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Oscillation of higher-order quasi-linear neutral differential equations

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Abstract

In this note, we establish some oscillation criteria for certain higher-order quasi-linear neutral differential equation. These criteria improve those results in the literature. Some examples are given to illustrate the importance of our results.
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1. Introduction

The neutral differential equations find numerous applications in natural science and technology. For example, they are frequently used for the study of distributed networks containing lossless transmission lines, see Hale [1]. In the past few years, many studies have been carried out on the oscillation and nonoscillation of solutions of various types of neutral functional differential equations. We refer the reader to the papers [2-22] and the references cited therein.

In this work, we restrict our attention to the oscillation of higher-order quasi-linear neutral differential equation of the form

$$\left\{r(t)\left[\left(x(t)+p(t)x(\tau(t))\right)^{(n-1)}\right]^{\gamma}\right\}'+q(t)x^{\gamma}(\sigma(t))=0,\quad n\geq 2. \tag{1.1}$$

Throughout this paper, we assume that:

(C_1) $\gamma \leq 1$ is the quotient of odd positive integers;
(C_2) $p \in C([t_0, \infty), [0, \infty))$;
(C_3) $q \in C([t_0, \infty), [0, \infty))$, and q is not eventually zero on any half line $[t_*, \infty)$ for $t_* \geq t_0$;
(C_4) $r, \tau, \sigma \in C^1([t_0, \infty), \mathbb{R})$, $r(t) > 0$, $r'(t) \geq 0$, $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$, σ^{-1} exists and σ^{-1} is continuously differentiable, where σ^{-1} denotes the inverse function of σ.

We consider only those solutions x of equation (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$. We assume that equation (1.1) possesses such a solution. As usual, a solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Regarding the oscillation of higher-order neutral differential equations, Agarwal et al. [3,4], Li et al. [13], Tang et al. [16], Zafer [19], Zhang et al. [21,22] studied the oscillatory behavior of even-order neutral differential equation

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)f(x(\sigma(t))) = 0.$$

Karpuz et al. [9] examined the oscillation of odd-order neutral differential equation

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)x(\sigma(t)) = 0, \quad 0 \le p(t) < 1.$$

Li and Thandapani [14], Yildiz and Öcalan [18] investigated the oscillatory behavior of the odd-order nonlinear neutral differential equations

$$[x(t) + p(t)x(a + bt)]^{(n)} + q(t)x^{\alpha}(c + dt) = 0, \quad 0 \le p(t) \le P_0 < \infty$$

and

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)x^{\alpha}(\sigma(t)) = 0, \quad 0 \le p(t) \le P_1 < 1,$$

respectively.

So far, there are few results on the oscillation of equation (1.1) under the condition $p(t) \ge 1$; see, e.g., [3,4,13-15]. In this note, we will use some different techniques for studying the oscillation of equation (1.1).

Remark 1.1. All functional inequalities considered in this paper are assumed to hold eventually; that is, they are satisfied for all t large enough.

Remark 1.2. Without loss of generality, we can deal only with the positive solutions of (1.1).

2. Main results

In this section, we will establish some new oscillation theorems for equation (1.1). Below, for the sake of convenience, f^{1} denotes the inverse function of f, and we let $z(t) := x(t) + p(t)x(\tau(t))$, and $Q(t) := \min\{q(\sigma^{-1}(t)), q(\sigma^{-1}(\tau(t)))\}$.

Lemma 2.1. (Kneser's theorem) [[2], Lemma 2.2.1] Let $f \in C^n([t_0, \infty), \mathbb{R})$ and its derivatives up to order $(n - 1)$ are of constant sign in $[t_0, \infty)$. If $f^{(n)}$ is of constant sign and not identically zero on a sub-ray of $[t_0, \infty)$, and then, there exist $m \in \mathbb{Z}$ and $t_1 \in [t_0, \infty)$ such that $0 \le m \le n - 1$, and $(-1)^{n+m} f f^{(n)} \ge 0$,

$$f f^{(j)} > 0 \text{ for } j = 0, 1, \ldots, m-1 \quad \text{when } m \ge 1$$

and

$$(-1)^{m+j} f f^{(j)} > 0 \text{ for } j = m, m+1, \ldots, n-1 \quad \text{when } m \le n-1$$

hold on $[t_1, \infty)$.

