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Stability of a nonlinear non-autonomous fractional order systems with different delays and non-local conditions

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Abstract

In this paper, we establish sufficient conditions for the existence of a unique solution for a class of nonlinear non-autonomous system of Riemann-Liouville fractional differential systems with different constant delays and non-local condition is. The stability of the solution will be proved. As an application, we also give some examples to demonstrate our results.

Keywords: Riemann-Liouville derivatives, nonlocal non-autonomous system, time-delay system, stability analysis

1 Introduction

Here we consider the nonlinear non-local problem of the form

$$D^{\alpha}x_i(t) = f_i(t, x_1(t), \ldots, x_n(t)) + g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n)), t \in (0, T), \quad T < \infty, \tag{1}$$

$$x(t) = \Phi(t) \quad \text{for} \quad t < 0 \text{ and } \lim_{t \to 0^-} \Phi(t) = 0, \tag{2}$$

$$I^{1-\alpha}x(t)|_{t=0} = 0, \tag{3}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$, $x(t) = (x_1(t), x_2(t), ..., x_n(t))'$, where ' denote the transpose of the matrix, and f_i, $g_i : [0, T] \times R^n \to R$ are continuous functions, $\Phi(t) = (\varphi_i(t))_{n \times 1}$ are given matrix and O is the zero matrix, $r_j \geq 0$, $j = 1, 2, ..., n$, are constant delays.

Recently, much attention has been paid to the existence of solution for fractional differential equations because they have applications in various fields of science and engineering. We can describe many physical and chemical processes, biological systems, etc., by fractional differential equations (see [1-9] and references therein).

In this work, we discuss the existence, uniqueness and uniform of the solution of stability non-local problem (1)-(3). Furthermore, as an application, we give some examples to demonstrate our results.

For the earlier work we mention: De la Sen [10] investigated the non-negative solution and the stability and asymptotic properties of the solution of fractional differential dynamic systems involving delayed dynamics with point delays.

El-Sayed [11] proved the existence and uniqueness of the solution $u(t)$ of the problem

$$\begin{aligned} &{}^cD_a^\alpha u(t) + C\,{}^cD_a^\beta u(t-r) = Au(t) + Bu(t-r), \quad 0 \le \beta \le \alpha \le 1, \\ &u(t) = g(t), \quad t \in [a-r, a], \quad r > 0 \end{aligned}$$

by the method of steps, where A, B, C are bounded linear operators defined on a Banach space X.

El-Sayed et al. [12] proved the existence of a unique uniformly stable solution of the non-local problem

$$D^\alpha x_i(t) = \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t-r_j) + h_i(t), \quad t > 0,$$
$$x(t) = \Phi(t) \text{ for } t < 0, \quad \lim_{t\to 0^-} \Phi(t) = O \quad \text{and} \quad I^\beta x(t)|_{t=0} = O, \quad \beta \in (0,1).$$

Sabatier et al. [6] delt with Linear Matrix Inequality (LMI) stability conditions for fractional order systems, under commensurate order hypothesis.

Abd El-Salam and El-Sayed [13] proved the existence of a unique uniformly stable solution for the non-autonomous system

$${}^cD_a^\alpha x(t) = A(t)x(t) + f(t), \quad x(0) = x^0, \ t > 0,$$

where ${}^cD_a^\alpha$ is the Caputo fractional derivatives (see [5-7,14]), $A(t)$ and $f(t)$ are continuous matrices.

Bonnet et al. [15] analyzed several properties linked to the robust control of fractional differential systems with delays. They delt with the BIBO stability of both retarded and neutral fractional delay systems. Zhang [16] established the existence of a unique solution for the delay fractional differential equation

$$D^\alpha x(t) = A_0x(t) + A_1x(t-r) + f(t), \quad t > 0, \quad x(t) = \phi(t), \quad t \in [-r, 0],$$

by the method of steps, where A_0, A_1 are constant matrices and studied the finite time stability for it.

2 Preliminaries

Let $L_1[a, b]$ be the space of Lebesgue integrable functions on the interval $[a, b]$, $0 \le a < b < \infty$ with the norm $||x||_{L_1} = \int_a^b |x(t)|dt$.

Definition 1 The fractional (arbitrary) order integral of the function $f(t) \in L_1[a, b]$ of order $\alpha \in R^+$ is defined by (see [5-7,14,17])

$$I_a^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,$$

where Γ (.) is the gamma function.

