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Bifurcation analysis of a diffusive model of pioneer and climax species interaction

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Abstract

A diffusive model of pioneer and climax species interaction is considered. We perform a detailed Hopf bifurcation analysis to the model, and derive conditions for determining the bifurcation direction and the stability of the bifurcating periodic solutions.

Keywords: pioneer and climax, Hopf bifurcation, diffusive model

1 Introduction

We consider the following model:

$$\begin{cases} u_t = d_1 \Delta u + uf(c_{11}u + v), \\ v_t = d_2 \Delta v + vg(u + c_{22}v), \end{cases} \tag{1.1}$$

where $x \in \Omega$, $t > 0$, and u, v represent a measure of a pioneer and a climax species, respectively. $f(z)$, the growth rate of the pioneer population, is generally assumed to be smoothly deceasing, and has a unique positive root at a value z_1 so that the crowding is particularly harmful for pioneer species. But for the climax population, it is different from pioneer population. Climax fitness increases at low total density but decreasing at higher densities. So that, it has an optimum value of density for growing. Hence, $g(z)$, the growth rate of the climax population, is assumed to be non-monotone, has a hump, and possesses two distinct positive roots at some values z_2 and z_3, with $z_2 < z_3$ and $g'(z_2) > 0 > g'(z_3)$. For the reason above, we set

$$\begin{aligned} f(c_{11}u + v) &= z_1 - c_{11}u - v, \\ g(u + c_{22}v) &= -(z_2 - u - c_{22}v)(z_3 - u - c_{22}v) \end{aligned} \tag{1.2}$$

in this article.

Equation (1.1) is often used to describe forestry models. Examples can be found in [1,2] and references therein. The dynamics of pioneer-climax models have been studied widely. Systems described by ordinary differential equations are under the hypothesis of homogeneous environment. The stability of positive equilibrium and bifurcation, especial Hopf bifurcation are the subject of many investigations. More recently, the environmental factors are introduced to the pioneer-climax systems. Models including diffusivity (i.e. systems described by reaction-diffusion equations) have been considered. The existence of positive steady state solutions are the subject of investigations.

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In addition, traveling wave solutions are the most interesting problem. The readers can get some results from [3]. In bifurcation problems, Buchanan [4] has studied Turing instability in a pioneer/climax population interaction model. He determined the values of the diffusional coefficients for which the model undergoes a Turing bifurcation, and he show that a Turing bifurcation occurs when an equilibrium solution becomes unstable to perturbations which are nonhomogeneous in space but remains stable to spatially homogeneous perturbations. Hopf bifurcation for diffusive pioneer-climax species interaction has not been studied. Our study will be performed in Hopf bifurcation.

The rest of this article are structured in the following way: in Section 2, the conditions of the existence of positive equilibrium are given. The critical values of the parameter for Hopf bifurcation occurring are also searched. And the stability and direction of the bifurcating periodic solutions at λ_1 are studied. In Section 3, some conclusions are stated.

2 Hopf bifurcation analysis

In this section, we consider the following model:

$$\begin{cases} u_t = d_1 \Delta u + u(z_1 - c_{11} u - v), \\ v_t = d_2 \Delta v - v(z_2 - u - c_{22} v)(z_3 - u - c_{22} v). \end{cases} \tag{2.1}$$

Clearly, it has one trivial equilibrium (0, 0), and three semitrivial equilibria (z_1/c_{11},0), (0, z_2/c_{22}), and (0, z_3/c_{22}). There also has two nontrivial equilibria E_1, E_2:

$$E_1 = \left(\frac{z_2 - c_{22} z_1}{1 - c_{11} c_{22}}, \frac{z_1 - c_{11} z_2}{1 - c_{11} c_{22}} \right), \quad E_2 = \left(\frac{z_3 - c_{22} z_1}{1 - c_{11} c_{22}}, \frac{z_1 - c_{11} z_3}{1 - c_{11} c_{22}} \right).$$

As in [4], in the following, we will limit our analysis to the case $z_3 > z_2$ and $z_1 > c_{11} z_2$, $z_2 > c_{22} z_1$. Immediately, the condition $c_{11}c_{22} < 1$ follows as a consequence, and then E_1 is a constant positive equilibrium. If there has additional condition that $z_1 > c_{11} z_3$, then E_2 is an another constant positive equilibrium. E_1, E_2 are also positive equilibria for Equation (2.1) without diffusion, and when E_2 exists, it is unstable. In fact, the linear system at $E_2 = (u^*, v^*)$ has the form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c_{11} u^* f'(c_{11} u^* + v^*) & u^* f'(c_{11} u^* + v^*) \\ v^* g'(z_3) & c_{22} v^* g'(z_3) \end{pmatrix}.$$

For $f'(c_{11} u^* + v^*) = -1$ and $g'(z_3) = z_2 - z_3$, then the trace and determinant of L are

$$\begin{aligned} \operatorname{tr} L &= -c_{11} u^* + c_{22} v^* (z_2 - z_3) < 0, \\ \det L &= (1 - c_{11} c_{22}) u^* v^* (z_2 - z_3) < 0, \end{aligned}$$

which imply that L has a positive eigenvalue, and then E_2 is unstable. Hence, the researchers are concerned more about the dynamics at E_1. In the corresponding diffusion system, the dynamics at E_1 is richer than that at E_2. Hence, we take our attention to the equilibrium E_1. In [4], Turing instability has been studied thoroughly. The effect on the stability due to the diffusion is analyzed. In this article, we pay attention to Hopf bifurcation bifurcated by E_1. We investigate on the effect on the stability due to the diffusion. In other words, diffusion driving Hopf bifurcation is studied.

Denote $\lambda = z_2 - c_{22}z_1$. With the conditions above, we have that $\lambda < z_3 - c_{22}z_1$ and $0 < \lambda < (1 - c_{11}c_{22})\, z_1/c_{11}$. Hence, the domain of the parameter λ is $0 < \lambda < \min\{z_3 - c_{22}z_1, (1 - c_{11}c_{22})\, z_1/c_{11}\}$. In this article, we choose λ as a main bifurcation parameter and consider the complicated dynamic behavior near the fixed point E_1 with the effect of diffusion.

For convenience, we first transform the equilibrium $E_1 = (u^*, v^*)$ to the origin via the translation $\hat{u} = u - \lambda/(1 - c_{11}c_{22})$, $\hat{v} = v - (z_1 - c_{11}\lambda/(1 - c_{11}c_{22}))$ and drop the hats for simplicity of notation, then system (2.1) is transformed into

$$\begin{cases} u_t = d_1 \Delta u + a_{11} u + a_{12} v + f(u, v), \\ v_t = d_2 \Delta v + a_{21} u + a_{22} v + g(u, v), \end{cases} \tag{2.2}$$

where

$$a_{11} = -c_{11}u^*, \quad a_{12} = -u^*, \quad a_{21} = \bar{z}v^*, \quad a_{22} = c_{22}\bar{z}v^*,$$

and

$$\begin{aligned} \bar{z} =& z_3 - u^* - c_{22}v^*, \\ f(u, v) =& - c_{11}u^2 - uv, \\ g(u, v) =& (\bar{z} - 2c_{22}v^*)uv + (c_{22}\bar{z} - c_{22}^2 v^*)v^2 \\ & - v^* u^2 - u^2 v - 2c_{22}uv^2 - c_{22}^2 v^3. \end{aligned}$$

In the following, we consider system (2.2) on spatial domain $\Omega = (0, \ell\pi)$, $\ell \in \mathbb{R}^+$ with Dirichlet boundary condition

$$u(0, t) = u(\ell\pi, t) = 0, \quad v(0, t) = v(\ell\pi, t) = 0, \quad t > 0.$$

Define the real-valued Sobolev space

$$X := \{(u, v) \mid u, v \in H^2(0, \ell\pi), (u, v) \mid_{x=0,\ell\pi} = 0\},$$

and the complexification of X by $X_{\mathbb{C}} = X + iX = \{x_1 + ix_2 | x_1, x_2 \in X\}$.

The linearized operator of system (2.2) evaluated at (0, 0) is

$$L := \begin{pmatrix} a_{11} + d_1 \partial^2/\partial x^2 & a_{12} \\ a_{21} & a_{22} + d_2 \partial^2/\partial x^2 \end{pmatrix}$$

and accordingly we define (denote μ_n, $n \in \mathbb{N}$ are the eigenvalues of the eigenvalue problem $-\Delta\varphi = \mu\varphi$, $\varphi(0) = \varphi(\ell\pi) = 0$)

$$L_n := \begin{pmatrix} a_{11} - d_1 \mu_n & a_{12} \\ a_{21} & a_{22} - d_2 \mu_n \end{pmatrix}.$$

Then, the characteristic equation of $L_n(\lambda)$ is

$$\beta^2 - \beta T_n + D_n = 0, \quad n = 1, 2, \ldots, \tag{2.3}$$

where

$$\begin{cases} T_n = a_{11} + a_{22} - (d_1 + d_2)\mu_n, \\ D_n = a_{11}a_{22} - a_{12}a_{21} - (d_1 a_{22} + d_2 a_{11})\mu_n + d_1 d_2 \mu_n^2. \end{cases}$$

More immediately, let T_n, D_n be expressed by expression with parameter λ:

$$\begin{cases} T_n(\lambda) = -(d_1 + d_2)\mu_n + \dfrac{c_{11}c_{22}}{1 - c_{11}c_{22}}\lambda^2 - \left(2c_{22}z_1 + \dfrac{c_{11}c_{22}z_3 - c_{22}z_1 + c_{11}}{1 - c_{11}c_{22}}\right)\lambda \\ \qquad + c_{22}z_1(z_3 - c_{22}z_1), \\ D_n(\lambda) = d_1 d_2 \mu_n^2 - \left[d_1 \dfrac{c_{11}c_{22}}{1 - c_{11}c_{22}}\lambda^2 - c_{22}d_1\left(2z_1 + \dfrac{c_{11}z_3 - z_1}{1 - c_{11}c_{22}}\right)\lambda \right. \\ \qquad \left. + c_{22}d_1 z_1(z_3 - c_{22}z_1) - d_2 \dfrac{c_{11}\lambda}{1 - c_{11}c_{22}}\right]\mu_n + \dfrac{c_{11}}{1 - c_{11}c_{22}}\lambda^3 \\ \qquad - \left(2z_1 + \dfrac{c_{11}z_3 - z_1}{1 - c_{11}c_{22}}\right)\lambda^2 + z_1(z_3 - c_{22}z_1)\lambda. \end{cases}$$

According to [5], we have

Lemma 2.1. *Hopf bifurcation occurs at a certain critical value λ_0 if there exists unique $n \in \mathbb{N}$ such that*

$$T_n(\lambda_0) = 0, \quad D_n(\lambda_0) > 0 \quad \text{and} \quad T_j(\lambda_0) \neq 0, D_j(\lambda_0) \neq 0 \quad \text{for } j \neq n; \tag{2.4}$$

and for the unique pair of complex eigenvalues near the imaginary axis $\alpha(\lambda) \pm i\omega(\lambda)$, the transversality condition $\alpha'(\lambda_0) \neq 0$ holds.

Let us consider the sign of $D_n(\lambda)$ first. Denote $\bar{\lambda} = \min\{z_3 - c_{22}z_1, (1 - c_{11}c_{22})z_1/c_{11}\}$. Clearly, $\bar{\lambda} = z_3 - c_{22}z_1$ if $c_{11}z_3 > z_1$ and $\bar{\lambda} = (1 - c_{11}c_{22})z_1/c_{11}$ if $c_{11}z_3 > z_1$. We will prove that there exists $N_1 \in \mathbb{N}$ such that $D_n(\lambda) > 0$ for all $\lambda \in (0, \bar{\lambda})$ and $n > N_1$ under some simple conditions.

Lemma 2.2. *If $z_1 \leq c_{11}z_3/2$ or $z_1 \geq 2c_{11}z_3$, then $D_n(\lambda) > 0$ for all $\lambda \in (0, \bar{\lambda})$ and $n > N_1$, where $N_1 \in \mathbb{N}$ such that $\mu_n > c_{22}z_1(z_3 - c_{22}z_1)/d_2$ for $n > N_1$.*

Proof. First, we claim that $D_n(0) > 0$, $D_n(\bar{\lambda}) > 0$ for all $n > N_1$. Directly calculating, we have

$$D_n(0) = d_1 d_2 \mu_n^2 - c_{22}d_1 z_1(z_3 - c_{22}z_1)\mu_n > 0,$$

$$D_n(\bar{\lambda}) = \begin{cases} d_1 d_2 \mu_n^2 + d_2 \mu_n \dfrac{c_{11}(z_3 - c_{22}z_1)}{1 - c_{11}c_{22}} > 0 & \text{if} \quad \bar{\lambda} = z_3 - c_{22}z_1, \\ d_1 d_2 \mu_n^2 + d_2 \mu_n z_1 > 0 & \text{if} \quad \bar{\lambda} = (1 - c_{11}c_{22})z_1/c_{11}. \end{cases}$$

Next, we prove that for all $\lambda \in (0, \bar{\lambda})$, $D_n(\lambda) > 0$ if $D_n(0) > 0$, $D_n(\bar{\lambda}) > 0$ satisfied. From the expression of $D_n(\lambda)$, we have $D_n(\lambda) \to +\infty$ when $\lambda \to +\infty$ and $D_n(\lambda) \to -\infty$ when $\lambda \to -\infty$, and $D_n(\lambda)$ has two inflection points for any fixed $n \in \mathbb{N}$. We only need to prove that 0 and $\bar{\lambda}$ are in the same side of the second inflection point. Differentiating $D_n(\lambda)$ with respect to λ for fixed n, we have

$$D'_n(\lambda) = a\lambda^2 + b\lambda + c,$$

where

$$a = \frac{3c_{11}}{1 - c_{11}c_{22}},$$
$$b = -2z_1 - \frac{2c_{11}(z_3 - c_{22}z_1)}{1 - c_{11}c_{22}} - 2d_1\mu_n \frac{c_{11}c_{22}}{1 - c_{11}c_{22}},$$
$$c = z_1(z_3 - c_{22}z_1) - c_{22}d_1\mu_n\left(2z_1 + \frac{c_{11}z_3 - z_1}{1 - c_{11}c_{22}}\right) + d_2\mu_n \frac{c_{11}}{1 - c_{11}c_{22}}.$$

The axis of symmetry of $D_n'(\lambda)$ is

$$\lambda_{\min} = \frac{1}{3}\left[(z_3 - c_{22}z_1) + \frac{1 - c_{11}c_{22}}{c_{11}}z_1 + c_{22}d_1\mu_n\right] > 0.$$

If $z_1 \le c_{11}z_3/2$, then $\lambda_{\min} \ge \bar{\lambda} = (1 - c_{11}c_{22})z_1/c_{11}$. Else if $z_1 \ge 2c_{11}z_3$, then $\lambda_{\min} \ge \bar{\lambda} = z_3 - c_{22}z_1$. That is, $0 < \bar{\lambda} \le \lambda_{\min}$, 0 and $\bar{\lambda}$ are in the same side of the second inflection point and the proof is complete.

Next, we seek the critical points $\lambda \in (0, \bar{\lambda})$ such that $T_n = 0$. Define

$$\mathscr{T}(\lambda, p) := -(d_1 + d_2)p + \frac{c_{11}c_{22}}{1 - c_{11}c_{22}}\lambda^2 - \left(2c_{22}z_1 + \frac{c_{11}c_{22}z_3 - c_{22}z_1 + c_{11}}{1 - c_{11}c_{22}}\right)\lambda + c_{22}z_1(z_3 - c_{22}z_1).$$

Then, $T_n(\lambda) = 0$ is equivalent to $\mathscr{T}(\lambda, p) = 0$. Solving p from $\mathscr{T}(\lambda, p) = 0$, we have

$$p(\lambda) = \frac{1}{d_1 + d_2}\left[\frac{c_{11}c_{22}}{1 - c_{11}c_{22}}\lambda^2 - \left(2c_{22}z_1 + \frac{c_{11}c_{22}z_3 - c_{22}z_1 + c_{11}}{1 - c_{11}c_{22}}\right)\lambda + c_{22}z_1(z_3 - c_{22}z_1)\right].$$

Immediately,

$$p(0) = \frac{1}{d_1 + d_2}c_{22}z_1(z_3 - c_{22}z_1) > 0,$$

$$p(\bar{\lambda}) = \begin{cases} -\dfrac{1}{d_1 + d_2}\cdot\dfrac{c_{11}(z_3 - c_{22}z_1)}{1 - c_{11}c_{22}} < 0 & \text{if } \bar{\lambda} = z_3 - c_{22}z_1, \\ -\dfrac{z_1}{d_1 + d_2} < 0 & \text{if } \bar{\lambda} = (1 - c_{11}c_{22})z_1/c_{11}. \end{cases}$$

Lemma 2.3. *Denote* $N_2 \in \mathbb{N}$ *be the number such that* $\mu_{N_2} \le p(0) < \mu_{N_2+1}$. *Then, there exists* N_2 *points* λ_i, $i = 1,2, \ldots, N_2$, *satisfying* $\bar{\lambda} > \lambda_1 > \lambda_2 > \cdots > \lambda_{N_2} \ge 0$, *such that* $T_i(\lambda_j) < 0$ *for* $i < j$, *and* $T_i(\lambda_j) > 0$ *for* $i < j$, $i = 1,2, \ldots, 1 \le j \le N_2$.

Lemma 2.4. *Suppose* λ_i, $1 \le i \le N_2$ *be defined as in Lemma 2.3. If* $\alpha(\lambda_i) \pm i\omega(\lambda_i)$be *the unique pair of complex eigenvalues near the imaginary axis, then* $\alpha'(\lambda_i) < 0$.

Theorem 2.5. *Suppose the condition of Lemma 2.2 is satisfied and* λ_i, $1 \le i \le N_2$ *be defined as in Lemma 2.3. Then, Hopf bifurcation occurs at* λ_i *if*

$$\mu_i < \frac{d_2 - d_1}{d_1(d_1 + d_2)}\cdot\frac{c_{11}\lambda_i}{1 - c_{11}c_{22}}, \quad 1 \le i \le \min\{N_1, N_2\}, \tag{2.5}$$

where N_1, N_2 *are defined as before.*

Proof. We need to show that $D_n(\lambda_i) > 0$, $n \in \mathbb{N}$, then Lemma 2.1 could be used. First, $T_i(\lambda_i) = 0$ gives

$$(d_1 + d_2)\mu_i + \frac{c_{11}\lambda_i}{1 - c_{11}c_{22}} = \frac{c_{11}c_{22}}{1 - c_{11}c_{22}}\lambda_i^2 - c_{22}\left(2z_1 + \frac{c_{11}z_3 - z_1}{1 - c_{11}c_{22}}\right)\lambda_i + c_{22}z_1(z_3 - c_{22}z_1).$$

Now, $D_n(\lambda_i)$ could be expressed as

$$D_n(\lambda_i) = d_1d_2\mu_n^2 - \left(d_1^2\mu_i + d_1d_2\mu_i + (d_1 - d_2)\frac{c_{11}\lambda_i}{1 - c_{11}c_{22}}\right)\mu_n + \frac{c_{11}}{1 - c_{11}c_{22}}\lambda_i^3 - \left(2z_1 + \frac{c_{11}z_3 - z_1}{1 - c_{11}c_{22}}\right)\lambda_i^2 + z_1(z_3 - c_{22}z_1)\lambda_i.$$

Define

$$\begin{aligned}\mathscr{D}(\lambda_i, p) =& d_1 d_2 p^2 - \left(d_1^2\mu_i + d_1 d_2 \mu_i + (d_1 - d_2)\frac{c_{11}\lambda_i}{1 - c_{11}c_{22}}\right)p \\ &+ \frac{c_{11}}{1 - c_{11}c_{22}}\lambda_i^3 - \left(2z_1 + \frac{c_{11}z_3 - z_1}{1 - c_{11}c_{22}}\right)\lambda_i^2 + z_1(z_3 - c_{22}z_1)\lambda_i.\end{aligned}$$

Clearly, $\mathscr{D}(\lambda_i, 0) > 0$ and the axis of symmetry of $\mathscr{D}(\lambda_i, p)$ is

$$p_{\min} = \frac{d_1^2\mu_i + d_1 d_2\mu_i + (d_1 - d_2)c_{11}\lambda_i/(1 - c_{11}c_{22})}{2d_1 d_2}.$$

The condition in the theorem ensure $p_{\min} < 0$, which lead to $\mathscr{D}(\lambda_i, p) > 0$ for $p > 0$. Hence, $D_n(\lambda_i) > 0$ and λ_i are Hopf bifurcation points.

Remark 2.6. Theorem 2.5 gives a sufficient condition for Hopf bifurcation occurring. From the proof of Theorem 2.5, we see that the inequality (2.5) is stringent. We consider that $\mathscr{D}(\lambda_i, p)$ is continuous with respect to p, but $D_n(\lambda_i)$ is a set of discrete values. Hence, we need not to ensure that the inequality (2.5) is always satisfied in some simple case. For instance, $N_2 = 1$. Example 2.8 exactly demonstrates this feature.

In the following, we take attention to the stability and direction of bifurcating periodic solutions bifurcated at λ_1.

We give the detail of the calculation process of the direction of Hopf bifurcation at λ_1 in the following. It is obvious that $\pm i\omega$, with $\omega = \sqrt{D_1(\lambda_1)}$, are the only pair of simple purely imaginary eigenvalues of $L(\lambda_1)$. We need to calculate the Poincaré norm form of (2.2) for $\lambda = \lambda_1$:

$$\dot{z} = i\omega z + z\sum_{j=1}^{M} c_j (z\bar{z})^j,$$

where z is a complex variable, $M \geq 1$and c_j are complex-valued coefficients. The direction of Hopf bifurcation at λ_1 is decided by the sign of $\mathrm{Re}(c_1)$, which has the following form:

$$c_1 = \frac{i}{2\omega}\left(g_{20}g_{11} - 2\,|\,g_{11}|^2 - \frac{1}{3}\,|\,g_{02}|^2\right) + \frac{1}{2}g_{21}.$$

In the following, we will calculate g_{20}, g_{11}, g_{02}, and g_{21}. We recall that

$$\begin{aligned} f(u, v) &= -c_{11}u^2 - uv, \\ g(u, v) &= (\bar{z} - 2c_{22}v^*)uv + (c_{22}\bar{z} - c_{22}^2 v^*)v^2 \\ &\quad - v^* u^2 - u^2 v - 2c_{22}uv^2 - c_{22}^2 v^3. \end{aligned}$$

Notice that the eigenvalues $\mu_n = n^2/\ell^2$, $n = 1,2, \ldots$, the corresponding eigenfunction are $\sin(nx/\ell)$ in our problem. Hence, we set $q = (a, b)^T \sin(x/\ell)$ be such that $L(\lambda_1)q = i\omega q$ and let $q^* = M(a^*, b^*)^T \sin(x/\ell)$ be such that $L(\lambda_1)^T q^* = -i\omega q^*$, and moreover, $\langle q^*, q\rangle = 1$ and $\langle q^*, \bar{q}\rangle = 0$. Here

$$\langle u, v\rangle = \int_0^{\ell\pi} \bar{u}^T v\,dx, \qquad u, v \in X_{\mathbb{C}}$$

be the inner dot and

$$a = b^* = 1, \quad b = \frac{i\omega + d_1\mu_1 - a_{11}}{a_{12}}, \quad a^* = \frac{-i\omega + d_2\mu_1 - a_{22}}{a_{12}}, \quad M = \frac{2\ell\pi\omega}{ia_{12}}.$$