Lemma 2.2. [[2], Lemma 2.2.3] Let f be a function as in Kneser's theorem and $f^{(n)}(t) \le 0$. If $\lim_{t\to\infty} f(t) \ne 0$, then for every $\lambda \in (0, 1)$, there exists $t_\lambda \in [t_1, \infty)$ such that

$$|f| \ge \frac{\lambda}{(n-1)!} t^{n-1} |f^{(n-1)}|$$

holds on $[t_\lambda, \infty)$.

In order to prove our theorems, we will use the following inequality.

Lemma 2.3. [23] Assume that $0 < \gamma \le 1$, $x_1, x_2 \in [0, \infty)$. Then,

$$x_1^{\gamma} + x_2^{\gamma} \ge (x_1 + x_2)^{\gamma}. \quad (2.1)$$

The following lemmas are very useful in the proofs of the main results.

Lemma 2.4. Assume that $r'(t) \geq 0$ and

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma}(t)} \mathrm{d}t = \infty. \tag{2.2}$$

If x is a positive solution of (1.1), then z satisfies

$$z(t) > 0, \quad (r(t)(z^{(n-1)}(t))^{\gamma})' \leq 0, \quad z^{(n-1)}(t) > 0, \quad z^{(n)}(t) \leq 0$$

eventually.

Proof. Due to $r'(t) \geq 0$, the proof is simple and so is omitted. □

Lemma 2.5. Assume that (2.2) holds, n is even and $r'(t) \geq 0$. If x is a positive solution of (1.1), then z satisfies

$$z(t) > 0,\ z'(t) > 0,\ (r(t)(z^{(n-1)}(t))^{\gamma})' \leq 0,\ z^{(n-1)}(t) > 0,\ z^{(n)}(t) \leq 0$$

eventually.

Proof. Due to $r'(t) \geq 0$ and Lemma 2.1, the proof is easy and hence is omitted.

Now, we give our results. Firstly, we establish some comparison theorems for the oscillation of (1.1).

Theorem 2.6. *Let n be odd, $0 \leq p(t) \leq p_0 < \infty$, $(\sigma^{-1}(t))' \geq \sigma_0 > 0$ and $\tau'(t) \geq \tau_0 > 0$. Assume that* (2.2) *holds. If the first-order neutral differential inequality*

$$\begin{aligned} &\left(\frac{y(\sigma^{-1}(t))}{\sigma_0} + \frac{p_0{}^{\gamma}}{\sigma_0\tau_0} y(\sigma^{-1}(\tau(t)))\right)' \\ &+Q(t)\left(\frac{\lambda_0 t^{n-1}}{(n-1)!r^{1/\gamma}(t)}\right)^{\gamma} y(t) \leq 0 \end{aligned} \tag{2.3}$$

has no positive solution for some $\lambda_0 \in (0, 1)$, then every solution of (1.1) *is oscillatory or tends to zero as $t \to \infty$.*

Proof. Let x be a nonoscillatory solution of (1.1) and $\lim_{t\to\infty} x(t) \neq 0$. Then $\lim_{t\to\infty} z(t) \neq 0$. It follows from (1.1) that

$$\frac{(r(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))^{\gamma})'}{(\sigma^{-1}(t))'} + q(\sigma^{-1}(t))x^{\gamma}(t) = 0. \tag{2.4}$$

Thus, for all sufficiently large t, we have

$$\begin{aligned} &\frac{(r(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))^{\gamma})'}{(\sigma^{-1}(t))'} \\ &+p_0{}^{\gamma}\frac{(r(\sigma^{-1}(\tau(t)))(z^{(n-1)}(\sigma^{-1}(\tau(t))))^{\gamma})'}{(\sigma^{-1}(\tau(t)))'} \\ &+q(\sigma^{-1}(t))x^{\gamma}(t) + p_0{}^{\gamma} q(\sigma^{-1}(\tau(t)))x^{\gamma}(\tau(t)) = 0. \end{aligned} \tag{2.5}$$

Note that

$$\begin{aligned} q(\sigma^{-1}(t))x^{\gamma}(t) + p_0{}^{\gamma} q(\sigma^{-1}(\tau(t)))x^{\gamma}(\tau(t)) &\geq Q(t)[x^{\gamma}(t) + p_0{}^{\gamma} x^{\gamma}(\tau(t))] \\ &\geq Q(t)[x(t) + p_0 x(\tau(t))]^{\gamma} \\ &\geq Q(t)z^{\gamma}(t) \end{aligned} \tag{2.6}$$

due to (2.1) and the definition of z and Q. It follows from (2.5) and (2.6) that

$$\begin{aligned}&\frac{(r(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))^{\gamma})'}{(\sigma^{-1}(t))'}\\&+p_0{}^{\gamma}\frac{(r(\sigma^{-1}(\tau(t)))(z^{(n-1)}(\sigma^{-1}(\tau(t))))^{\gamma})'}{(\sigma^{-1}(\tau(t)))'}+Q(t)z^{\gamma}(t)\le 0.\end{aligned}\tag{2.7}$$

In view of $(\sigma^{-1}(t))' \geq \sigma_0 > 0$ and $\tau'(t) \geq \tau_0 > 0$, we get

$$\begin{aligned}&\frac{(r(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))^{\gamma})'}{\sigma_0}\\&+p_0{}^{\gamma}\frac{(r(\sigma^{-1}(\tau(t)))(z^{(n-1)}(\sigma^{-1}(\tau(t))))^{\gamma})'}{\sigma_0\tau_0}+Q(t)z^{\gamma}(t)\le 0.\end{aligned}\tag{2.8}$$

On the other hand, by Lemma 2.2 and Lemma 2.4, we have

$$z(t)\ge\frac{\lambda}{(n-1)!r^{1/\gamma}(t)}t^{n-1}r^{1/\gamma}(t)z^{(n-1)}(t).\tag{2.9}$$

Therefore, setting $r(t)(z^{(n-1)}(t))^{\gamma} = y(t)$ in (2.8) and utilizing (2.9), one can see that y is a positive solution of (2.3). This contradicts our assumptions, and the proof is complete.

Applying additional conditions on the coefficients of (2.3), we can deduce from Theorem 2.6 various oscillation criteria for (1.1).

Theorem 2.7. *Let n be odd, $0 \leq p(t) \leq p_0 < \infty$, $(\sigma^{-1}(t))' \geq \sigma_0 > 0$, $\tau'(t) \geq \tau_0 > 0$ and $\tau(t) \leq t$. Assume that* (2.2) *holds. If the first-order differential inequality*

$$w'(t)+\frac{1}{\frac{1}{\sigma_0}+\frac{p_0{}^{\gamma}}{\sigma_0\tau_0}}Q(t)\left(\frac{\lambda_0t^{n-1}}{(n-1)!r^{1/\gamma}(t)}\right)^{\gamma}w(\tau^{-1}(\sigma(t)))\le 0\tag{2.10}$$

has no positive solution for some $\lambda_0 \in (0, 1)$, then every solution of (1.1) *is oscillatory or tends to zero as $t \to \infty$.*

Proof. We assume that x is a positive solution of (1.1) and $\lim_{t\to\infty} x(t) \neq 0$. Then Lemma 2.4 and the proof of Theorem 2.6 imply that $y(t) = r(t)(z^{(n-1)}(t))^{\gamma} > 0$ is nonincreasing and it satisfies (2.3). Let us denote

$$w(t)=\frac{y(\sigma^{-1}(t))}{\sigma_0}+\frac{p_0{}^{\gamma}}{\sigma_0\tau_0}y(\sigma^{-1}(\tau(t))).$$

It follows from $\tau(t) \leq t$ that

$$w(t)\le y(\sigma^{-1}(\tau(t)))\left(\frac{1}{\sigma_0}+\frac{p_0{}^{\gamma}}{\sigma_0\tau_0}\right).$$

Substituting these terms into (2.3), we get that w is a positive solution of (2.10). This contradiction completes the proof.

Corollary 2.8. Let n be odd, $0 \leq p(t) \leq p_0 < \infty$, $(\sigma^{-1}(t))' \geq \sigma_0 > 0$, $\tau'(t) \geq \tau_0 > 0$ and $\tau(t) \leq t$. Assume that (2.2) holds. If $\tau^{-1}(\sigma(t)) < t$ and

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\sigma(t))}^{t}\frac{Q(s)(s^{n-1})^{\gamma}}{r(s)}\mathrm{d}s>\frac{\left(\frac{1}{\sigma_0}+\frac{p_0{}^{\gamma}}{\sigma_0\tau_0}\right)((n-1)!)^{\gamma}}{\mathrm{e}},\tag{2.11}$$

then every solution of (1.1) is oscillatory or tends to zero as $t \to \infty$.

Proof. According to [[10], Theorem 2.1.1], the condition (2.11) guarantees that (2.10) has no positive solution. The proof of the corollary is complete.

Theorem 2.9. *Let n be odd,* $0 \le p(t) \le p_0 < \infty$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$ *and* $\tau(t) \le t$. *Assume that* (2.2) *holds. If the first-order differential inequality*

$$w'(t) + \frac{1}{\frac{1}{\sigma_0} + \frac{p_0{}^{\gamma}}{\sigma_0\tau_0}}\left(\frac{\lambda_0 t^{n-1}}{(n-1)!r^{1/\gamma}(t)}\right)^{\gamma} w(\sigma(t)) \le 0 \tag{2.12}$$

has no positive solution for some $\lambda_0 \in (0, 1)$, *then every solution of* (1.1) *is oscillatory or tends to zero as* $t \to \infty$.

Proof. We assume that x is a positive solution of (1.1) and $\lim_{t\to\infty} x(t) \ne 0$. Then Lemma 2.4 and the proof of Theorem 2.6 imply that $y(t) = r(t)(z^{(n-1)}(t))^{\gamma} > 0$ is nonincreasing and it satisfies (2.3). We denote

$$w(t) = \frac{y(\sigma^{-1}(t))}{\sigma_0} + \frac{p_0{}^{\gamma}}{\sigma_0\tau_0} y(\sigma^{-1}(\tau(t))).$$

In view of $\tau(t) \ge t$, we obtain

$$w(t) \le y(\sigma^{-1}(t))\left(\frac{1}{\sigma_0} + \frac{p_0{}^{\gamma}}{\sigma_0\tau_0}\right).$$

Substituting these terms into (2.3), we get that w is a positive solution of (2.12). This is a contradiction, and the proof is complete.

Corollary 2.10. Let n be odd, $0 \le p(t) \le p_0 < \infty$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$ and $\tau(t) \le t$. Assume that (2.2) holds. If $\sigma(t) < t$ and

$$\liminf_{t\to\infty} \int\limits_{\sigma(t)}^{t} \frac{Q(s)(s^{n-1})^{\gamma}}{r(s)} \mathrm{d}s > \frac{\left(\frac{1}{\sigma_0} + \frac{p_0{}^{\gamma}}{\sigma_0\tau_0}\right)((n-1)!)^{\gamma}}{\mathrm{e}}, \tag{2.13}$$

then every solution of (1.1) is oscillatory or tends to zero as $t \to \infty$.

Proof. The proof of the corollary is similar to the proof of Corollary 2.8 and so it is omitted.

Example 2.11. Consider the odd-order neutral differential equation

$$\left[x(t) + \frac{17}{18}x\left(\frac{t}{\mathrm{e}}\right)\right]^{(n)} + \frac{q_0}{t^n}x\left(\frac{t}{\mathrm{e}^2}\right) = 0, \quad n \ge 3, \quad q_0 > 0, \quad t \ge 1. \tag{2.14}$$

Using result of [[9], Example 1], every solution of (2.14) is oscillatory or tends to zero as $t \to \infty$, if

$$q_0 > 9(n-1)!\mathrm{e}^{2n-3}.$$

Applying Corollary 2.8, we have that every solution of (2.14) is oscillatory or tends to zero as $t \to \infty$, when

$$q_0 > (n-1)!\left(\mathrm{e}^{2n-3} + \frac{17\mathrm{e}^{2n-2}}{18}\right).$$

It is easy to see that our result improves those of [9].

From the above results on the oscillation of odd-order differential equation and Lemma 2.5, we can easily obtain the following results regarding the oscillation of even-order neutral differential equations.

Theorem 2.12. *Let n be even,* $0 \le p(t) \le p_0 < \infty$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$ *and* $\tau'(t) \ge \tau_0 > 0$. *Assume that* (2.2) *holds. If the first-order neutral differential inequality* (2.3) *has no positive solution for some* $\lambda_0 \in (0, 1)$, *then every solution of* (1.1) *is oscillatory.*

Theorem 2.13. *Let n be even,* $0 \le p(t) \le p_0 < \infty$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$ *and* $\tau(t) \le t$. *Assume that* (2.2) *holds. If the first-order differential inequality* (2.10) *has no positive solution for some* $\lambda_0 \in (0, 1)$, *then every solution of* (1.1) *is oscillatory.*

Corollary 2.14. Let n be even, $0 \le p(t) \le p_0 < \infty$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$ and $\tau(t) \le t$. Assume that (2.2) holds. If (2.11) holds and $\tau^{-1}(\sigma(t)) < t$, then every solution of (1.1) is oscillatory.

Theorem 2.15. *Let n be even,* $0 \le p(t) \le p_0 < \infty$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$ *and* $\tau(t) \le t$. *Assume that* (2.2) *holds. If the first-order differential inequality* (2.12) *has no positive solution for some* $\lambda_0 \in (0, 1)$, *then every solution of* (1.1) *is oscillatory.*

Corollary 2.16. Let n be even, $0 \le p(t) \le p_0 < \infty$, $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$ and $\tau(t) \le t$. Assume that (2.2) holds. If (2.13) holds and $\sigma(t) < t$, then every solution of (1.1) is oscillatory.

Example 2.17. Consider the even-order neutral differential equation

$$\left[x(t) + \frac{7}{8}x\left(\frac{t}{\mathrm{e}}\right)\right]^{(n)} + \frac{q_0}{t^n}x\left(\frac{t}{\mathrm{e}^2}\right) = 0, \quad n \ge 4, \quad q_0 > 0, \quad t \ge 1. \tag{2.15}$$

Using results of [[9], Example 1], [[21,22], Corollary 1], we find that every solution of (2.15) is oscillatory if

$$q_0 > 4(n-1)!\mathrm{e}^{2n-3}.$$

Using [[19], Theorem 2], we can obtain that (2.15) is oscillatory when

$$q_0 > 4(n-1)2^{(n-1)(n-2)}\mathrm{e}^{2n-3}.$$

Applying Corollary 2.14 in this paper, we see that (2.15) is oscillatory when

$$q_0 > (n-1)!\left(\mathrm{e}^{2n-3} + \frac{7\mathrm{e}^{2n-2}}{8}\right).$$

Hence, we can see that our results are better than [9,19,21,22].

3. Further results

In Section 2, we establish some oscillation criteria for (1.1) for the case when $(\sigma^{-1}(t))' \ge \sigma_0 > 0$, $\tau'(t) \ge \tau_0 > 0$ and $0 \le p(t) \le p_0 < \infty$, which can restrict our applications. For example, if $\tau(t) = \sqrt{t}$, then results in Section 2 fail to apply. Below, we try to weak the above restrictions. In the following, we shall continue use the notation Q as in Section 2, and we let $H(t) := \max\{1/(\sigma^{-1}(t))', p^{\gamma}(t)/(\sigma^{-1}(\tau(t)))'\}$.

Theorem 3.1. *Let n be odd,* $(\sigma^{-1}(t))' > 0$ *and* $\tau'(t) > 0$. *Assume that* (2.2) *holds. If the first-order neutral differential inequality*

$$\left(y(\sigma^{-1}(t)) + y(\sigma^{-1}(\tau(t)))\right)' + \frac{Q(t)}{H(t)}\left(\frac{\lambda_0 t^{n-1}}{(n-1)!r^{1/\gamma}(t)}\right)^{\gamma} y(t) \le 0 \tag{3.1}$$

has no positive solution for some $\lambda_0 \in (0, 1)$, *then every solution of* (1.1) *is oscillatory or tends to zero as* $t \to \infty$.

Proof. Let x be a nonoscillatory solution of (1.1) and $\lim_{t\to\infty} x(t) \neq 0$. Then $\lim_{t\to\infty} z(t) \neq 0$. From (1.1), we obtain (2.4). Thus, for all sufficiently large t, we have

$$\begin{aligned}&\frac{(r(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))^{\gamma})'}{(\sigma^{-1}(t))'}\\&+p^{\gamma}(t)\frac{(r(\sigma^{-1}(\tau(t)))(z^{(n-1)}(\sigma^{-1}(\tau(t))))^{\gamma})'}{(\sigma^{-1}(\tau(t)))'}\\&+q(\sigma^{-1}(t))x^{\gamma}(t)+p^{\gamma}(t)q(\sigma^{-1}(\tau(t)))x^{\gamma}(\tau(t))=0.\end{aligned} \tag{3.2}$$

