Ran *Advances in Difference Equations* 2011, **2011**:53
http://www.advancesindifferenceequations.com/content/2011/1/53

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Galilean invariance and the conservative difference schemes for scalar laws

Zheng Ran

Correspondence: zran@staff.shu.edu.cn
Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, P. R. China

Abstract

Galilean invariance for general conservative finite difference schemes is presented in this article. Two theorems have been obtained for first- and second-order conservative schemes, which demonstrate the necessity conditions for Galilean preservation in the general conservative schemes. Some concrete application has also been presented.

Keywords: difference scheme, symmetry, shock capturing method

1. Introduction

For gas dynamics, the non-invariance relative to Galilean transformation of a difference scheme which approximates the equations results in non-physical fluctuations, that has been marked in the 1960s of the past century [1]. In 1970, Yanenko and Shokin [2] developed a method of differential approximations for the study of the group properties of difference schemes for hyperbolic systems of equations. They used the first differential approximation to perform a group analysis. A more recent series of articles was devoted to the Lie point symmetries of differential difference equations on [3]. In a series of more recent articles, the author of this article has used Lie symmetry analysis method to investigate some noteworthy properties of several difference schemes for nonlinear equations in shock capturing [4,5].

It is well known that as for Navier-Stokes equations, the intrinsic symmetries, except for the scaling symmetries, are just macroscopic consequences of the basic symmetries of Newton's equations governing microscopic molecular motion (in classical approximation). Any physical difference scheme should inherit the elementary symmetries (at least for Galilean symmetry) from the Navier-Stokes equations. This means that Galilean invariance has been an important issue in computational fluid dynamics (CFD). Furthermore, we stress that Galilean invariance is a basic requirement that is demanded for any physical difference scheme. The main purpose of this article is to make differential equations discrete while preserving their Galilean symmetries.

Two important questions on numerical analysis, especially important for shock capturing methods, are discussed from the point view of group theory below.

(1) Galilean preservation in first- second-order conservative schemes;

(2) Galilean symmetry preservation and Harten's entropy enforcement condition [6].

The structure of this article is as follows. First, the general remarks on scalar conservation law and its numerical approximation are very briefly discussed in Section 2, while Section 3 is devoted to the theory of symmetries of differential equations. The following sections are devoted to a complete development of Lie symmetry analysis method proposed here and its application to some special cases of interest. The final section contains concluding remarks.

2. Scalar conservation laws and its numerical approximation

In this article, we consider numerical approximations to weak solutions of the initial value problem (IVP) for hyperbolic systems of conservation laws [6,7]

$$u_t + f(u)_x = 0,\ u(x,0) = u_0(x), -\infty < x < +\infty. \tag{2.1}$$

where $u(x, t)$ is a column vector of m unknowns, and $f(u)$, the flux, is a scalar valued function. Equation 2.1 can be written as

$$u_t + a(u)u_x = 0,\ a(u) = \frac{df}{du}, \tag{2.2}$$

which asserts that u is constant along the characteristic curves $x = x(t)$, where

$$\frac{dx}{dt} = a\,(u)\,. \tag{2.3}$$

The constancy of u along the characteristic combined with (2.3) implies that the characteristics are straight lines. Their slope, however, depends upon the solution and therefore they may intersect, and where they do, no continuous solution can exist. To get existence in the large, i.e., for all time, we admit weak solutions which satisfy an integral version of (2.1)

$$\int_0^\infty \int_{-\infty}^\infty \left[w_t u + w_x f\,(u)\right] dxdt + \int_{-\infty}^\infty w\,(x,0)\,u_0\,(x)\,dx = 0. \tag{2.4}$$

for every smooth test function $w(x, t)$ of compact support.

If u is piecewise continuous weak solution, then it follows from (2.4) that across the line of discontinuity the Rankine-Hugoniot relation

$$f\,(u_{\mathrm{R}}) - f\,(u_{\mathrm{L}}) = s\,(u_{\mathrm{R}} - u_{\mathrm{L}})\,. \tag{2.5}$$

holds, where s is the speed of propagation of the discontinuity, and u_{L} and u_{R} are the states on the left and on the right of the discontinuity, respectively.

The class of all weak solutions is too wide in the sense that there is no uniqueness for the IVP, and an additional principle is needed for determining a physically relevant solution. Usually this principle identifies the physically relevant solution as a limit of solutions with some dissipation, namely

$$u_t + f(u)_x = \varepsilon[\beta\,(u)\,u_x]_x. \tag{2.6}$$

Oleinik [8] has shown that discontinuities of such admissible solutions can be characterized by the following condition:

$$\frac{f\,(u) - f\,(u_L)}{u - u_L} \geq s \geq \frac{f\,(u) - f\,(u_R)}{u - u_R}. \tag{2.7}$$

for all u between u_{L} and u_{R}; this is called the entropy condition, or Condition E. