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Stability of an additive functional equation in the spaces of generalized functions

Young-Su Lee

Correspondence: masuri@sogang.ac.kr
Department of mathematics, Sogang university, Seoul 121-741, Republic of Korea

Abstract

We reformulate the following additive functional equation with n-independent variables

$$nf\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{1 \le i < j \le n} f(x_i + x_j)$$

as the equation for the spaces of generalized functions. Making use of the fundamental solution of the heat equation we solve the general solutions and the stability problems of this equation in the spaces of tempered distributions and Fourier hyperfunctions. Moreover, using the regularizing functions, we extend these results to the space of distributions.

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1. Introduction

A function $f : \mathbb{R} \to \mathbb{R}$ is called an additive function if and only if it satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \tag{1.1}$$

for all $x, y \in \mathbb{R}$. It is well-known that every measurable solution of (1.1) is of the form $f(x) = ax$ for some constant a. In 1941, Hyers proved the stability theorem of (1.1) as follows:

Theorem 1.1 [1]. *Let E_1 be a normed vector space, E_2 a Banach space. Suppose that $f : E_1 \to E_2$ satisfies the inequality*

$$\| f(x + y) - f(x) - f(y) \| \le \varepsilon$$

for all $x, y \in E_1$. Then, there exists the unique additive mapping $g : E_1 \to E_2$ such that

$$\| f(x) - g(x) \| \le \varepsilon$$

for all $x \in E_1$.

The above stability theorem was motivated by Ulam [2]. Forti [3] noticed that the theorem of Hyers is still true if E_1 is replaced by an arbitrary semigroup. In 1978,

Rassias [4] generalized Hyers' result to the unbounded Cauchy difference. Thereafter, many authors studied the stability problems of (1.1) in various settings (see [5-7]).

During the last decades, stability problems of various functional equations have been extensively studied and generalized by a number of authors (see [8-13]). Among them, the following additive functional equation with n-independent variables:

$$nf\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{1 \leq i < j \leq n} f(x_i + x_j) \tag{1.2}$$

was proposed by Nakmahachalasint [14], where n is a positive integer with $n > 1$. He proved that (1.2) is equivalent to (1.1). For that reason, we say that (1.2) is a generalization of the Cauchy functional equation. The stability theorem of (1.2) was also proved.

In this article, in a similar manner as in [15-19], we solve the general solutions and the stability problems of (1.2) in the spaces of generalized functions such as the space $\mathcal{S}'(\mathbb{R}^m)$ of tempered distributions, the space $\mathcal{F}'(\mathbb{R}^m)$ of Fourier hyperfunctions and the space $\mathcal{D}'(\mathbb{R}^m)$ of distributions. Using the notions as in [15-19], we first reformulate (1.2) and the related inequality in the spaces of generalized functions as follows:

$$n\, u \circ A = \sum_{i=1}^{n} u \circ P_i + \sum_{1 \leq i < j \leq n} u \circ B_{ij}, \tag{1.3}$$

$$\left\| n\, u \circ A = \sum_{i=1}^{n} u \circ P_i - \sum_{1 \leq i < j \leq n} u \circ B_{ij}, \right\| \leq \varepsilon, \tag{1.4}$$

where A, P_i and B_{ij} are the functions defined by

$$A(x_1, ..., x_n) = x_1 + \cdots + x_n,$$
$$P_i(x_1, ..., x_n) = x_i, \quad 1 \leq i \leq n,$$
$$B_{ij}(x_1, ..., x_n) = x_i + x_j, \quad 1 \leq i < j \leq n.$$

Here, $\circ$ denotes the pullback of generalized functions and the inequality $||v|| \leq \varepsilon$ in (1.4) means that $|\langle v, \varphi \rangle| \leq \varepsilon \| \varphi \|_{L^1}$ for all test functions ϕ.

