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Fractional nonlocal impulsive quasilinear multi-delay integro-differential systems

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Abstract

In this article, sufficient conditions for the existence result of quasilinear multi-delay integro-differential equations of fractional orders with nonlocal impulsive conditions in Banach spaces have been presented using fractional calculus, resolvent operators, and Banach fixed point theorem. As an application that illustrates the abstract results, a nonlocal impulsive quasilinear multi-delay integro-partial differential system of fractional order is given.

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Introduction

Many fractional models can be represented by the following system

$$\frac{d^{\alpha}u(t)}{dt^{\alpha}} + A(t, u(t))u(t) = f(t, u(t), u(\beta(t))) + \int_0^t g(t, s, u(s), u(\gamma(s)))\, ds, \tag{1.1}$$

$$u(0) + h(u) = u_0, \tag{1.2}$$

$$\Delta u(t_i) = I_i(u(t_i)), \tag{1.3}$$

in a Banach space X, where $0 < \alpha \leq 1$, $t \in [0, a]$, $u_0 \in X$, $i = 1, 2, \dots, m$ and $0 < t_1 < t_2 < \cdots < t_m < a$. We assume that $-A(t,.)$ is a closed linear operator defined on a dense domain $D(A)$ in X into X such that $D(A)$ is independent of t. It is assumed also that $-A(t,.)$ generates an evolution operator in the Banach space X. The functions $f : J\, X^{r+1} \to X$, $g : \Lambda \times X^{k+1} \to X$, $h : PC(J, X) \to X$, $u(\beta) = (u(\beta_1), \dots, u(\beta_r))$, $u(\gamma) = (u(\gamma_1), \dots, u(\gamma_k))$, and β_p, γ_q $: J \to J$ are given, where $p = 1, 2, \dots, r$ and $q = 1, 2, \dots,$ k. Here $J = [0, a]$ and $\Lambda = \{(t, s).\ 0 \leq s \leq t \leq a\}$. Let $PC\,(J, X)$ consist of functions u from J into X, such that $u(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists for $i = 1, 2, \dots, m$. Clearly $PC(J, X)$ is a Banach space with the norm $||u||_{PC} = \sup_{t \in J} ||u(t)||$, and let $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitutes an impulsive condition. Fractional differential equations have proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [1-5]). They involve a wide area of applications by bringing into a broader paradigm concepts of physics and mathematics [6-8]. There has been a significant development in fractional differential and partial differential equations in recent years, see Kilbas et al. [9,10], also in fractional nonlinear systems with delay and fractional variational principles with delay, see Baleanu et al. [11,12].

The existence results to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [13,14], subsequently, many authors were pointed in the same field, see reference therein. Deng [15] indicated that, using the nonlocal condition $u(0) + h(u) = u_0$ to describe for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy problem $u(0) = u_0$. Let us observe also that since Deng's papers, the function h is considered

$$h(u) = \sum_{k=1}^{p} c_k u(t_k),$$

where c_k, $k = 1, 2, \ldots, p$ are given constants and $0 \le t_1 < \cdots < t_p \le a$. However, among the previous research on nonlocal cauchy problems, few are concerned with mild solutions of fractional semilinear differential equations, see Mophou and N'Guérékata [16], and others with fractional nonlocal boundary value problems, for instance, Ahmad et al. [17,18].

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years, because all the structures of its emergence have deep physical background and realistic mathematical model. The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. It has seen considerable development in the last decade, see the monographs of Bainov and Simeonov [19], Lakshmikantham et al. [20], and Samoilenko and Perestyuk [21] where numerous properties of their solutions are studied, and detailed bibliographies are given.

Recently, the existence of solutions of fractional abstract differential equations with nonlocal initial condition was investigated by N'Guérékata [22] and Li [23]. Much attention has been paid to existence results for the impulsive differential and integrodifferential equations of fractional order in abstract spaces, see Benchohra et al. [2,24]. Several authors have studied the existence of solutions of abstract quasilinear evolution equations in Banach space [25-27].

Regarding this article, it generalizes previous results concerned the existence of solutions to nonlocal and impulsive integrodifferential equations of quasilinear type with delays of arbitrary orders. Section "Preliminaries" is devoted to a review of some essential results. In next section, we state and prove our main results, the last section deals to giving an example to illustrate the abstract results.