Note that

$$\begin{aligned}&q(\sigma^{-1}(t))x^{\gamma}(t)+p^{\gamma}(t)q(\sigma^{-1}(\tau(t)))x^{\gamma}(\tau(t))\\&\geq Q(t)[x^{\gamma}(t)+p^{\gamma}(t)x^{\gamma}(\tau(t))]\\&\geq Q(t)[x(t)+p(t)x(\tau(t))]^{\gamma}\\&=Q(t)z^{\gamma}(t)\end{aligned} \tag{3.3}$$

due to (2.1) and the definition of z. It follows from (3.2) and (3.3) that

$$\begin{aligned}&\frac{(r(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))^{\gamma})'}{(\sigma^{-1}(t))'}+p^{\gamma}(t)\frac{(r(\sigma^{-1}(\tau(t)))(z^{(n-1)}(\sigma^{-1}(\tau(t))))^{\gamma})'}{(\sigma^{-1}(\tau(t)))'}\\&+Q(t)z^{\gamma}(t)\leq 0.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\left(r(\sigma^{-1}(t))(z^{(n-1)}(\sigma^{-1}(t)))^{\gamma}+r(\sigma^{-1}(\tau(t)))(z^{(n-1)}(\sigma^{-1}(\tau(t))))^{\gamma}\right)'\\&+\frac{Q(t)}{H(t)}z^{\gamma}(t)\leq 0.\end{aligned} \tag{3.4}$$

On the other hand, by Lemma 2.2 and Lemma 2.4, we have (2.9). Thus, setting $r(t)(z^{(n-1)}(t))^{\gamma} = y(t)$ in (3.4) and utilizing (2.9), one can see that y is a positive solution of (3.1). This contradicts our assumptions and the proof is complete.

Applying additional conditions on the coefficients of (3.1), we can deduce from Theorem 3.1 various oscillation criteria for (1.1).

Theorem 3.2. *Let n be odd,* $(\sigma^{-1}(t))' > 0$, $\tau'(t) > 0$ *and* $\tau(t) \leq t$. *Assume that* (2.2) *holds. If the first-order differential inequality*

$$w'(t)+\frac{Q(t)}{2H(t)}\left(\frac{\lambda_0 t^{n-1}}{(n-1)!r^{1/\gamma}(t)}\right)^{\gamma}w(\tau^{-1}(\sigma(t)))\leq 0 \tag{3.5}$$

has no positive solution for some $\lambda_0 \in (0, 1)$, *then* (1.1) *is oscillatory or tends to zero as* $t \to \infty$.

Proof. We assume that x is a positive solution of (1.1) and $\lim_{t\to\infty} x(t) \neq 0$. Then Lemma 2.4 and the proof of Theorem 3.1 imply that $y(t) = r(t)(z^{(n-1)}(t))^{\gamma} > 0$ is nonincreasing and it satisfies (3.1). Let us denote

$$w(t)=y(\sigma^{-1}(t))+y(\sigma^{-1}(\tau(t))).$$

It follows from $\tau(t) \leq t$ that

$$w(t)\leq 2y(\sigma^{-1}(\tau(t))).$$

Substituting these terms into (3.1), we get that w is a positive solution of (3.5). This contradiction completes the proof.

Corollary 3.3. Let n be odd, $(\sigma^{-1}(t))' > 0$, $\tau'(t) > 0$ and $\tau(t) \le t$. Assume that (2.2) holds. If $\tau^{-1}(\sigma(t)) < t$ and

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} \frac{Q(s)}{H(s)} \frac{(s^{n-1})^{\gamma}}{r(s)} \mathrm{d}s > \frac{2((n-1)!)^{\gamma}}{\mathrm{e}}, \tag{3.6}$$

then every solution of (1.1) is oscillatory or tends to zero as $t \to \infty$.

Proof. According to [[10], Theorem 2.1.1] the condition (3.6) guarantees that (3.5) has no positive solution. The proof of the corollary is complete.

Theorem 3.4. *Let n be odd, $(\sigma^{-1}(t))' > 0$, $\tau'(t) > 0$ and $\tau(t) \ge t$. Assume that* (2.2) *holds. If the first-order differential inequality*

$$w'(t) + \frac{Q(t)}{2H(t)} \left(\frac{\lambda_0 t^{n-1}}{(n-1)! r^{1/\gamma}(t)} \right)^{\gamma} w(\sigma(t)) \le 0 \tag{3.7}$$

has no positive solution for some $\lambda_0 \in (0, 1)$, then every solution of (1.1) *is oscillatory or tends to zero as $t \to \infty$.*

Proof. We assume that x is a positive solution of (1.1) and $\lim_{t\to\infty} x(t) \neq 0$. Then Lemma 2.4 and the proof of Theorem 3.1 imply that $y(t) = r(t)(z^{(n-1)}(t))^{\gamma} > 0$ is nonincreasing and it satisfies (3.1). We denote

$$w(t) = y(\sigma^{-1}(t)) + y(\sigma^{-1}(\tau(t))).$$

In view of $\tau(t) \ge t$, we obtain

$$w(t) \le 2y(\sigma^{-1}(t)).$$

Substituting these terms into (3.1), we get that w is a positive solution of (3.7). This is a contradiction and the proof is complete.

