_1 + x_2) + f(x_1 + (n-1)x_2) + 2(n-1)f(x_1 - x_2)\\ &\quad -f(nx_1) - f(nx_2) \| \le \varphi(nx_1, -nx_2, 0, \ldots, 0)\end{aligned} \tag{2.26}$$

for all $x_1, x_2 \in G$. So we obtain from (2.25) and (2.26) that

$$\begin{aligned}&\| f((n-1)(x_1 - x_2)) - (n-1)^2 f(x_1 - x_2) \|\\ &\quad \le \frac{1}{|2|}\max\{n\varphi(nx_1, -nx_2, 0, \ldots, 0),\\ &\quad \varphi(nx_1, nx_2, \ldots, nx_2), \varphi(nx_2, nx_1, \ldots, nx_1)\}\end{aligned} \tag{2.27}$$

for all $x_1, x_2 \in G$. Setting $x_1 = x$, $x_2 = 0$ in (2.27), we obtain

$$\begin{aligned}&\| f((n-1)x) - (n-1)^2 f(x) \|\\ &\quad \le \frac{1}{|2|}\max\{n\varphi(nx, 0, \ldots, 0), \varphi(nx, 0, \ldots, 0), \varphi(0, nx, \ldots, nx)\}\end{aligned} \tag{2.28}$$

for all $x \in G$. Putting $x_1 = nx$, $x_i = 0$ (i = 2,..., n) in (2.4), one obtains

$$\| f(nx) - f((n-1)x) - (2n-1)f(x) \| \le \varphi(nx, 0, \ldots, 0) \tag{2.29}$$

for all $x \in G$. It follows from (2.28) and (2.29) that

$$\begin{aligned}\| f(nx) - n^2 f(x) \| \le \max\Bigg\{&\varphi(nx, 0, \ldots, 0), \frac{n}{|2|}\varphi(nx, 0, \ldots, 0),\\ &\frac{1}{|2|}\varphi(nx, 0, \ldots, 0), \frac{1}{|2|}\varphi(0, nx, \ldots, nx)\Bigg\}\end{aligned} \tag{2.30}$$

for all $x \in G$. Letting $x_2 = -(n-1)x_1$ and replacing x_1 by $\frac{x}{n}$ in (2.26), we get

$$\| f((n-1)x) - f((n-2)x) - (2n-3)f(x) \| \le \varphi(x, (n-1)x, 0, \ldots, 0) \tag{2.31}$$

for all $x \in G$. It follows from (2.28) and (2.31) that

$$\begin{aligned} &\| f((n-2)x) - (n-2)^2 f(x) \| \le \max\Big\{\varphi(x, (n-1)x, 0, \ldots, 0), \\ &\frac{n}{|2|}\varphi(nx, 0, \ldots, 0), \frac{1}{|2|}\varphi(nx, 0, \ldots, 0), \frac{1}{|2|}\varphi(0, nx, \ldots, nx)\Big\} \end{aligned} \tag{2.32}$$

for all $x \in G$. It follows from (2.30) and (2.32) that

$$\begin{aligned} &\| f(nx) - f((n-2)x) - 4(n-1)f(x) \| \\ &\le \max\{\varphi(nx, 0, \ldots, 0), \varphi(x, (n-1)x, 0, \ldots, 0), \Psi(x)\} \end{aligned} \tag{2.33}$$

for all $x \in G$. Setting $x_1 = x_2 = n_x$, $x_i = 0$ $(i = 3, \ldots, n)$ in (2.4), we obtain

$$\| f((n-2)x) + (n-1)f(2x) - f(nx) \| \le \frac{1}{|2|}\varphi(nx, nx, 0, \ldots, 0) \tag{2.34}$$

for all $x \in G$. It follows from (2.33) and (2.34) that

$$\begin{aligned} &\| f(2x) - 4f(x) \| \le \frac{1}{|n-1|} \max\Big\{\frac{1}{|2|}\varphi(nx, nx, 0, \ldots, 0), \\ &\quad \varphi(nx, 0, \ldots, 0), \varphi(x, (n-1)x, 0, \ldots, 0), \Psi(x)\Big\} \end{aligned} \tag{2.35}$$

for all $x \in G$. Thus,

$$\left\| f(x) - \frac{f(2x)}{2^2} \right\| \le \frac{1}{|2|^2}\tilde{\varphi}(x) \tag{2.36}$$

for all $x \in G$. Replacing x by $2^{m-1}x$ in (2.36), we have

$$\left\| \frac{f(2^{m-1}x)}{2^{2(m-1)}} - \frac{f(2^m x)}{2^{2m}} \right\| \le \frac{1}{|2|^{2m}}\tilde{\varphi}(2^{m-1}x) \tag{2.37}$$