1 Preliminaries

Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X. For any Banach space Z, the norm of Z is denoted by $||\cdot||_Z$. The space of all bounded linear operators from X to Y is denoted by $B(X, Y)$ and $B(X, X)$ is written as $B(X)$. We recall some definitions in fractional calculus from Gelfand-Shilov [28] and Podlubny [29], then some known facts of the theory of semigroups from Pazy [30].

Definition 2.1 The fractional integral of order with the lower limit zero for a function $f \in C([0, \infty))$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{f(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t > 0,\ 0 < \alpha < 1,$$

provided the right side is pointwise defined on $[0, \infty)$, where Γ is the gamma function. Riemann-Liouville derivative of order α with the lower limit zero for a function $f \in C([0, \infty))$ can be written as

$${}^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{0}^{t} \frac{f(s)}{(t-s)^{\alpha}} \mathrm{d}s, \quad t > 0,\ 0 < \alpha < 1.$$

The Caputo derivative of order for a function $f \in C([0, \infty))$ can be written as

$${}^{C}D^{\alpha} f(t) = {}^{L}D^{\alpha}(f(t) - f(0)), \quad t > 0,\ 0 < \alpha < 1.$$

Remark 2.1

(1) If $f \in C^{1}([0, \infty))$, then

$${}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{f'(s)}{(t-s)^{\alpha}} \mathrm{d}s = I^{1-\alpha} f'(t), \quad t > 0,\ 0 < \alpha < 1.$$

(2) The Caputo derivative of a constant is equal to zero.

(3) If f is an abstract function with values in X, then integrals which appear in Definition 2.1 are taken in Bochner's sense.

Definition 2.2 A two parameter family of bounded linear operators $U(t, s)$, $0 \le s \le t \le a$, on X is called an evolution system if the following two conditions are satisfied

(i) $U(t, t) = I$, $U(t, r)U(r, s) = U(t, s)$ for $0 \le s \le r \le t \le a$,

(ii) $(t, s) \to U(t, s)$ is strongly continuous for $0 \le s \le t \le a$.

More detail about evolution system and quasilinear equation of evolution can be found in [30, Chap. 5 and Sect. 6.4, respectively].

Let E be the Banach space formed from $D(A)$ with the graph norm. Since $-A(t)$ is a closed operator, it follows that $-A(t)$ is in the set of bounded operators from E to X.

Definition 2.3 [31-33] A resolvent operators for problem (1.1)-(1.3) is a bounded operators valued function $R_u(t, s) \in B(X)$, $0 \le s \le t \le a$, the space of bounded linear operators on X, having the following properties:

(i) $R_u(t, s)$ is strongly continuous in s and t, $R_u(s, s) = I$, $0 \le s \le a$, $\|R_u(t, s)\| \le Me^{N(t, s)}$ for some constants M and N.

(ii) $R_u(t, s)E \subset E$, $R_u(t, s)$ is strongly continuous in s and t on E.

(iii) For $x \in X$, $R_u(t, s)x$ is continuously differentiable in $s \in [0, a]$ and

$$\frac{\partial R_u}{\partial s}(t, s)x = R_u(t, s)A(s, u(s))x.$$

(iv) For $x \in X$ and $s \in [0, a]$, $R_u(t, s)x$ is continuously differentiable in $t \in [s, a]$ and

$$\frac{\partial R_u}{\partial t}(t, s)x = -A(t, u(t))R_u(t, s)x,$$

with $\frac{\partial R_u}{\partial s}(t,s)x$ and $\frac{\partial R_u}{\partial t}(t,s)x$ are strongly continuous on $0 \le s \le t \le a$. Here $R_u(t, s)$ can be extracted from the evolution operator of the generator $-A(t, u)$. The resolvent operator is similar to the evolution operator for nonautonomous differential equations in a Banach space. Let Ω be a subset of X.