Express the partial derivatives of $f(u, v)$ and $g(u, v)$ at $(u, v) = (0, 0)$ with respect to λ when λ_1, we have

$$\begin{aligned}
&f_{uu} = -c_{11}, \quad f_{uv} = -1, \quad g_{uv} = z_3 - 3c_{22}z_1 + \frac{\lambda_1(3c_{11}c_{22} - 1)}{1 - c_{11}c_{22}},\\
&g_{uu} = -z_1 + \frac{c_{11}\lambda_1}{1 - c_{11}c_{22}}, \quad g_{vv} = c_{22}(z_3 - 2c_{22}z_1) + \frac{c_{22}\lambda_1(2c_{11}c_{22} - 1)}{1 - c_{11}c_{22}},\\
&g_{vvv} = -c_{22}^2, \quad g_{uuv} = -1, \quad g_{uvv} = -2c_{22},
\end{aligned}$$

and the others are equal to zero. As stated in [5,6], we need to calculate $Q_{qq}, Q_{q\bar{q}}$, and $C_{qq\bar{q}}$, which are defined as

$$Q_{qq} = \sin^2(x/\ell)\begin{pmatrix} c \\ d \end{pmatrix}, \quad Q_{q\bar{q}} = \sin^2(x/\ell)\begin{pmatrix} e \\ f \end{pmatrix}, \quad C_{qq\bar{q}} = \sin^3(x/\ell)\begin{pmatrix} g \\ h \end{pmatrix},$$

where

$$\begin{cases}
c = f_{uu}a^2 + 2f_{uv}ab + f_{vv}b^2, \ d = g_{uu}a^2 + 2g_{uv}ab + g_{vv}b^2,\\
e = f_{uu} \mid a|^2 + f_{uv}(a\bar{b} + \bar{a}b) + f_{vv} \mid b|^2, \ f = g_{uu} \mid a|^2 + g_{uv}(a\bar{b} + \bar{a}b) + g_{vv} \mid b|^2,\\
g = f_{uuu} \mid a|^2 a + f_{uuv}(2 \mid a|^2 b + a^2\bar{b}) + f_{uvv}(2 \mid b|^2 a + b^2\bar{a}) + f_{vvv} \mid b|^2 b,\\
h = g_{uuu} \mid a|^2 a + g_{uuv}(2 \mid a|^2 b + a^2\bar{b}) + g_{uvv}(2 \mid b|^2 a + b^2\bar{a}) + g_{vvv} \mid b|^2 b.
\end{cases}$$

From direct calculation, we have

$$\begin{aligned}
\langle q^*, Q_{qq}\rangle &= \frac{4\ell\bar{M}}{3}(\bar{a}^*c + d), \quad \langle q^*, Q_{q\bar{q}}\rangle = \frac{4\ell\bar{M}}{3}(\bar{a}^*e + f),\\
\langle \bar{q}^*, Q_{qq}\rangle &= \frac{4\ell M}{3}(a^*c + d), \quad \langle \bar{q}^*, Q_{q\bar{q}}\rangle = \frac{4\ell M}{3}(a^*e + f).
\end{aligned} \tag{2.6}$$

Then, we have (the detail meaning of the following parameters are stated in [6,5])

$$\begin{aligned}
H_{20} &= Q_{qq} - \langle q^*, Q_{qq}\rangle q - \langle \bar{q}^*, Q_{qq}\rangle \bar{q}\\
&= \frac{1}{2}(1 - \cos(2x/\ell))\begin{pmatrix} c \\ d \end{pmatrix} - \left[\langle q^*, Q_{qq}\rangle \begin{pmatrix} a \\ b \end{pmatrix} - \langle \bar{q}^*, Q_{qq}\rangle \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}\right]\sin(x/\ell)\\
&= \sum_{k=1}^{\infty} \frac{-8}{(2k-1)(2k+1)(2k-3)\pi}\begin{pmatrix} c \\ d \end{pmatrix}\sin((2k-1)x/\ell)\\
&\quad - \left[\langle q^*, Q_{qq}\rangle \begin{pmatrix} 1 \\ b \end{pmatrix} - \langle \bar{q}^*, Q_{qq}\rangle \begin{pmatrix} 1 \\ \bar{b} \end{pmatrix}\right]\sin(x/\ell)
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
H_{11} &= Q_{q\bar{q}} - \langle q^*, Q_{q\bar{q}}\rangle q - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \bar{q}\\
&= \frac{1}{2}(1 - \cos(2x/\ell))\begin{pmatrix} e \\ f \end{pmatrix} - \left[\langle q^*, Q_{q\bar{q}}\rangle \begin{pmatrix} a \\ b \end{pmatrix} - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}\right]\sin(x/\ell)\\
&= \sum_{k=1}^{\infty} \frac{-8}{(2k-1)(2k+1)(2k-3)\pi}\begin{pmatrix} e \\ f \end{pmatrix}\sin((2k-1)x/\ell)\\
&\quad - \left[\langle q^*, Q_{q\bar{q}}\rangle \begin{pmatrix} 1 \\ b \end{pmatrix} - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \begin{pmatrix} 1 \\ \bar{b} \end{pmatrix}\right]\sin(x/\ell).
\end{aligned} \tag{2.8}$$

Therefore, we can obtain w_{20}, w_{11} as

$$w_{20} = [2i\omega I - L(\lambda_1)]^{-1}H_{20} \quad \text{and} \quad w_{11} = -[L(\lambda_1)]^{-1}H_{11}.$$

Clearly, the calculation of $(2i\omega I - L(\lambda_1))^{-1}$ and $[L(\lambda_1)]^{-1}$ are restricted to the subspaces spanned by the eigenmodes $\sin(kx/\ell)$, $k = 1,2, \ldots$. One can compute that

$$\begin{aligned}
&(2i\omega I - L_k(\lambda_1))^{-1} \\
&\quad = (\alpha_1^k + i\alpha_2^k)^{-1} \begin{pmatrix} 2i\omega - a_{22} + d_2\mu_k & a_{12} \\ a_{21} & 2i\omega - a_{11} + d_1\mu_k \end{pmatrix}, \\
&L_k^{-1}(\lambda_1) = \frac{1}{\alpha_3^k} \begin{pmatrix} a_{22} - d_2\mu_k & -a_{12} \\ -a_{21} & a_{11} - d_1\mu_k \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1^k &= -4\omega^2 + a_{11}a_{22} - a_{12}a_{21} - (d_1a_{22} + d_2a_{11})\mu_k + d_1d_2\mu_k^2, \\
\alpha_2^k &= -2\omega(a_{11} + a_{22}) + 2\omega(d_1 + d_2)\mu_k, \\
\alpha_3^k &= a_{11}a_{22} - a_{12}a_{21} - (d_2a_{11} + d_1a_{22})\mu_k + d_1d_2\mu_k^2.
\end{aligned}$$