for all $x \in G$. It follows from (2.17) and (2.37) that the sequence $\{\frac{f(2^m x)}{2^{2m}}\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{f(2^m x)}{2^{2m}}\}$ is convergent. So one can define the mapping $Q : G \to X$ by $Q(x) := \lim_{m\to\infty} \frac{f(2^m x)}{2^{2m}}$ for all $x \in G$. By using induction, it follows from (2.36) and (2.37) that

$$\left\| f(x) - \frac{f(2^m x)}{2^{2m}} \right\| \le \frac{1}{|2|^2} \max\left\{ \frac{1}{|2|^{2k}}\tilde{\varphi}(2^k x) : 0 \le k < m \right\} \tag{2.38}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking m to approach infinity in (2.38) and using (2.18), one gets (2.21).

The rest of proof is similar to proof of Theorem 2.2. □

Corollary 2.6. *Let* $\eta : [0, \infty) \to [0, \infty)$ *be a function satisfying*

(*i*) $\eta(|l|t) \le \eta(|l|)\eta(t)$ *for all* $t \ge 0$,
(*ii*) $\eta(|l|) < |l|^2$ *for* $l \in \{2, n-1, n\}$.

Let $\varepsilon > 0$ *and let G be a normed space. Suppose that an even mapping* $f : G \to X$ *with* *f*(0) = 0 *satisfies the inequality*

$$\| \Delta f(x_1, \ldots, x_n) \| \leq \varepsilon \sum_{i=1}^{n} \eta(\| x_i \|)$$

for all $x_1,..., x_n \in G$. *Then there exists a unique quadratic mapping* $Q : G \to X$ *such that*

$$\| f(x) - Q(x) \| \leq \begin{cases} \frac{2}{|2|^2} \varepsilon \eta(\| x \|), & \text{if} \quad n = 2; \\ \frac{n}{|2|^3 |n-1|} \varepsilon \eta(\| nx \|), & \text{if} \quad n > 2, \end{cases}$$

for all $x \in G$.

Proof. Defining $\phi : G^n \to [0, \infty)$ by $\varphi(x_1, \ldots, x_n) := \varepsilon \sum_{i=1}^{n} \eta(\| x_i \|)$, we have

$$\lim_{m\to\infty} \frac{1}{|2|^{2m}} \varphi(2^m x_1, \ldots, 2^m x_n) \leq \lim_{m\to\infty} \left(\frac{\eta(|2|)}{|2|^2} \right)^m \varphi(x_1, \ldots, x_n) = 0$$

for all $x_1,..., x_n \in G$. We have

$$\tilde{\varphi}_q(x) := \lim_{m\to\infty} \max \left\{ \frac{1}{|2|^{2k}} \tilde{\varphi}(2^k x) : 0 \leq k < m \right\} = \tilde{\varphi}(x)$$

and

$$\lim_{\ell\to\infty} \lim_{m\to\infty} \max \left\{ \frac{1}{|2|^{2k}} \tilde{\varphi}(2^k x) : \ell \leq k < m + \ell \right\} = \lim_{\ell\to\infty} \frac{1}{|2|^{2\ell}} \tilde{\varphi}(2^\ell x) = 0$$

for all $x \in G$. It follows from (2.20) that

$$\begin{aligned} \Psi(x) &= \frac{1}{|2|} \max\{n\varepsilon\eta(\| nx \|), \varepsilon\eta(\| nx \|), (n-1)\varepsilon\eta(\| nx \|)\} \\ &= \frac{1}{|2|} \max\{n\varepsilon\eta(\| nx \|), (n-1)\varepsilon\eta(\| nx \|)\} \\ &= \frac{n}{|2|} \varepsilon\eta(\| nx \|) \end{aligned}$$

Hence, by using (2.19), we obtain

$$\begin{aligned} \tilde{\varphi}(x) &= \frac{1}{|n-1|} \max \left\{ \frac{2}{|2|} \varepsilon\eta(\| nx \|), \right. \\ &\quad \left. \varepsilon\eta(\| nx \|), \frac{n}{|2|} \varepsilon\eta(\| nx \|), \varepsilon(\eta(\| x \|) + \eta(\| (n-1)x \|)) \right\} \\ &= \begin{cases} 2\varepsilon\eta(\| x \|), \text{ if} \quad n = 2; \\ \qquad \frac{n}{|2||n-1|} \varepsilon\eta(\| nx \|), \text{ if} \quad n > 2, \end{cases} \end{aligned}$$

for all $x \in G$.

Applying Theorem 2.5, we conclude the required result. □

Lemma 2.7. [17]. *Let* V_1 *and* V_2 *be real vector spaces. A mapping* $f : V_1 \to V_2$ *satisfies* (1.2) *if and only if there exist a symmetric bi-additive mapping* $B : V_1 \times V_1 \to V_2$ *and an additive mapping* $A : V_1 \to V_2$ *such that* $f(x) = B(x, x) + A(x)$ *for all* $x \in V_1$.