In Section 2, we prove that every solution u in $\mathcal{F}'(\mathbb{R}^m)$ or $\mathcal{S}'(\mathbb{R}^m)$ of the equation (1.3) has the form

$$u = a \cdot x,$$

where $a \in \mathbb{C}^m$. Also, we prove that every solution u in $\mathcal{S}'(\mathbb{R}^m)$ or $\mathcal{F}'(\mathbb{R}^m)$ of the inequality (1.4) can be written uniquely in the form

$$u = a \cdot x + \mu(x),$$

where $a \in \mathbb{C}^m$ and μ is a bounded measurable function such that $\| \mu \|_{L^\infty} \leq \frac{2\varepsilon}{n}$. Subsequently, in Section 3, these results are extended to the space $\mathcal{D}'(\mathbb{R}^m)$.

2. Stability in $\mathcal{F}'$

We first introduce the spaces of tempered distributions and Fourier hyperfunctions. Here, we use the m-dimensional notations, $|\alpha| = \alpha_1 + ... + \alpha_m$, $\alpha! = \alpha_1! ... \alpha_m!$,

$\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$, for $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{R}^m$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial \zeta_j}$.

Definition 2.1 [20,21]. We denote by $\mathcal{S}(\mathbb{R}^m)$ the Schwartz space of all infinitely differentiable functions ϕ in $\mathbb{R}^m$ satisfying

$$\| \varphi \|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^m} | x^\alpha \partial^\beta \varphi(x) | < \infty \tag{2.1}$$

for all $\alpha, \beta \in \mathbb{N}_0^m$. A linear functional u on $\mathcal{S}(\mathbb{R}^m)$ is said to be tempered distribution if there exists constant $C \geq 0$ and nonnegative integer N such that

$$| \langle u, \varphi \rangle | \leq C \sum_{|\alpha|,|\beta| \leq N} \sup_{x \in \mathbb{R}^m} | x^\alpha \partial^\beta \varphi |$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^m)$. The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^m)$.

Note that tempered distributions are generalizations of L^p-functions. These are very useful for the study of Fourier transforms in generality, since all tempered distributions have a Fourier transform. Imposing the growth condition on $||\cdot||_{\alpha,\beta}$ in (2.1) a new space of test functions has emerged as follows:

Definition 2.2 [22]. We denote by $\mathcal{F}(\mathbb{R}^m)$ the set of all infinitely differentiable functions ϕ in $\mathbb{R}^m$ such that

$$\| \varphi \|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|} B^{|\beta|} \alpha! \beta!} < \infty \tag{2.2}$$

for some positive constants A, B depending only on ϕ. The strong dual of $\mathcal{F}(\mathbb{R}^m)$, denoted by $\mathcal{F}'(\mathbb{R}^m)$, is called the Fourier hyperfunction.

It can be verified that the seminorm (2.2) is equivalent to

$$\| \varphi \|_{h,k} = \sup_{x,\alpha} \frac{| \partial^\alpha \varphi(x) | \exp k | x |}{h^{|\alpha|} \alpha!} < \infty$$

for some constants h, $k > 0$. It is easy to see the following topological inclusions:

$$\mathcal{F}(\mathbb{R}^m) \hookrightarrow \mathcal{S}(\mathbb{R}^m), \qquad \mathcal{S}'(\mathbb{R}^m) \hookrightarrow \mathcal{F}'(\mathbb{R}^m). \tag{2.3}$$

In order to solve the general solutions and the stability problems of (1.2) in the spaces $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$ we employ the m-dimensional heat kernel, fundamental solution of the heat equation,

$$E_t(x) = E(x, t) = \begin{cases} (4\pi t)^{-m/2} \exp(-|x|^2/4t), & x \in \mathbb{R}^m, \quad t > 0, \\ 0, & x \in \mathbb{R}^m, \quad t \leq 0. \end{cases}$$

Since for each $t > 0$, $E(\cdot, t)$ belongs to the space $\mathcal{F}(\mathbb{R}^m)$, the convolution

$$\tilde{u}(x, t) = (u * E)(x, t) = \langle u_y, E_t(x - y) \rangle, \qquad x \in \mathbb{R}^m, \ t > 0$$

is well-defined for all u in $\mathcal{F}'(\mathbb{R}^m)$, which is called the Gauss transform of u. Subsequently, the semigroup property

$$(E_t * E_s)(x) = E_{t+s}(x), \quad x \in \mathbb{R}^m, \quad t, s > 0$$

of the heat kernel is very useful to convert Equation (1.3) into the classical functional equation defined on upper-half plane. We also use the following famous result, so-called heat kernel method, which states as follows:

Theorem 2.3 [23]. *Let $u \in \mathcal{S}'(\mathbb{R}^m)$. Then, its Gauss transform $\tilde{u}$ is a C^∞-solution of the heat equation*

$$(\partial/\partial t - \Delta)\tilde{u}(x, t) = 0$$

satisfying

(i) There exist positive constants C, M and N such that

$$|\tilde{u}(x, t)| \leq Ct^{-M}(1 + |x|)^N \text{ in } \mathbb{R}^m \times (0, \delta). \tag{2.4}$$

(ii) $\tilde{u}(x, t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}(\mathbb{R}^m)$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x, t)\varphi(x)dx.$$

Conversely, every C^∞-solution $U(x, t)$ of the heat equation satisfying the growth condition (2.4) *can be uniquely expressed as $U(x, t) = \tilde{u}(x, t)$ for some $u \in \mathcal{S}'(\mathbb{R}^m)$.*

Similarly, we can represent Fourier hyperfunctions as a special case of the results as in [24]. In this case, the estimate (2.4) is replaced by the following:

For every $\varepsilon > 0$, there exists a positive constant C_ε such that

$$|\tilde{u}(x, t)| \leq C_\varepsilon \exp(\varepsilon(|x| + 1/t)) \text{ in } \mathbb{R}^m \times (0, \delta).$$

We are now going to solve the general solutions and the stability problems of (1.2) in the spaces of $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$. Here, we need the following lemma.

Lemma 2.4. *Suppose that $f : \mathbb{R}^m \times (0, \infty) \to \mathbb{C}$ is a continuous function satisfying*

$$nf\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i\right) = \sum_{i=1}^{n} f(x_i, t_i) + \sum_{1 \leq i < j \leq n} f(x_i + x_j, t_i + t_j) \tag{2.5}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^m$, $t_1, \ldots, t_n > 0$. Then, the solution f has the form

$$f(x, t) = a \cdot x + bt,$$

where $a \in \mathbb{C}^m$, $b \in \mathbb{C}$.

Proof. Putting $(x_1, \ldots, x_n) = (0, \ldots, 0)$ in (2.5) gives

$$nf\left(0, \sum_{i=1}^{n} t_i\right) = \sum_{i=1}^{n} f(0, t_i) + \sum_{1 \leq i < j \leq n} f(0, t_i + t_j) \tag{2.6}$$

for all $t_1, \ldots, t_n > 0$. In view of (2.6), we see that

$$c := \limsup_{t \to 0^+} f(0, t)$$

exists. Letting $t_1 = \ldots = t_n \to 0^+$ in (2.6) we have $c = 0$. Replacing $(x_3, \ldots, x_n) = (0, \ldots, 0)$ and letting $t_3 = \ldots = t_n \to 0^+$ in (2.5) yields

$$f(x_1 + x_2, t_1 + t_2) = f(x_1, t_1) + f(x_2, t_2) \tag{2.7}$$

for all $x_1, x_2 \in \mathbb{R}^m$, $t_1, t_2 > 0$. Given the continuity, the solution f of (2.7) has the form

$$f(x, t) = a \cdot x + bt,$$

where $a \in \mathbb{C}^m$, $b \in \mathbb{C}$. □

From the above lemma, we can find the general solutions of (1.2) in the spaces of $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$. Taking the inclusions of (2.3) into account, it suffices to consider the space $\mathcal{F}'(\mathbb{R}^m)$.

Theorem 2.5. *Every solution u in $\mathcal{F}'(\mathbb{R}^m)$ (or $\mathcal{S}'(\mathbb{R}^m)$, resp.) of the equation* (1.3) *has the form*

$$u = a \cdot x,$$

where $a \in \mathbb{C}^m$.