Definition 2.4 (Compare [31] with [7,22,34]) By a mild solution of (1.1)-(1.3) we mean a function $u \in PC(J : X)$ with values in Ω satisfying the integral equation

$$\begin{aligned} u(t) &= R_u(t,0)u_0 - R_u(t,0)h(u) \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}R_u(t,s)[f(s,u(s),u(\beta(s))) + \int_0^s g(s,\eta,u(\eta),u(\gamma(\eta)))\mathrm{d}\eta]\mathrm{d}s \\ &+ \sum_{0<t_i<t} R_u(t,t_i)I_i(u(t_i)),\ t \in J \end{aligned} \tag{2.1}$$

for all $u_0 \in X$.

Definition 2.5 (Compare [35,36] with [2]) By a classical solution of (1.1)-(1.3) on J, we mean a function u with values in X such that:

(1) u is continuous function on $J \setminus \{t_1, t_2,..., t_m\}$ and $u(t) \in D(A)$,

(2) $\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}$ exists and continuous on J_0, $0 < \alpha < 1$,

(3) u satisfies (1.1) on J_0, the nonlocal condition (1.2) and the impulsive condition (1.3), where $J_0 = (0, a]\setminus\{t_1, t_2,..., t_m\}$. We assume the following conditions

(H_1) $h : PC(J : \Omega) \to Y$ is Lipschitz continuous in X and bounded in Y, i.e., there exist constants $k_1 > 0$ and $k_2 > 0$ such that

$$||h(u)||_Y \le k_1,$$
$$||h(u) - h(v)||_Y \le k_2 \max_{t\in J}||u - v||_{PC}, \quad u,\ v \in PC(J : X).$$

For the conditions (H_2) and (H_3) let Z be taken as both × and Y.

(H_2) $g : \Lambda \times Z^{k+1} \to Z$ is continuous and there exist constants $k_3 > 0$ and $k_4 > 0$ such that

$$\int_0^t ||g(t,s,u_1,\ldots,u_{k+1}) - g(t,s,v_1,\ldots,v_{k+1})||_Z\,\mathrm{d}s \le k_3\sum_{q=1}^{k+1}||u_q - v_q||_Z, \quad u_q,\ v_q \in X, \quad q = 1,\ \ldots,\ k+1,$$
$$k_4 = \max\left\{\int_0^t ||g(t,s,0,\ldots,0)||_Z\,\mathrm{d}s \ :\ (t,s) \in \Lambda\right\}.$$

(H_3) $f : J \times Z^{r+1} \to Z$ is continuous and there exist constants $k_5 > 0$ and $k_6 > 0$ such that

$$||f(t,u_1,\ldots,u_{r+1}) - f(t,v_1,\ldots,v_{r+1})||_Z \le k_5\sum_{p=1}^{r+1}||u_p - v_p||_Z, \quad u_p,\ v_p \in X, \quad p = 1,\ \ldots,\ r+1,$$
$$k_6 = \max_{t\in J}||f(t,0,\ldots,0)||_Z.$$

(H_4) β_p, $\gamma_q : J \to J$ are bijective absolutely continuous and there exist constants $c_p > 0$ and $b_q > 0$ such that $\beta_p'(t) \ge c_p$ and $\gamma_q'(t) \ge b_q$, respectively, for $t \in J$, $p = 1,..., r$ and $q = 1,..., k$.

(H_5) $I_i : X \to X$ are continuous and there exist constants $l_i > 0$, $i = 1, 2,..., m$ such that

$$||I_i(u) - I_i(v)|| \le l_i||u - v||, \quad u,\ v \in X.$$

Let us take $M_0 = \max ||R_u(t, s)||_{B(Z)}$, $0 \le s \le t \le a$, $u \in \Omega$.

(H_6) There exist positive constants $\delta_1, \delta_2, \delta_3 \in (0, \delta/3]$ and $\lambda_1, \lambda_2, \lambda_3 \in [0, \frac{1}{3})$ such that

$$\delta_1 = M_0||u_0||_Y + M_0k_1, \quad \delta_2 = M_0\theta, \quad \delta_3 = M_0\xi,$$

and

$$\begin{aligned}\lambda_1 &= Ka||u_0||_Y + k_1Ka + M_0k_2,\\ \lambda_2 &= Ka\theta + M_0\sigma[k_5(1 + 1/c_1 + \cdots + 1/c_r) + k_3(1 + 1/b_1 + \cdots + 1/b_k)],\\ \lambda_3 &= Ka\xi + M_0\sum_{i=1}^{m} l_i,\end{aligned}$$

where $\rho = \sigma[k_5(1/c_1 + \cdots + 1/c_r) + k_3(1/b_1 + \cdots + 1/b_k)]$, $\theta = \sigma\delta(k_3 + k_5) + \rho\delta + \sigma(k_4 + k_6)$, $\sigma = \frac{a^\alpha}{\Gamma(1+\alpha)}$ and $\xi = \sum_{i=1}^{m}(l_i\delta + ||I_i(0)||)$.