Then,

$$\begin{aligned}
w_{20} &= \sum_{k=1}^{\infty} \frac{-8\sin((2k-1)x/\ell)}{(2k-1)(2k+1)(2k-3)\pi}(2i\omega I - L_{2k-1}(\lambda_1))^{-1}\begin{pmatrix} c \\ d \end{pmatrix} \\
&\quad - (2i\omega I - L_1(\lambda_1))^{-1}\left[\langle q^*, Q_{qq}\rangle \begin{pmatrix} a \\ b \end{pmatrix} - \langle \bar{q}^*, Q_{qq}\rangle \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}\right]\sin(x/\ell) \\
&= \sum_{k=1}^{\infty} \frac{-8\sin((2k-1)x/\ell)(2k-3)^{-1}}{(4k^2-1)(\alpha_1^{2k-1} + i\alpha_2^{2k-1})\pi} \begin{pmatrix} (2i\omega - a_{22} + d_2\mu_{2k-1})c + a_{12}d \\ a_{21}c + (2i\omega - a_{11} + d_1\mu_{2k-1})d \end{pmatrix} \\
&\quad - \frac{1}{\alpha_1^1 + i\alpha_2^1} \begin{pmatrix} (2i\omega - a_{22} + d_2\mu_1)\xi_1 + a_{12}\xi_2 \\ a_{21}\xi_1 + (2i\omega - a_{11} + d_1\mu_1)\xi_2 \end{pmatrix}\sin(x/\ell), \\
w_{11} &= \sum_{k=1}^{\infty} \frac{-8\sin((2k-1)x/\ell)}{\alpha_3^{2k-1}(4k^2-1)(2k-3)\pi} - \begin{pmatrix} (a_{22} - d_2\mu_{2k-1})e + a_{12}f \\ a_{21}e - (a_{11} - d_1\mu_{2k-1})f \end{pmatrix} \\
&\quad - \frac{1}{\alpha_3^1} \begin{pmatrix} -(a_{22} - d_2\mu_1)\xi_3 + a_{12}\xi_4 \\ a_{21}\xi_3 - (a_{11} - d_1\mu_1)\xi_4 \end{pmatrix}\sin(x/\ell),
\end{aligned}$$

where

$$\begin{aligned}
\xi_1 &= \langle q^*, Q_{qq}\rangle a - \langle \bar{q}^*, Q_{qq}\rangle \bar{a} = \frac{4c\ell}{3}(\bar{a}^*\bar{M} - a^*M) + \frac{4d\ell}{3}(\bar{M} - M), \\
\xi_2 &= \langle q^*, Q_{qq}\rangle b - \langle \bar{q}^*, Q_{qq}\rangle \bar{b} = \frac{4c\ell}{3}(b\bar{a}^*\bar{M} - \bar{b}a^*M) + \frac{4d\ell}{3}(b\bar{M} - \bar{b}M), \\
\xi_3 &= \langle q^*, Q_{q\bar{q}}\rangle a - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \bar{a} = \frac{4e\ell}{3}(\bar{a}^*\bar{M} - a^*M) + \frac{4f\ell}{3}(\bar{M} - M), \\
\xi_4 &= \langle q^*, Q_{q\bar{q}}\rangle b - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \bar{b} = \frac{4e\ell}{3}(b\bar{a}^*\bar{M} - \bar{b}a^*M) + \frac{4f\ell}{3}(b\bar{M} - \bar{b}M).
\end{aligned}$$

Then,

$$\begin{aligned}
Q_{w_{20}\bar{q}} &= \sum_{k=1}^{\infty} \begin{pmatrix} Q^{1k}_{w_{20}\bar{q}} \\ Q^{2k}_{w_{20}\bar{q}} \end{pmatrix} \sin\frac{x}{\ell}\sin\frac{(2k-1)x}{\ell} + \begin{pmatrix} Q^{10}_{w_{20}\bar{q}} \\ Q^{20}_{w_{20}\bar{q}} \end{pmatrix}\sin^2\frac{x}{\ell} \\
&= \sum_{k=1}^{\infty} \begin{pmatrix} f_{uu}w_{20}^{1k} + f_{uv}\bar{b}w_{20}^{1k} + f_{uv}w_{20}^{2k} \\ g_{uu}w_{20}^{1k} + g_{uv}\bar{b}w_{20}^{1k} + g_{uv}w_{20}^{2k} + g_{vv}\bar{b}w_{20}^{2k} \end{pmatrix}\sin\frac{x}{\ell}\sin\frac{(2k-1)x}{\ell} \\
&\quad + \begin{pmatrix} f_{uu}w_{20}^{10} + f_{uv}\bar{b}w_{20}^{10} + f_{uv}w_{20}^{20} \\ g_{uu}w_{20}^{10} + g_{uv}\bar{b}w_{20}^{10} + g_{uv}w_{20}^{20} + g_{vv}\bar{b}w_{20}^{20} \end{pmatrix}\sin^2\frac{x}{\ell}, \\
Q_{w_{11}q} &= \sum_{k=1}^{\infty} \begin{pmatrix} Q^{1k}_{w_{11}q} \\ Q^{2k}_{w_{11}q} \end{pmatrix} \sin\frac{x}{\ell}\sin\frac{(2k-1)x}{\ell} + \begin{pmatrix} Q^{10}_{w_{11}q} \\ Q^{20}_{w_{11}q} \end{pmatrix}\sin^2\frac{x}{\ell}, \\
&= \sum_{k=1}^{\infty} \begin{pmatrix} f_{uu}w_{11}^{1k} + f_{uv}bw_{11}^{1k} + f_{uv}w_{11}^{2k} \\ g_{uu}w_{11}^{1k} + g_{uv}bw_{11}^{1k} + g_{uv}w_{11}^{2k} + g_{vv}bw_{11}^{2k} \end{pmatrix}\sin\frac{x}{\ell}\sin\frac{(2k-1)x}{\ell} \\
&\quad + \begin{pmatrix} f_{uu}w_{11}^{10} + f_{uv}bw_{11}^{10} + f_{uv}w_{11}^{20} \\ g_{uu}w_{11}^{10} + g_{uv}bw_{11}^{10} + g_{uv}w_{11}^{20} + g_{vv}bw_{11}^{20} \end{pmatrix}\sin^2\frac{x}{\ell},
\end{aligned}$$