Definition 2 The Caputo fractional (arbitrary) order derivatives of order α, $n < \alpha < n + 1$, of the function $f(t)$ is defined by (see [5-7,14]),

$${}^cD_a^\alpha f(t) = I_a^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f(s)ds, \quad t \in [a, b],$$

Definition 3 The Riemann-liouville fractional (arbitrary) order derivatives of order α, $n < \alpha < n + 1$ of the function $f(t)$ is defined by (see [5-7,14,17])

$$D_a^{\alpha} f(t) = \frac{d^n}{dt^n} I_a^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds, \quad t \in [a, b],$$

The following theorem on the properties of fractional order integration and differentiation can be easily proved.

Theorem 1 *Let α, $\beta \in R^+$. Then we have*

(i) $I_a^{\alpha} : L_1 \to L_1$, and if $f(t) \in L_1$ then $I_a^{\alpha} I_a^{\beta} f(t) = I_a^{\alpha+\beta} f(t)$.
(ii) $\lim_{\alpha \to n} I_a^{\alpha} = I_a^n$, $n = 1,2,3,...$ uniformly.
(iii) ${}^c D^{\alpha} f(t) = D^{\alpha} f(t) - \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a)$, $\alpha \in (0,1)$, $f(t)$ is absolutely continuous.
(iv) $\lim_{\alpha \to 1} {}^c D_a^{\alpha} f(t) = \frac{df}{dt} \neq \lim_{\alpha \to 1} D^{\alpha} f(t)$, $\alpha \in (0,1)$, $f(t)$ is absolutely continuous.

3 Existence and uniqueness

Let $X = (C_n(I), || \cdot ||_1)$, where $C_n(I)$ is the class of all continuous column n-vectors function. For $x \in C_n[0, T]$, the norm is defined by $||x||_1 = \sum_{i=1}^n \sup_{t \in [0,T]} \{e^{-Nt} |x_i(t)|\}$, where $N > 0$.

Theorem 2 *Let f_i, $g_i : [0, T] \times R^n \to R$ be continuous functions and satisfy the Lipschitz conditions*

$$|f_i(t, u_1, \ldots, u_n) - f_i(t, v_1, \ldots, v_n)| \leq \sum_{j=1}^n h_{ij} |u_j - v_j|,$$

$$|g_i(t, u_1, \ldots, u_n) - g_i(t, v_1, \ldots, v_n)| \leq \sum_{j=1}^n k_{ij} |u_j - v_j|,$$

and $h = \sum_{i=1}^n |h_i| = \sum_{i=1}^n max_{\forall j} |h_{ij}|$, $k = \sum_{i=1}^n |k_i| = \sum_{i=1}^n \max_{\forall j} |k_{ij}|$.
Then there exists a unique solution $\times \in X$ *of the problem (1)-(3).*

Proof Let $t \in (0, T)$. Then equation (1) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} x_i(t) = f_i(t, x_1(t), \ldots, x_n(t)) + g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n)).$$

Integrating both sides, we obtain

$$I^{1-\alpha} x_i(t) - I^{1-\alpha} x_i(t)|_{t=0} = \int_0^t \{f_i(t, x_1(t), \ldots, x_n(t)) + g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n))\} \mathrm{d}s.$$

From (3), we get

$$I^{1-\alpha} x_i(t) = \int_0^t \{f_i(t, x_1(t), \ldots, x_n(t)) + g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n))\} \mathrm{d}s.$$

Operating by I^{α} on both sides, we obtain

$$I x_i(t) = I^{\alpha+1} \{f_i(t, x_1(t), \ldots, x_n(t)) + g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n))\}.$$

Differentiating both side is, we get

$$x_i(t) = I^\alpha\{f_i(t, x_1(t), \ldots, x_n(t)) + g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n))\}, \quad i = 1, 2, \ldots, n. \tag{4}$$

Now let $F : X \to X$, defined by

$$Fx_i = I^\alpha\{f_i(t, x_1(t), \ldots, x_n(t)) + g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n))\}.$$

then

$$\begin{aligned}
|Fx_i - Fy_i| &= |I^\alpha\{f_i(t, x_1(t), \ldots, x_n(t)) - f_i(t, y_1(t), \ldots, y_n(t)) \\
&\quad + g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n)) - g_i(t, y_1(t - r_1), \ldots, y_n(t - r_n))\}| \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f_i(s, x_1(s), \ldots, x_n(s)) - f_i(s, y_1(s), \ldots, y_n(s))| \mathrm{d}s \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |g_i(s, x_1(s - r_1), \ldots, x_n(s - r_n)) - g_i(s, y_1(s - r_1), \ldots, y_n(s - r_n))| \mathrm{d}s \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^n h_{ij} |x_j(s) - y_j(s)| \mathrm{d}s \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^n k_{ij} |x_j(s - r_j) - y_j(s - r_j)| \mathrm{d}s
\end{aligned}$$

and

$$\begin{aligned}
e^{-Nt}|Fx_i - Fy_i| &\le h_i \sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns} |x_j(s) - y_j(s)| \mathrm{d}s \\
&\quad + k_i \sum_{j=1}^n \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+r_j)} e^{-N(s-r_j)} |x_j(s - r_j) - y_j(s - r_j)| \mathrm{d}s \\
&\le h_i \sum_{j=1}^n \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} \mathrm{d}s \\
&\quad + k_i \sum_{j=1}^n \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} e^{-Nr_j} \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} \mathrm{d}s \\
&\le h_i \sum_{j=1}^n \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \frac{1}{N^\alpha} \int_0^{Nt} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} \mathrm{d}u \\
&\quad + k_i \sum_{j=1}^n \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \frac{e^{-Nr_j}}{N^\alpha} \int_0^{N(t-r_j)} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} \mathrm{d}u \\
&\le \frac{h_i}{N^\alpha} ||x - y||_1 + \frac{k_i}{N^\alpha} \sum_{j=1}^n \sup_t \{e^{-Nt}|x_j(t) - y_j(t)|\} \\
&\le \frac{h_i + k_i}{N^\alpha} ||x - y||_1
\end{aligned}$$

and

$$\begin{aligned}
||Fx - Fy||_1 &= \sum_{i=1}^n \sup_t e^{-Nt} |Fx_i - Fy_i| \le \sum_{i=1}^n \frac{h_i + k_i}{N^\alpha} ||x - y||_1 \\
&\le \frac{h + k}{N^\alpha} ||x - y||_1.
\end{aligned}$$

Now choose N large enough such that $\frac{h+k}{N^\alpha} < 1$, so the map $F : X \to X$ is a contraction and hence, there exists a unique column vector $x \in X$ which is the solution of the integral equation (4).

Now we complete the proof by proving the equivalence between the integral equation (4) and the non-local problem (1)-(3). Indeed:

Since $x \in C_n$ and $I^{1-\alpha} x(t) \in C_n(I)$, and $f_i, g_i \in C(I)$ then $I^{1-\alpha} f_i(t)$, $I^{1-\alpha} g_i(t) \in C(I)$. Operating by $I^{1-\alpha}$ on both sides of (4), we get

$$\begin{aligned} I^{1-\alpha}x_i(t) &= I^{1-\alpha}I^{\alpha}\{f_i(t, x_1(t), \dots, x_n(t)) + g_i(t, x_1(t-r_1), \dots, x_n(t-r_n))\} \\ &= I\{f_i(t, x_1(t), \dots, x_n(t)) + g_i(t, x_1(t-r_1), \dots, x_n(t-r_n))\}. \end{aligned}$$

Differentiating both sides, we obtain

$$DI^{1-\alpha}x_i(t) = DI\{f_i(t, x_1(t), \dots, x_n(t)) + g_i(t, x_1(t-r_1), \dots, x_n(t-r_n))\},$$

which implies that

$$D^{\alpha}x_i(t) = f_i(t, x_1(t), \dots, x_n(t)) + g_i(t, x_1(t-r_1), \dots, x_n(t-r_n)), \quad t > 0,$$

which completes the proof of the equivalence between (4) and (1).

Now we prove that $\lim_{t\to 0^+} x_i = 0$. Since $f_i(t, x_1(t), ..., x_n(t))$, $g_i(t, x_1(t - r_1), ..., x_n(t - r_n))$ are continuous on $[0, T]$ then there exist constants l_i, L_i, m_i, M_i such that $l_i \le f_i(t, x_1(t), ..., x_n(t)) \le L_i$ and $m_i \le g_i(t, x_1(t - r_1)$), ..., $x_n(t - r_n)) \le M_i$, and we have

$$I^{\alpha}f_i(t, x_1(t), \dots, x_n(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_i(s, x_1(s), \dots, x_n(s))\mathrm{d}s,$$

which implies

$$\begin{gathered} l_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s \le I^{\alpha}f_i(t, x_1(t), \dots, x_n(t)) \le L_i \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s \Rightarrow \\ \frac{l_i t^{\alpha}}{\Gamma(\alpha+1)} \le I^{\alpha}f_i(t, x_1(t), \dots, x_n(t)) \le \frac{L_i t^{\alpha}}{\Gamma(\alpha+1)} \end{gathered}$$

and

$$\lim_{t\to 0^+} I^{\alpha}f_i(t, x_1(t), \dots, x_n(t)) = 0.$$

Similarly, we can prove

$$\lim_{t\to 0^+} I^{\alpha}g_i(t, x_1(t-r_1), \dots, x_n(t-r_n)) = 0.$$

Then from (4),$\lim_{t\to 0^+} x_i(t) = 0$. Also from (2), we have $\lim_{t\to 0^-} \Phi(t) = O$.

Now for $t \in (-\infty, T]$, $T < \infty$, the continuous solution $x(t) \in (-\infty, T]$ of the problem (1)-(3) takes the form

$$x_i(t) = \begin{cases} \phi_i(t), & t < 0 \\ 0, & t = 0 \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\{f_i(s, x_1(s), \dots, x_n(s)) + g_i(s, x_1(s-r_1), \dots, x_n(s-r_n))\}\mathrm{d}s, & t > 0. \end{cases}$$

4 Stability

In this section we study the stability of the solution of the non-local problem (1)-(3)

Definition 5 The solution of the non-autonomous linear system (1) is stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any two solutions $x(t) = (x_1(t), x_2(t), ..., x_n(t))'$ and $\tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t), \dots, \tilde{x}_n(t))'$ with the initial conditions (2)-(3) and

$||x(t) - \tilde{x}(t)||_1 < \varepsilon$respectively, one has $||\Phi(t) - \tilde{\Phi}(t)||_1 \le \delta$, then $||x(t) - \tilde{x}(t)||_1 < \varepsilon$ for all $t \geq 0$.

Theorem 3 *The solution of the problem* (1)-(3) *is uniformly stable.*

Proof Let $x(t)$ and $\tilde{x}(t)$ be two solutions of the system (1) under conditions (2)-(3) and $\{I^{\beta}\tilde{x}(t)|_{t=0} = 0, \tilde{x}(t) = \tilde{\Phi}(t), t < 0 \text{ and} \lim_{t\to 0} \tilde{\Phi}(t) = O\}$, respectively. Then for $t > 0$, we have from (4)

$$
\begin{aligned}
|x_i - \tilde{x}_i| &= |I^{\alpha}\{f_i(t, x_1(t), \ldots, x_n(t)) - f_i(t, \tilde{x}_1(t), \ldots, \tilde{x}_n(t)) \\
&\quad + g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n)) - g_i(t, \tilde{x}_1(t - r_1), \ldots, \tilde{x}_n(t - r_n))\}| \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |\, f_i(s, x_1(s), \ldots, x_n(s)) - f_i(s, y_1(s), \ldots, y_n(s))| ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |g_i(s, x_1(s - r_1), \ldots, x_n(s - r_n)) - g_i(s, \tilde{x}_1(s - r_1), \ldots, \tilde{x}_n(s - r_n))| \mathrm{d}s \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} h_{ij} |x_j(s) - \tilde{x}_j(s)| \mathrm{d}s \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \sum_{j=1}^{n} k_{ij} |x_j(s - r_j) - \tilde{x}_j(s - r_j)| \mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
e^{-Nt}|x_i - \tilde{x}_i| &\le h_i \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns} |x_j(s) - \tilde{x}_j(s)| \mathrm{d}s \\
&\quad + k_i \sum_{j=1}^{n} \int_0^{r_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+r_j)} e^{-N(s-r_j)} |\phi_j(s - r_j) - \tilde{\phi}_j(s - r_j)| \mathrm{d}s \\
&\quad + k_i \sum_{j=1}^{n} \int_{r_j}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+r_j)} e^{-N(s-r_j)} |x_j(s - r_j) - \tilde{x}_j(s - r_j)| \mathrm{d}s \\
&\le \frac{h_i}{N^{\alpha}} ||x_j(t) - \tilde{x}_j(t)||_1 \int_0^{Nt} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} \mathrm{d}u \\
&\quad + k_i \sum_{j=1}^{n} \sup_t \{e^{-Nt} |\phi_j(t) - \tilde{\phi}_j(t)|\} \frac{e^{-Nr_j}}{N^{\alpha}} \int_{N(t-r_j)}^{Nt} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} \mathrm{d}u \\
&\quad + k_i \sum_{j=1}^{n} \sup_t \{e^{-Nt} |x_j(t) - \tilde{x}_j(t)|\} \frac{e^{-Nr_j}}{N^{\alpha}} \int_0^{N(t-r_j)} \frac{u^{\alpha-1} e^{-u}}{\Gamma(\alpha)} \mathrm{d}u \\
&\le \frac{h_i}{N^{\alpha}} ||x_j(t) - \tilde{x}_j(t)||_1 + \frac{k_i}{N^{\alpha}} \sum_{j=1}^{n} e^{-Nr_j} \sup_t \{e^{-Nt} |x_j(t) - \tilde{x}_j(t)|\} \\
&\quad + \frac{k_i}{N^{\alpha}} \sum_{j=1}^{n} e^{-Nr_j} \sup_t \{e^{-Nt} |\varphi_j(t) - \tilde{\phi}_j(t)|\} \\
&\le \frac{h_i + k_i}{N^{\alpha}} ||x - \tilde{x}||_1 + \frac{k_i}{N^{\alpha}} ||\Phi - \tilde{\Phi}||_1.
\end{aligned}
$$