Main results

Lemma 3.1 Let $R_u(t, s)$ the resolvent operators for the fractional problem (1.1)-(1.3). There exists a constant $K > 0$ such that

$$||R_u(t,s)\omega - R_v(t,s)\omega|| \le K||\omega||_Y \int_s^t ||u(\tau) - v(\tau)||d\tau,$$

for every $u, v \in PC(J : X)$ with values in Ω and every $\omega \in Y$, see [30, lemma 4.4, p. 202].

Let $S_\delta = \{u : u \in PC(J : X), u(0) + h(u) = u_0, \Delta u(t_i) = I_i(u(t_i)), ||u|| \le \delta\}$, for $t \in J$, $\delta > 0$, $u_0 \in X$ and $i = 1,..., m$.

Lemma 3.2

$$||\varphi(t)||_Y \le \theta,$$

where

$$\varphi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[f(s, u(s), u(\beta(s))) + \int_0^s g(s, \tau, u(\tau), u(\gamma(\tau)))d\tau\right]ds.$$

Proof We have

$$\begin{aligned}&||\varphi(t)||_Y\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[||f(s, u(s), u(\beta_1(s)), \ldots, u(\beta_r(s)) - f(s, 0, \ldots, 0)|| + ||f(s, 0, \ldots, 0)||\\ &\quad + \int_0^s ||g(s, \tau, u(\tau), u(\gamma_1(\tau)), \ldots, u(\gamma_k(\tau)) - g(s, \tau, 0, \ldots, 0)||d\tau + \int_0^s ||g(s, \tau, 0, \ldots, 0)||d\tau\Bigg]ds.\end{aligned}$$

Using H_2, H_3, and H_4, we get

$$\begin{aligned}
&\|\varphi(t)\|_Y \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[k_5(\|u(s)\| + \|u(\beta_1(s))\| + \cdots + \|u(\beta_r(s))\|) + k_6 \\
&\quad + k_3(\|u(s)\| + \|u(\gamma_1(s))\| + \cdots + \|u(\gamma_k(s))\|) + k_4]\mathrm{d}s \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[k_5\{\delta + \|u(\beta_1(s))\|(\beta_1'(s)/c_1) + \cdots + \|u(\beta_r(s))\|(\beta_r'(s)/c_r)\} + k_6 \\
&\quad + k_3\{\delta + \|u(\gamma_1(s))\|(\gamma_1'(s)/b_1) + \cdots + \|u(\gamma_k(s))\|(\gamma_k'(s)/b_k)\} + k_4]\,\mathrm{d}s \\
&\le \sigma\delta(k_3+k_5) + \sigma(k_4+k_6) \\
&\quad + \frac{k_5}{c_1\Gamma(\alpha)}\int_{\beta_1(0)}^{\beta_1(t)} (t-\beta_1^{-1}(\tau))^{\alpha-1}\|u(\tau)\|d\tau + \cdots + \frac{k_5}{c_r\Gamma(\alpha)}\int_{\beta_r(0)}^{\beta_r(t)} (t-\beta_r^{-1}(\tau))^{\alpha-1}\|u(\tau)\|\mathrm{d}\tau \\
&\quad + \frac{k_3}{b_1\Gamma(\alpha)}\int_{\gamma_1(0)}^{\gamma_1(t)} (t-\gamma_1^{-1}(\eta))^{\alpha-1}\|u(\eta)\|\mathrm{d}\eta + \cdots + \frac{k_3}{b_k\Gamma(\alpha)}\int_{\gamma_k(0)}^{\gamma_k(t)} (t-\gamma_k^{-1}(\eta))^{\alpha-1}\|u(\eta)\|\mathrm{d}\eta.
\end{aligned}$$

Hence the required result.