where

$$
\begin{aligned}
w_{20}^{1k} &= \sum_{k=1}^{\infty} \frac{-8(2i\omega - a_{22} + d_2\mu_{2k-1})c + a_{12}d)}{(4k^2-1)(2k-3)(\alpha_1^{2k-1} + i\alpha_2^{2k-1})\pi}, \quad k = 1, 2, \dots, \\
w_{20}^{2k} &= \sum_{k=1}^{\infty} \frac{-8(a_{21}c + (2i\omega - a_{11} + d_1\mu_{2k-1})d)}{(4k^2-1)(2k-3)(\alpha_1^{2k-1} + i\alpha_2^{2k-1})\pi}, \quad k = 1, 2, \dots, \\
w_{11}^{1k} &= \sum_{k=1}^{\infty} \frac{-8(-(a_{22} - d_2\mu_{2k-1})e + a_{12}f)}{\alpha_3^{2k-1}(4k^2-1)(2k-3)\pi}, \quad k = 1, 2, \dots, \\
w_{11}^{2k} &= \sum_{k=1}^{\infty} \frac{-8(a_{21}e - (a_{11} - d_1\mu_{2k-1})f)}{\alpha_3^{2k-1}(4k^2-1)(2k-3)\pi}, \quad k = 1, 2, \dots,
\end{aligned}
$$

and

$$
\begin{aligned}
w_{20}^{10} &= \frac{(2i\omega - a_{22} + d_2\mu_1)\xi_1 + a_{12}\xi_2}{\alpha_1^1 + i\alpha_2^1}, \quad w_{20}^{20} = \frac{a_{21}\xi_1 + (2i\omega - a_{11} + d_1\mu_1)\xi_2}{\alpha_1^1 + i\alpha_2^1}, \\
w_{11}^{10} &= \frac{-(a_{22} - d_2\mu_1)\xi_3 + a_{12}\xi_4}{\alpha_3^1}, \quad w_{11}^{20} = \frac{a_{21}\xi_3 - (a_{11} - d_1\mu_1)\xi_4}{\alpha_3^1}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\int_0^{\ell\pi} \sin^4(x/\ell)dx = \frac{3\ell\pi}{8}, \\
&\int_0^{\ell\pi} \sin^2(x/\ell)\sin((2k-1)x/\ell)dx = \frac{-4\ell}{(2k-1)(2k+1)(2k-3)},
\end{aligned}
$$

we have

$$
\begin{aligned}
\langle q^*, C_{qq\bar{q}} \rangle &= \frac{3\ell\bar{M}h\pi}{8}, \\
\langle q^*, Q_{w_{20}\bar{q}} \rangle &= \sum_{k=1}^{\infty} \frac{-4\ell\bar{M}}{(2k-1)(2k+1)(2k-3)} (\bar{a}^* Q_{w_{20}\bar{q}}^{1k} + Q_{w_{20}\bar{q}}^{2k}) \\
&\quad + \frac{4\ell M}{3} (\bar{a}^* Q_{w_{20}\bar{q}}^{10} + Q_{w_{20}\bar{q}}^{20}), \\
\langle q^*, Q_{w_{11}q} \rangle &= \sum_{k=1}^{\infty} \frac{-4\ell\bar{M}}{(2k-1)(2k+1)(2k-3)} (\bar{a}^* Q_{w_{11}q}^{1k} + Q_{w_{11}q}^{2k}) \\
&\quad + \frac{4\ell M}{3} (\bar{a}^* Q_{w_{11}\bar{q}}^{10} + Q_{w_{11}\bar{q}}^{20}).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
g_{20} &= \langle q^*, Q_{qq} \rangle = \frac{4\ell\bar{M}}{3} (\bar{a}^* c + d), \\
g_{11} &= \langle q^*, Q_{q\bar{q}} \rangle = \frac{4\ell\bar{M}}{3} (\bar{a}^* e + f), \\
g_{02} &= \langle q^*, Q_{\bar{q}q} \rangle = \frac{4\ell\bar{M}}{3} (\bar{a}^* \bar{c} + \bar{d}),
\end{aligned}
$$

and

$$\begin{aligned} g_{21} &= 2\langle q^*, Q_{w_{11}q}\rangle + \langle q^*, Q_{w_{20}\bar{q}}\rangle + \langle q^*, C_{qq\bar{q}}\rangle \\ &= \sum_{k=1}^{\infty} \frac{-4\ell\bar{M}((2Q^{1k}_{w_{11}q} + Q^{1k}_{w_{20}\bar{q}})\bar{a}^*_n + (2Q^{2k}_{w_{11}q} + Q^{2k}_{w_{20}\bar{q}}))}{(2k-1)(2k+1)(2k-3)} \\ &\quad + \frac{4\ell\bar{M}((2Q^{10}_{w_{11}q} + Q^{10}_{w_{20}\bar{q}})\bar{a}^* + (2Q^{20}_{w_{11}q} + Q^{20}_{w_{20}\bar{q}}))}{3} + \frac{3\ell\bar{M}h\pi}{8}. \end{aligned}$$