Now, we prove the main theorem concerning the Hyers-Ulam stability problem for the functional equation (1.2) in non-Archimedean spaces.

Theorem 2.8. *Let $\phi : G^n \to [0, \infty)$ be a function satisfying* (2.1) *and* (2.17) *for all x, $x_1, x_2,..., x_n \in G$, and $\tilde{\varphi}_a(x)$and $\tilde{\varphi}_q(x)$exist for all $x \in G$, where $\tilde{\varphi}_a(x)$and $\tilde{\varphi}_q(x)$are defined as in Theorems* 2.2 *and* 2.5. *Suppose that a mapping $f : G \to X$ with $f(0) = 0$ satisfies the inequality* (2.4) *for all $x_1, x_2,..., x_n \in G$. Then there exist an additive mapping $A : G \to X$ and a quadratic mapping $Q : G \to X$ such that*

$$\begin{aligned} &\| f(x) - A(x) - Q(x) \| \\ &\leq \frac{1}{|2|^2} \max\left\{\tilde{\varphi}_a(x), \tilde{\varphi}_a(-x), \frac{1}{|2|}\tilde{\varphi}_q(x), \frac{1}{|2|}\tilde{\varphi}_q(-x)\right\} \end{aligned} \tag{2.39}$$

for all $x \in G$. If

$$\begin{aligned} &\lim_{\ell\to\infty}\lim_{m\to\infty} \max\left\{\frac{1}{|2|^k}\Phi(2^k x) : \ell \leq k < m + \ell\right\} = 0 \\ &= \lim_{\ell\to\infty}\lim_{m\to\infty} \max\left\{\frac{1}{|2|^{2k}}\tilde{\varphi}(2^k x) : \ell \leq k < m + \ell\right\} \end{aligned}$$

then A is a unique additive mapping and Q is a unique quadratic mapping satisfying (2.39).

Proof. Let $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in G$. Then

$$\begin{aligned} \| \Delta f_e(x_1, \ldots, x_n) \| &= \left\| \frac{1}{2}(\Delta f(x_1, \ldots, x_n) + \Delta f(-x_1, \ldots, -x_n)) \right\| \\ &\leq \frac{1}{|2|} \max\{\varphi(x_1, \ldots, x_n), \varphi(-x_1, \ldots, -x_n)\} \end{aligned}$$

for all $x_1, x_2,..., x_n \in G$. By Theorem 2.5, there exists a quadratic mapping $Q : G \to X$ such that

$$\| f_e(x) - Q(x) \| \leq \frac{1}{|2|^3} \max\{\tilde{\varphi}_q(x), \tilde{\varphi}_q(-x)\} \tag{2.40}$$

for all $x \in G$. Also, let $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ for all $x \in G$. By Theorem 2.2, there exists an additive mapping $A : G \to X$ such that

$$\| f_o(x) - A(x) \| \leq \frac{1}{|2|^2} \max\{\tilde{\varphi}_a(x), \tilde{\varphi}_a(-x)\} \tag{2.41}$$

for all $x \in G$. Hence (2.39) follows from (2.40) and (2.41).

The rest of proof is trivial. □

Corollary 2.9. *Let $\gamma : [0, \infty) \to [0, \infty)$ be a function satisfying*

(*i*) $\gamma(|l|t) \leq \gamma(|l|)\ \gamma(t)$ *for all* $t \geq 0$,
(*ii*) $\gamma(|l|) < |l|^2$ *for* $l \in \{2, n - 1, n\}$.

Let $\varepsilon > 0$, G a normed space and let $f : G \to X$ satisfy

$$\| \Delta f(x_1, \ldots, x_n) \| \leq \varepsilon \sum_{i=1}^{n} \gamma(\| x_i \|)$$

for all $x_1,..., x_n \in G$ and $f(0) = 0$. Then there exist a unique additive mapping $A : G \to X$ and a unique quadratic mapping $Q : G \to X$ such that

$$\| f(x) - A(x) - Q(x) \| \leq \frac{2n}{|2|^3} \varepsilon \gamma (\| x \|)$$

for all $x \in G$.

Proof. The result follows by Corollaries 2.6 and 2.3. □

Acknowledgements

Dong Yun Shin was supported by the Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792).

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Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

Received: 29 March 2011 Accepted: 29 September 2011 Published: 29 September 2011

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doi:10.1186/1687-1847-2011-37
Cite this article as: Gordji *et al.*: A functional equation related to inner product spaces in non-Archimedean normed spaces. *Advances in Difference Equations* 2011 **2011**:37.