Then we have,

$$
\begin{aligned}
||x - \tilde{x}||_1 &\le \sum_{i=1}^{n} \frac{h_i + k_i}{N^{\alpha}} ||x - \tilde{x}||_1 + \sum_{i=1}^{n} \frac{k_i}{N^{\alpha}} ||\Phi - \tilde{\Phi}||_1 \\
&\le \frac{h + k}{N^{\alpha}} ||x - \tilde{x}||_1 + \frac{k}{N^{\alpha}} ||\Phi - \tilde{\Phi}||_1
\end{aligned}
$$

i.e. $\left(1 - \frac{h+k}{N^{\alpha}}\right) ||x - \tilde{x}||_1 \le \frac{k}{N^{\alpha}} ||\Phi - \tilde{\Phi}||_1$and $\quad ||x - \tilde{x}||_1 \le \frac{k}{N^{\alpha}} \left(1 - \frac{h+k}{N^{\alpha}}\right)^{-1} ||\Phi - \tilde{\Phi}||_1$

Therefore, for $\delta > 0$ s.t.$||\Phi - \tilde{\Phi}||_1 < \delta$, we can find $\varepsilon = \frac{k}{N^\alpha}\left(1 - \frac{h+k}{N^\alpha}\right)^{-1}\delta$ s.t. $||x - \tilde{x}||_1 \leq \varepsilon$which proves that the solution $x(t)$ is uniformly stable.

5 Applications

Example 1 Consider the problem

$$D^\alpha x_i(t) = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} g_{ij}(t, x_j(t - r_j)), \quad t > 0$$

$$x(t) = \Phi(t)\text{for}t < 0\text{and} \lim_{t \to 0^-} \Phi(t) = O$$

$$I^{1-\alpha}x(t)|_{t=0} = O,$$

where $A(t) = (a_{ij}(t))_{n \times n}$ and $(g_i(t, x_1(t - r_1), \ldots, x_n(t - r_n)))' = (\sum_{j=1}^{n} g_{ij}(t, x_j(t - r_j))'$ are given continuous matrix, then the problem has a unique uniformly stable solution $x \in X$ on (-∞, T], $T < \infty$

Example 2 Consider the problem

$$D^\alpha x_i(t) = \sum_{j=1}^{n} f_{ij}(t, x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)x_j(t - r_j), \quad t > 0$$

$$x(t) = \Phi(t)\text{for } t < 0\text{and } \lim_{t \to 0^-} \Phi(t) = O$$

$$I^{1-\alpha}x(t)|_{t=0} = O,$$

where $B(t) = (b_{ij}(t))_{n \times n}$, and $(f_i(t, x_1(t), \ldots, x_n(t)))' = (\sum_{j=1}^{n} f_{ij}(t, x_j(t)))'$ are given continuous matrices, then the problem has a unique uniformly stable solution $x \in X$ on (-∞, T], $T < \infty$

Example 3 Consider the problem (see [12])

$$D^\alpha x_i(t) = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t - r_j) + h_i(t), \quad t > 0$$

$$x(t) = \Phi(t) \text{ for } t < 0\text{and } \lim_{t \to 0^-} \Phi(t) = O$$

$$I^{1-\alpha}x(t)|_{t=0} = O,$$

where $A(t) = (a_{ij}(t))_{n \times n}$ $B(t) = (b_{ij}(t))_{n \times n}$, and $H(t) = (h_i(t))_{n \times 1}$ are given continuous matrices, then the problem has a unique uniformly stable solution $x \in X$ on (-∞, T], $T < \infty$.

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Authors' contributions section
All authors contributed equally to the manuscript and read and approved the final draft.

Competing interests
The authors declare that they have no competing interests.

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