Then, it follows that

$$\begin{aligned} c_1 &= \frac{i}{2\omega}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{1}{2}g_{21} \\ &= \frac{8\ell^2 i}{9\omega}[\bar{M}^2(\bar{a}^*c + d)(\bar{a}^*e + f) - 2|M|^2|\bar{a}^*e + f|^2 - \frac{1}{3}|M|^2|\bar{a}^*c + d|^2] \\ &\quad + \sum_{k=1}^{\infty} \frac{-2\ell\bar{M}((2Q^{1k}_{w_{11}q} + Q^{1k}_{w_{20}\bar{q}})\bar{a}^*_n + (2Q^{2k}_{w_{11}q} + Q^{2k}_{w_{20}\bar{q}}))}{(2k-1)(2k+1)(2k-3)} \\ &\quad + \frac{2\ell\bar{M}((2Q^{10}_{w_{11}q} + Q^{10}_{w_{20}\bar{q}})\bar{a}^* + (2Q^{20}_{w_{11}q} + Q^{20}_{w_{20}\bar{q}}))}{3} + \frac{3\ell\bar{M}h\pi}{16}. \end{aligned}$$

Theorem 2.7. *Suppose the conditions in Theorem 2.7 are satisfied. Then, the positive constant equilibrium E_1 is asymptotically stable when $\lambda \in (\lambda_1, \bar{\lambda})$. Hopf bifurcation occurs at λ_1, and the bifurcating periodic solutions are in the left(right) neighborhood of λ_1 and stable(unstable) if $Re(c_1) < 0 (> 0)$.*

Example 2.8. *Suppose $\ell = 1$(i. e. $\Omega = (0, \pi)$). $d_1 = 1/10$, $d_2 = 3/10$, $z_1 = z_2 = 1$, $z_3 = 3/2$ and $c_{11} = 1/3$. Let c_{22} be the bifurcation parameter. We found that there has only one Hopf bifurcation point $\lambda = 0.0833$. E_1 is stable for $0.0833 < \lambda < 1.1667$. For $\lambda < 0.0833$, Hopf bifurcation occurs and the bifurcating periodic solutions are stable. In other words, $c_{22} = 0.9167$ is the critical value for Hopf bifurcation. We give the simulation for c_{22} 0.9167 ± 0.02 in the follows. If $c_{22} = 0.9167 - 0.02$, E_1 is stable* (Figure 1). *If $c_{22} = 0.9167 + 0.02$, there exists periodic solution, which is stable* (Figure 2).

3 Conclusion

In this article, we take λ as a main bifurcation parameter, study stability of the constant positive equilibrium E_1, which exists for $\lambda \in (0, \bar{\lambda})$. The critical values for Hopf

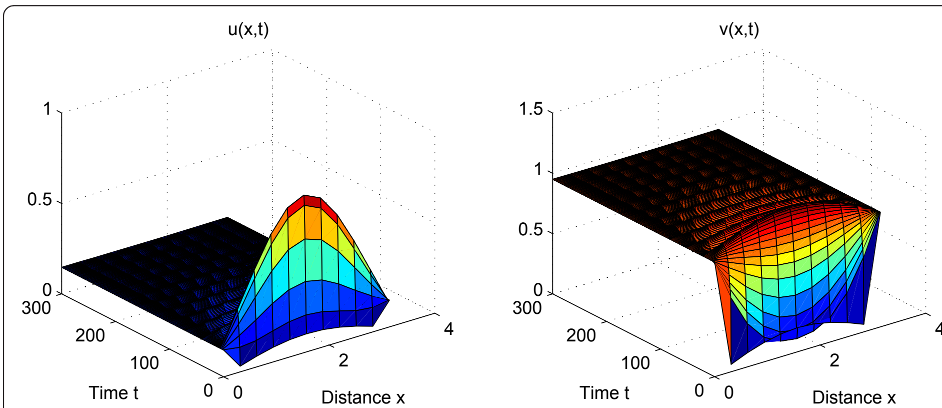

Figure 1 E_1 is asymptotically stable for c_{22} = 0.9167 - 0.02. The initial value is $(u_0, v_0) = (0.1, 0.2) * \sin x$.

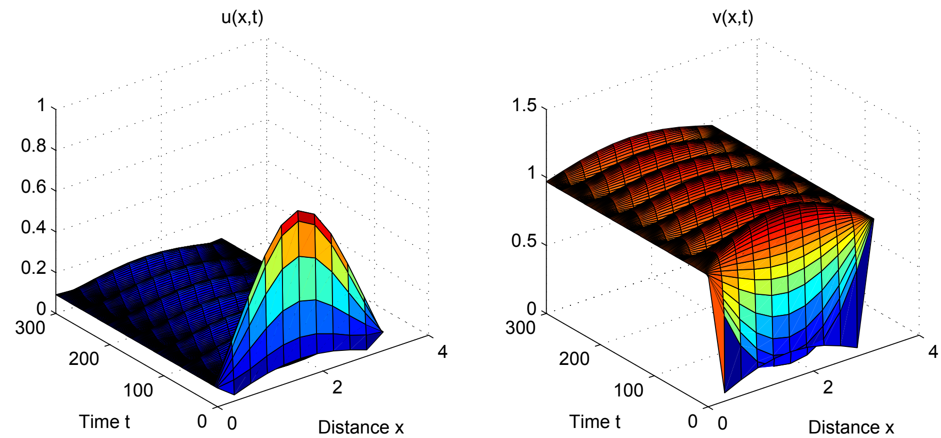

Figure 2 Periodic solution appear for c_{22} = 0.9167 + 0.02. The initial value is $(u_0, v_0) = (0.1, 0.2) * \sin x$.

bifurcation occurring are found out, and the stability and direction of bifurcating periodic solutions bifurcated at λ_1 are studied. By the method of the reference [5] and our early work [6], we give the detail of the calculation of the norm form for system (2.2). In addition, we claim that the bifurcating periodic solutions are all spatially nonhomogeneous, since the problem is subject to Dirichlet fixed boundary conditions.

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Authors' contributions
JL carried out the theoretical analysis and simulation, and drafted the manuscript. JW conceived of the study, and participated in its design and coordination and helped to draft the manuscript. All authors read and approved the final manuscript.

Competing interests
We declare that we have no significant competing financial, professional, or personal interests that might have influenced the performance or presentation of the work described in this manuscript.

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