Proof. Convolving the tensor product $E_{t_1}(x_1) \cdots E_{t_n}(x_n)$ of the heat kernels on both sides of (1.3) we have

$$
\begin{aligned}
&[(u \circ A) * (E_{t_1}(x_1) \cdots E_{t_n}(x_n))](\xi_1, \ldots, \xi_n) \\
&\quad = \langle u \circ A, E_{t_1}(\xi_1 - x_1) \cdots E_{t_n}(\xi_n - x_n) \rangle \\
&\quad = \left\langle u, \int \cdots \int E_{t_1}(\xi_1 - x_1 + x_2 + \cdots + x_n) E_{t_2}(\xi_2 - x_2) \cdots E_{t_n}(\xi_n - x_n)\, dx_2 \cdots dx_n \right\rangle \\
&\quad = \left\langle u, \int \cdots \int E_{t_1}(\xi_1 + \cdots + \xi_n - x_1 - \cdots - x_n) E_{t_2}(x_2) \cdots E_{t_n}(x_n)\, dx_2 \cdots dx_n \right\rangle \\
&\quad = \langle u, (E_{t_1} * \cdots * E_{t_n})(\xi_1 + \cdots + \xi_n - x_1) \rangle \\
&\quad = \langle u, E_{t_1 + \cdots + t_n}(\xi_1 + \cdots + \xi_n) \rangle \\
&\quad = \tilde{u}(\xi_1 + \cdots + \xi_n, t_1 + \cdots + t_n), \\
&[(u \circ P_i) * (E_{t_1}(x_1) \cdots E_{t_n}(x_n))](\xi_1, \ldots, \xi_n) = \tilde{u}(\xi_i, t_i), \\
&[(u \circ B_{ij}) * (E_{t_1}(x_1) \cdots E_{t_n}(x_n))](\xi_1, \ldots, \xi_n) = \tilde{u}(\xi_i + \xi_j, t_i + t_j),
\end{aligned}
$$

where $\tilde{u}$ is the Gauss transform of u. Thus, (1.3) is converted into the following classical functional equation:

$$n\tilde{u}\left(\sum_{i=1}^n x_i, \sum_{i=1}^n t_i\right) = \sum_{i=1}^n \tilde{u}(x_i, t_i) + \sum_{1 \leq i < j \leq n} \tilde{u}(x_i + x_j, t_i + t_j) \tag{2.8}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^m$, $t_1, \ldots, t_n > 0$. It follows from Lemma 2.4 that the solution $\tilde{u}$ of (2.8) has the form

$$\tilde{u}(x, t) = a \cdot x + bt, \tag{2.9}$$

where $a \in \mathbb{C}^m$, $b \in \mathbb{C}$. Letting $t \to 0^+$ in (2.9) we obtain

$$u = a \cdot x.$$

This completes the proof. □

In what follows, we denote B, P and Q are the functions defined by

$$B(x, y) = x + y, \quad P(x_1, x_2) = x_1, \quad Q(x_1, x_2) = x_2, \quad x, y \in \mathbb{R}^m.$$

From the above theorem, we have the general solution of (1.1) in the spaces of $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$ immediately.

Corollary 2.6. *Every solution u in $\mathcal{F}'(\mathbb{R}^m)$ (or $\mathcal{S}'(\mathbb{R}^m)$, resp.) of the equation*

$$u \circ B = u \circ P + u \circ Q \tag{2.10}$$

has the form

$$u = a \cdot x,$$

where $a \in \mathbb{C}^m$.

We are going to prove the stability theorem of (1.2) in the spaces of $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$ as follows:

Theorem 2.7. *Suppose that u in $\mathcal{F}'(\mathbb{R}^m)$ (or $\mathcal{S}'(\mathbb{R}^m)$, resp.) satisfies the inequality (1.4). Then, there exists a unique $a \in \mathbb{C}^m$ such that*

$$\| u - a \cdot x \| \le \frac{2\varepsilon}{n}.$$

Proof. Convolving the tensor product $E_{t_1}(x_1) \cdots E_{t_n}(x_n)$ of the heat kernels on both sides of (1.4) we have