Theorem 3.3 Suppose that the operator $-A(t, u)$ generates the resolvent operator $R_u(t, s)$ with $\|R_u(t, s)\| \le Me^{N(t-s)}$. If the hypotheses (H_1)-(H_6) are satisfied, then the fractional integro-differential equation (1.1) with nonlocal condition (1.2) and impulsive condition (1.3) has a unique mild solution on J for all $u_0 \in X$.

Proof Consider a mapping P on S_δ defined by

$$\begin{aligned}
(Pu)(t) &= R_u(t,0)u_0 - R_u(t,0)h(u) \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}R_u(t,s)\left[f(s,u(s),u(\beta(s))) + \int_0^s g(s,\eta,u(\eta),\, u(\gamma(\eta)))\mathrm{d}\eta\right]\mathrm{d}s \\
&\quad + \sum_{0<t_i<t} R_u(t,t_i)I_i(u(t_i)).
\end{aligned}$$

We shall show that $P: S_\delta \to S_\delta$. For $u \in S_\delta$, we have

$$\begin{aligned}
\|Pu(t)\|_Y \le\ & \|R_u(t,0)u_0\| + \|R_u(t,0)h(u)\| \\
&+ \left\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}R_u(t,s)\left[f(s,u(s),u(\beta(s))) + \int_0^s g(s,\eta,u(\eta),u(\gamma(\eta)))\mathrm{d}\eta\right]\mathrm{d}s\right\| \\
&+ \sum_{0<t_i<t} \|R_u(t,t_i)\|(\|I_i(u(t_i)) - I_i(0)\| + \|I_i(0)\|).
\end{aligned}$$

Using H_1, Lemma 3.2 and H_5, we get

$$\|Pu(t)_Y\| \le M_0\left\{\|u_0\| + k_1 + \theta + \sum_{i=1}^m (l_i\delta + \|I_i(0)\|)\right\}.$$

From assumption H_6, one gets $\|(Pu_\mu)(t)\|_Y \le \delta$. Thus, P maps S_δ into itself. Now for $u, v \in S_\delta$, we have

$$\|Pu(t) - Pv(t)\| \le I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \|R_u(t,0)u_0 - R_v(t,0)u_0\| + \|R_u(t,0)h(u) - R_v(t,0)h(v)\|, \\
I_2 &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|R_u(t,s)\left[f(s,u(s),u(\beta(s))) + \int_0^s g(s,\eta,u(\eta),u(\gamma(\eta)))\mathrm{d}\eta\right] \\
&\quad - R_v(t,s)[f(s,v(s),v(\beta(s))) + \int_0^s g(s,\eta,v(\eta),v(\gamma(\eta)))\mathrm{d}\eta]\|\mathrm{d}s
\end{aligned}$$

and

$$I_3 = \sum_{i=1}^{m} ||R_u(t, t_i)I_i(u(t_i)) - R_v(t, t_i)I_i(v(t_i))||.$$

Applying Lemma 3.1 and H_1, we get

$$\begin{aligned} I_1 \leq\ & ||R_u(t,0)u_0 - R_v(t,0)u_0|| + ||R_u(t,0)h(u) - R_v(t,0)h(u)|| \\ & + ||R_v(t,0)h(u) - R_v(t,0)h(v)|| \\ \leq\ & \{Ka||u_0||_Y + k_1 Ka + M_0 k_2\} \max_{\tau\in J} ||u(\tau) - v(\tau)||. \end{aligned}$$