Corollary 3.5. Let n be odd, $(\sigma^{-1}(t))' > 0$, $\tau'(t) > 0$ and $\tau(t) \ge t$. Assume that (2.2) holds. If $\sigma(t) < t$ and

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} \frac{Q(s)}{H(s)} \frac{(s^{n-1})^{\gamma}}{r(s)} \mathrm{d}s > \frac{2((n-1)!)^{\gamma}}{\mathrm{e}}, \tag{3.8}$$

then (1.1) is oscillatory or tends to zero as $t \to \infty$.

Proof. The proof of the corollary is similar to the proof of Corollary 3.3 and so it is omitted.

From the above results on the oscillation of odd-order differential equation and Lemma 2.5, we can easily derive the following results on the oscillation of even-order neutral differential equations.

Theorem 3.6. *Let n be even, $(\sigma^{-1}(t))' > 0$ and $\tau'(t) > 0$. Assume that* (2.2) *holds. If the first-order neutral differential inequality* (3.1) *has no positive solution for some $\lambda_0 \in (0, 1)$, then every solution of* (1.1) *is oscillatory.*

Theorem 3.7. *Let n be even, $(\sigma^{-1}(t))' > 0$, $\tau'(t) > 0$ and $\tau(t) \le t$. Assume that* (2.2) *holds. If the first-order differential inequality* (3.5) *has no positive solution for some $\lambda_0 \in (0, 1)$, then* (1.1) *is oscillatory.*

Corollary 3.8. Let n be even, $(\sigma^{-1}(t))' > 0$, $\tau'(t) > 0$ and $\tau(t) \le t$. Assume that (2.2) holds. If (3.6) holds and $\tau^{-1}(\sigma(t)) < t$, then every solution of (1.1) is oscillatory.

Theorem 3.9. *Let n be even, $(\sigma^{-1}(t))' > 0$, $\tau'(t) > 0$ and $\tau(t) \ge t$. Assume that* (2.2) *holds. If the first-order differential inequality* (3.7) *has no positive solution for some $\lambda_0 \in (0, 1)$, then every solution of* (1.1) *is oscillatory.*

Corollary 3.10. Let n be even, $(\sigma^{-1}(t))' > 0$, $\tau'(t) > 0$ and $\tau(t) \ge t$. Assume that (2.2) holds. If (3.8) holds and $\sigma(t) < t$, then (1.1) is oscillatory.

For some applications of the above results, we give the following examples.

Example 3.11. Consider the odd-order neutral differential equation

$$\left[x(t) + t^2 x(t^2)\right]^{(n)} + \frac{q_0}{t^{(n-1)/4}} x(\sqrt{t}) = 0, \quad n \ge 3, \quad t \ge 1. \tag{3.9}$$

It is easy to verify that all conditions of Corollary 3.5 are satisfied. Hence, every solution of (3.9) is oscillatory or tends to zero as $t \to \infty$.

Example 3.12. Consider the even-order neutral differential equation (2.15).

Applying Corollary 3.8, we know that (2.15) is oscillatory when

$$q_0 > \frac{7}{4} e^{2n-2} (n-1)!.$$

Note that result in the section 2 is better than this. However, they are different in some cases. Therefore, they are significative for theirs existence.

4. Summary

In this note, we consider the oscillatory behavior of higher-order quasi-linear neutral differential equation (1.1) for the case when $\gamma \le 1$. Regarding the results for the case when $\gamma \ge 1$, we can replace $Q(t)$ with $Q(t)/2^{\gamma-1}$. Since

$$x_1{}^{\gamma} + x_2{}^{\gamma} \ge \frac{1}{2^{\gamma-1}} (x_1 + x_2)^{\gamma}, \quad x_1, x_2 \in [0, \infty)$$

for $\gamma \ge 1$.

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Authors' contributions
All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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