$$\left| nf\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i \right) - \sum_{i=1}^{n} f(x_i, t_i) - \sum_{1 \le i < j \le n} f(x_i + x_j, t_i + t_j) \right| \le \varepsilon \tag{2.11}$$

for all $x_1, ..., x_n \in \mathbb{R}^m$, $t_1, ..., t_n > 0$, where f is the Gauss transform of u. Putting $(x_1, ..., x_n) = (0, ..., 0)$ in (2.11) yields

$$\left| nf\left(0, \sum_{i=1}^{n} t_i \right) - \sum_{i=1}^{n} f(0, t_i) - \sum_{1 \le i < j \le n} f(0, t_i + t_j) \right| \le \varepsilon \tag{2.12}$$

for all $t_1, ..., t_n > 0$. In view of (2.12), we see that

$$c := \limsup_{t \to 0^+} f(0, t)$$

exists. Letting $t_1 = ... = t_n \to 0^+$ in (2.12) gives

$$| c | \le \frac{2\varepsilon}{n^2 - n}. \tag{2.13}$$

Setting $(x_3, ..., x_n) = (0, ..., 0)$, $t_3 = ... = t_n \to 0^+$ in (2.11) and using (2.13) we obtain

$$| f(x_1 + x_2, t_1 + t_2) - f(x_1, t_1) - f(x_2, t_2) | \le \frac{2\varepsilon}{n} \tag{2.14}$$

for all $x_1, x_2 \in \mathbb{R}^m$, $t_1, t_2 > 0$. Putting $(x_1, x_2) = (x, x)$, $(t_1, t_2) = (t, t)$ in (2.14) and dividing the result by 2 we have

$$\left| \frac{f(2x, 2t)}{2} - f(x, t) \right| \le \frac{\varepsilon}{n}$$

for all $x \in \mathbb{R}^m$, $t > 0$. Making use of the induction argument yields

$$\left| \frac{f(2^k x, 2^k t)}{2^k} - f(x, t) \right| \le \frac{2\varepsilon}{n}\left(1 - \frac{1}{2^k} \right) \tag{2.15}$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}^m$, $t > 0$. Replacing x, t by $2^l x$, $2^l t$ in (2.15), respectively, and dividing the result by 2^l we can see that $2^{-k} f(2^k x, 2^k t)$ is a Cauchy sequence which converges uniformly. Now let

$$A(x, t) := \lim_{k \to \infty} 2^{-k} f(2^k x, 2^k t).$$

Then, we verify that $A(x, t)$ is the unique mapping in $\mathbb{R}^m \times (0, \infty)$ satisfying the equation

$$nA\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i\right) = \sum_{i=1}^{n} A(x_i, t_i) + \sum_{1 \le i < j \le n} A(x_i + x_j, t_i + t_j) \tag{2.16}$$

for all $x_1, ..., x_n \in \mathbb{R}^m$, $t_1, ..., t_n > 0$ and the inequality

$$\mid f(x, t) - A(x, t) \mid \le \frac{2\varepsilon}{n} \tag{2.17}$$

for all $x \in \mathbb{R}^m$, $t > 0$. Given the continuity, the solution $A(x, t)$ of (2.16) is of the form

$$A(x, t) = a \cdot x + bt,$$

where $a \in \mathbb{C}^m$, $b \in \mathbb{C}$. Letting $t \to 0^+$ in (2.17) we obtain

$$\| u - a \cdot x \| \le \frac{2\varepsilon}{n}. \tag{2.18}$$

Now inequality (2.18) implies that $u - a \cdot x$ belongs to $(L^1)' = L^\infty$. Thus, all the solution u in $\mathcal{F}'$ can be written uniquely in the form $u = a \cdot x + h(x)$, where $\| h \|_{L^\infty} \le \frac{2\varepsilon}{n}$. □

From the above theorem, we shall prove the stability theorem of (1.1) in the spaces of $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$ as follows:

Corollary 2.8. *Suppose that u in $\mathcal{F}'(\mathbb{R}^m)$ (or $\mathcal{S}'(\mathbb{R}^m)$, resp.) satisfies the inequality*

$$\| u \circ B - u \circ P - u \circ Q \| \le \varepsilon. \tag{2.19}$$

Then, there exists a unique $a \in \mathbb{C}^m$ such that

$$\| u - a \cdot x \| \le \varepsilon.$$

3. Stability in $\mathcal{D}'$

In this section, we shall extend the previous results to the space of distributions. Recall that a distribution u is a linear functional on $C_c^\infty(\mathbb{R}^m)$ of infinitely differentiable functions on $\mathbb{R}^m$ with compact supports such that for every compact set $K \subset \mathbb{R}^m$ there exist constants $C > 0$ and $N \in \mathbb{N}_0$ satisfying

$$\mid \langle u, \varphi \rangle \mid \le C \sum_{|\alpha| \le N} \sup \mid \partial^\alpha \varphi \mid$$

for all $\varphi \in C_c^\infty(\mathbb{R}^m)$ with supports contained in K. The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^m)$. It is well-known that the following topological inclusions hold:

$$C_c^\infty(\mathbb{R}^m) \hookrightarrow \mathcal{S}(\mathbb{R}^m), \quad \mathcal{S}'(\mathbb{R}^m) \hookrightarrow \mathcal{D}'(\mathbb{R}^m).$$

As we seen in [15,16,19], by the semigroup property of the heat kernel, Equation (1.3) can be controlled easily in the spaces $\mathcal{F}'(\mathbb{R}^m)$ and $\mathcal{S}'(\mathbb{R}^m)$. But we cannot employ the heat kernel in the space $\mathcal{D}'(\mathbb{R}^m)$. Instead of the heat kernel, we use the function $\psi_t(x) := t^{-m}\psi(\frac{x}{t})$, $x \in \mathbb{R}^m$, $t > 0$, where $\psi(x) \in C_c^\infty(\mathbb{R}^m)$ such that

$$\psi(x) \geq 0, \quad \operatorname{supp} \psi(x) \subset \{x \in \mathbb{R}^m : \mid x \mid \leq 1\}, \quad \int \psi(x)dx = 1.$$

For example, let

$$\psi(x) = \begin{cases} A \exp(-(1 - \mid x \mid^2)^{-1}), & \mid x \mid < 1, \\ 0, & \mid x \mid \geq 1, \end{cases}$$

where

$$A = \left(\int_{|x|<1} \exp(-(1 - \mid x \mid^2)^{-1})dx \right)^{-1},$$

then it is easy to see $\psi(x)$ is an infinitely differentiable function with support $\{x : |x| \leq 1\}$. Usually, we call $\psi(x)$ to the regularizing function. If $u \in \mathcal{D}'(\mathbb{R}^m)$, then for each $t > 0$, $(u * \psi_t)(x) = \langle u_y, \psi_t(x - y)\rangle$ is a smooth function in $\mathbb{R}^m$ and $(u * \psi_t)(x) \to u$ as $t \to 0^+$ in the sense of distributions, that is, for every $\varphi \in C_c^\infty(\mathbb{R}^m)$

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int (u * \psi_t)(x)\varphi(x)dx.$$

Making use of the regularizing functions we can find the general solution of (1.2) in the space $\mathcal{D}'(\mathbb{R}^m)$ as follows:

Theorem 3.1. *Every solution u in $\mathcal{D}'(\mathbb{R}^m)$ of Equation (1.3) has the form*

$$u = a \cdot x,$$

where $a \in \mathbb{C}^m$.

Proof. Convolving the tensor product $\psi_{t_1}(x_1) \cdots \psi_{t_n}(x_n)$ of the regularizing functions on both sides of (1.3) we have