Also, we apply Lemmas 3.1,3.2, H_2, H_3, H_4, and H_6, we obtain

$$\begin{aligned} I_2 \leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ \left\| R_u(t,s) \left[f(s, u(s), u(\beta(s))) + \int_0^s g(s, \eta, u(\eta), u(\gamma(\eta)))\mathrm{d}\eta \right] \right.\right. \\ & \left. - R_v(t,s) \left[f(s, u(s), u(\beta(s))) + \int_0^s g(s, \eta, u(\eta), u(\gamma(\eta)))\mathrm{d}\eta \right] \right\| \\ & + \left\| R_v(t,s) \left[f(s, u(s), u(\beta(s))) + \int_0^s g(s, \eta, u(\eta), u(\gamma(\eta)))\mathrm{d}\eta \right] \right. \\ & \left.\left. - R_v(t,s) \left[f(s, v(s), v(\beta(s))) + \int_0^s g(s, \eta, v(\eta), v(\gamma(\eta)))\mathrm{d}\eta \right] \right\| \right\} \mathrm{d}s \\ \leq\ & Ka\theta \max_{\tau\in J} ||u(\tau) - v(\tau)|| \\ & + M_0 \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\{ k_5 \left[||u(s) - v(s)|| + \sum_{p=1}^{r} ||u(\beta_p(s)) - v(\beta_p(s))|| (\beta_p'(s)/c_p) \right] \right. \\ & \left. + k_3 \left[||u(s) - v(s)|| + \sum_{q=1}^{k} ||u(\gamma_q(s)) - v(\gamma_q(s))|| (\gamma_q'(s)/b_q) \right] \right\} \mathrm{d}s \\ \leq\ & Ka\theta \max_{\tau\in J} ||u(\tau) - v(\tau)|| \\ & + M_0 \sigma [k_5(1 + 1/c_1 + \cdots + 1/c_r) + k_3(1 + 1/b_1 + \cdots + 1/b_k)] \max_{\tau\in J} ||u(\tau) - v(\tau)||. \end{aligned}$$

Again, Lemma 3.1, H_5 and H_6, we have

$$\begin{aligned} I_3 \leq\ & \sum_{i=1}^{m} \{||R_u(t,t_i)I_i(u(t_i)) - R_v(t,t_i)I_i(u(t_i))|| + ||R_v(t,t_i)I_i(u(t_i)) - R_v(t,t_i)I_i(v(t_i))||\} \\ \leq\ & \left\{ K \sum_{i=1}^{m} (l_i\delta + ||I_i(0)||)\, a + M_0 \sum_{i=1}^{m} l_i \right\} \max_{\tau\in J} ||u(\tau) - v(\tau)||. \end{aligned}$$

It follows from these estimations that

$$||Pu(t) - Pv(t)|| \leq \lambda \max_{\tau\in J} ||u(\tau) - v(\tau)||,$$

where $0 \leq \lambda < 1$. Thus P is a contraction on S_δ. From the contraction mapping theorem, P has a unique fixed point $u \in S_\delta$ which is the mild solution of (1.1)-(1.3) on J.

Theorem 3.4 Assume that

(i) Conditions (H_1)-(H_6) hold,
(ii) Y is a reflexive Banach space with norm $||\cdot||$,
(iii) The functions f and g are uniformly Hölder continuous in $t \in J$.

Then the problem (1.1)-(1.3) has a unique classical solution on J.

Proof From (i), applying Theorem 3.3, the problem (1.1)-(1.3) has a unique mild solution $u \in S_\delta$.. Set

$$\omega(t) = f(t, u(t), u(\beta(t))) + \int_0^t g(t, s, u(s), u(\gamma(s)))\, ds.$$

In order to prove the regularity of the mild solution, we use the further assumptions, it is easy to conclude that the function $\omega(t)$ is also uniformly Hölder continuous in $t \in J$. Consider the following fractional differential equation

$$\frac{d^\alpha v(t)}{dt^\alpha} + A(t, u)u(t) = \omega(t), \tag{3.1}$$

with the nonlocal condition (1.2) and impulsive condition (1.3).

According to Pazy [30], the late problem has a unique solution v on J into$X$ given by

$$\begin{aligned} v(t) \;=\; & R_u(t,0)u_0 - R_u(t,0)h(u) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}R_u(t,s)\omega(s)ds \\ & + \sum_{0<t_i<t} R_u(t,t_i)I_i(u(t_i)). \end{aligned}$$

Noting that, each term on the right-hand side belongs to $D(A)$, using the uniqueness of v(t), we have that $u(t) \in D(A)$. It follows that u is a unique classical solution of (1.1)-(1.3) on J.