$$
\begin{aligned}
&[(u \circ A) * (\psi_{t_1}(x_1) \cdots \psi_{t_n}(x_n))](\xi_1, \ldots, \xi_n) \\
&\quad = \langle u \circ A, \psi_{t_1}(\xi_1 - x_1) \cdots \psi_{t_n}(\xi_n - x_n)\rangle \\
&\quad = \left\langle u, \int \cdots \int \psi_{t_1}(\xi_1 - x_1 + x_2 + \cdots + x_n)\psi_{t_2}(\xi_2 - x_2) \cdots \psi_{t_n}(\xi_n - x_n)\, dx_2 \cdots dx_n \right\rangle \\
&\quad = \left\langle u, \int \cdots \int \psi_{t_1}(\xi_1 + \cdots + \xi_n - x_1 - \cdots - x_n)\psi_{t_2}(x_2) \cdots \psi_{t_n}(x_n)\, dx_2 \cdots dx_n \right\rangle \\
&\quad = \langle u, (\psi_{t_1} * \cdots * \psi_{t_n})(\xi_1 + \cdots + \xi_n - x_1)\rangle \\
&\quad = (u * \psi_{t_1} * \cdots * \psi_{t_n})(\xi_1 + \cdots + \xi_n), \\
&[(u \circ P_i) * (\psi_{t_1}(x_1) \cdots \psi_{t_n}(x_n))](\xi_1, \ldots, \xi_n) = (u * \psi_{t_i})(\xi_i), \\
&[(u \circ B_{ij}) * (\psi_{t_1}(x_1) \cdots \psi_{t_n}(x_n))](\xi_1, \ldots, \xi_n) = (u * \psi_{t_i} * \psi_{t_j})(\xi_i + \xi_j).
\end{aligned}
$$

Thus, (1.3) is converted into the following functional equation:

$$n(u * \psi_{t_1} * \cdots * \psi_{t_n})(x_1 + \cdots + x_n)$$
$$= \sum_{i=1}^{n} (u * \psi_{t_i})(x_i) + \sum_{1 \leq i < j \leq n} (u * \psi_{t_i} * \psi_{t_j})(x_i + x_j) \tag{3.1}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^m$, $t_1, \ldots, t_n > 0$. In view of (3.1), it is easy to see that

$$f(x) := \limsup_{t \to 0^+} (u * \psi_t)(x)$$

exists. Putting $(x_1, \ldots, x_n) = (0, \ldots, 0)$ and letting $t_1 = \ldots = t_n \to 0^+$ in (3.1) yields $f(0) = 0$. Setting $(x_1, x_2, x_3, \ldots x_n) = (x, y, 0, \ldots, 0)$, $(t_1, t_2) = (t, s)$ and letting $t_3 = \ldots = t_n \to 0^+$ in (3.1) we have

$$(u * \psi_t * \psi_s)(x + y) = (u * \psi_t)(x) + (u * \psi_s)(y) \tag{3.2}$$

for all $x, y \in \mathbb{R}^m$, $t, s > 0$. Letting $t \to 0^+$ in (3.2) gives

$$(u * \psi_s)(x + y) = f(x) + (u * \psi_s)(y) \tag{3.3}$$

for all $x, y \in \mathbb{R}^m$, $s > 0$. Putting $y = 0$ in (3.3) yields

$$f(x) = (u * \psi_s)(x) - (u * \psi_s)(0) \tag{3.4}$$

for all $x \in \mathbb{R}^m$, $s > 0$. Applying (3.4) to (3.3) we see that f satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}^m$. Since f is a smooth function in view of (3.4), it follows that $f(x) = a \cdot x$, where $a \in \mathbb{C}^m$. Thus, from (3.4), we have

$$(u * \psi_s)(x) = a \cdot x + (u * \psi_s)(0). \tag{3.5}$$

Letting $s \to 0^+$ in (3.5) we finally obtain

$$u = a \cdot x.$$

This completes the proof. □

In a similar manner, we have the following corollary immediately.

Corollary 3.2. *Every solution u in $\mathcal{D}'(\mathbb{R}^m)$ of Equation (2.10) has the form*

$$u = a \cdot x,$$

where $a \in \mathbb{C}^m$.

Using the regularizing functions, Chung [17] extended the stability theorem of the Cauchy functional equation (1.1) to the space $\mathcal{D}'(\mathbb{R}^m)$. Similarly, we shall extend the stability theorem of (1.2) mentioned in the previous section to the space $\mathcal{D}'(\mathbb{R}^m)$.