Application

Consider the nonlinear integro-partial differential equation of fractional order

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \sum_{|q|\le 2m} a_q(x,t)u(x,t)D_x^q u(x,t) = F(x,t,u,w_1) + \int_0^t G(x,t,s,u(x,s),w_2(s))ds, \tag{4.1}$$

$$u(x,0) + \sum_{k=1}^{p} c_k u(x,t_k) = g(x), \tag{4.2}$$

$$\Delta u(x,t_k) = \int_{R^n} \rho k(y,x)u(y,t_k)dy, \tag{4.3}$$

where $0 < \alpha \le 1$, $0 \le t_1 < \cdots < t_p \le a$, $x \in R^n$, $D_x^q = D_{x_1}^{q_1} \ldots D_{x_n}^{q_n}$, $D_{x_i} = \dfrac{\partial}{\partial x_i}$, $q = (q_1,\ldots,q_n)$ is an n-dimensional multi-index, $|q| = q_1 + \cdots + q_n$, and w_i, $i = 1, 2$, is given by

$$w_i(x,t) \;=\; \sum_{|q|\le 2m-1} b_{qi}(x,t)D_x^q u(x,\sin t) \;+ \int_\Omega \sum_{|q|\le 2m-1} c_{q_i}(x,t)D_y^q u(y,\sin t)dy.$$

Let $L_2(R^n)$ be the set of all square integrable functions on R^n. We denote by $C^m(R^n)$ the set of all continuous real-valued functions defined on R^n which have continuous partial derivatives of order less than or equal to m. By $C_0^m(R^n)$ we denote the set of all functions $f \in C^m(R^n)$ with compact supports. Let $H^m(R^n)$ be the completion of $C_0^m(R^n)$ with respect to the norm

$$||f||_m^2 = \sum_{|q| \leq m} \int_{R^n} |D_x^q f(x)|^2 \mathrm{d}x.$$

It is supposed that

(i) The operator $A(t, u) = -\sum_{|q| \leq 2m} a_q(x, t)u(x, t)D_x^q$ is uniformly elliptic on R^n. In other words, all the coefficients a_q, $|q| = 2m$, are continuous and bounded on R^n and there is a positive number c such that

$$(-1)^{m+1} \sum_{|q|=2m} a_q(x, t)u(x, t)\xi^q \geq c|\xi|^{2m},$$

for all $x \in R^n$ and all $\xi \neq 0$, $\xi \in R^n$, $\xi^q = \xi_1^{q1} \ldots \xi_{n^n}^q$ and $|\xi|^2 = \xi_1^2 + \ldots + \xi_n^2$.

(ii) All the coefficients a_q, $|q| = 2m$, satisfy a uniform Hölder condition on R^n. Under these conditions the operator A with domain of definition $D(A) = H^{2m}(R^n)$ generates an evolution operator defined on $L_2(R^n)$, and it is well known that $H^{2m}(R^n)$ is dense in $X = L_2(R^n)$ and the initial function $g(x)$ is an element in Hilbert space $H^{2m}(R^n)$, see [14,15,35]. Applying Theorem 3.3, this achieves the proof of the existence of mild solutions of the system (4.1)-(4.3). In addition,

(iii) If the coefficients b_q, c_q, $|q| \leq 2m - 1$ satisfy a uniform Hölder condition on R^n and the operators F and G satisfy

There are numbers $L_1, L_2 \geq 0$ and $\lambda_1, \lambda_2 \in (0, 1)$ such that

$$\sum_{|q| \leq 2m-1} \int_{R^n} |F(x, t, u, D_x^q w_1) - F(x, s, u, D_x^q w_1^*)|^2 \mathrm{d}x \leq L_1(|t - s|^{\lambda_1} + |w_1 - w_1^*|^2 \mathrm{d}x).$$

and

$$\sum_{|q| \leq 2m-1} \int_{R^n} |\, G(x, t, \eta, u, D_x^q w_2) - G(x, s, \eta, u, D_x^q w_2)|^2 \mathrm{d}x \leq L_2 |t - s|^{\lambda_2}.$$

for all $t, s \in I$, (t, η), $(s, \eta) \in \Delta$, and all $x \in R^n$. Applying Theorem 3.4, we deduce that (4.1)-(4.3) has a unique strong solution.

Competing interests

The author declare that he has no competing interests.

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