Theorem 3.3. *Suppose that u in $\mathcal{D}'(\mathbb{R}^m)$ satisfies the inequality (1.4). Then, there exists a unique $a \in \mathbb{C}^m$ such that*

$$\| u - a \cdot x \| \leq \frac{2\varepsilon}{n}.$$

Proof. It suffices to show that every distribution satisfying (1.4) belongs to the space $\mathcal{S}'(\mathbb{R}^m)$. Convolving the tensor product $\psi_{t_1}(x_1) \cdots \psi_{t_n}(x_n)$ on both sides of (1.4) we have

$$
\left| n(u * \psi_{t_1} * \cdots * \psi_{t_n})(x_1 + \cdots + x_n) \right.
$$
$$
\left. - \sum_{i=1}^{n} (u * \psi_{t_i})(x_i) - \sum_{1 \le i < j \le n} (u * \psi_{t_i} * \psi_{t_j})(x_i + x_j) \right| \le \epsilon
\tag{3.6}
$$

for all $x_1, ..., x_n \in \mathbb{R}^m$, $t_1, ..., t_n > 0$. In view of (3.6), it is easy to see that for each fixed x,

$$
f(x) := \limsup_{t \to 0^+} (u * \psi_t)(x)
$$

exists. Putting $(x_1, ..., x_n) = (0, ..., 0)$ and letting $t_1 = ... = t_n \to 0^+$ in (3.6) yields

$$
| f(0) | \le \frac{2\varepsilon}{n^2 - n}.
\tag{3.7}
$$

Setting $(x_1, x_2, x_3, ..., x_n) = (x, y, 0, ..., 0)$, $(t_1, t_2) = (t, s)$, $t_3 = ... = t_n \to 0^+$ in (3.6), and using (3.7), we have

$$
| (u * \psi_t * \psi_s)(x + y) - (u * \psi_t)(x) - (u * \psi_s)(y) | \le \frac{2\varepsilon}{n}
\tag{3.8}
$$

for all $x, y \in \mathbb{R}^m$, $t, s > 0$. Putting $y = 0$ in (3.8) we obtain

$$
| (u * \psi_t * \psi_s)(x) - (u * \psi_t)(x) - (u * \psi_s)(0) | \le \frac{2\varepsilon}{n}
\tag{3.9}
$$

for all $x \in \mathbb{R}^m$, $t, s > 0$. Letting $t \to 0^+$ in (3.9) gives

$$
| (u * \psi_s)(x) - f(x) - (u * \psi_s)(0) | \le \frac{2\varepsilon}{n}
\tag{3.10}
$$

for all $x \in \mathbb{R}^m$, $s > 0$. From (3.8) and (3.10), we have

$$
| f(x + y) - f(x) - f(y) | \le \frac{6\varepsilon}{n}
$$

for all $x, y \in \mathbb{R}^m$. According to the result as in [1], there exists a unique function $g : \mathbb{R}^m \to \mathbb{C}$ satisfying the equation

$$
g(x + y) = g(x) + g(y)
$$

for all $x, y \in \mathbb{R}^m$ such that

$$
| f(x) - g(x) | \le \frac{6\varepsilon}{n}
\tag{3.11}
$$

for all $x \in \mathbb{R}^m$. It follows from (3.10) and (3.11) that

$$
| (u * \psi_s)(x) - g(x) - (u * \psi_s)(0) | \le \frac{12\varepsilon}{n}
\tag{3.12}
$$

for all $x \in \mathbb{R}^m$, $s > 0$. Letting $s \to 0^+$ in (3.12) we obtain

$$\| u - g(x) \| \leq \frac{2(6n - 5)\varepsilon}{n^2 - n}. \tag{3.13}$$

Inequality (3.13) implies that $h(x) := u - g(x)$ belongs to $(L^1)' = L^\infty$. Thus, we conclude that $u = g(x) + h(x) \in \mathcal{S}'(\mathbb{R}^m)$. □

From the above theorem, we have the following corollary.

Corollary 3.4. *Suppose that u in $\mathcal{D}'(\mathbb{R}^m)$ satisfies the inequality* (2.19). *Then, there exists a unique $a \in \mathbb{C}^m$ such that*

$$\| u - a \cdot x \| \leq \varepsilon.$$

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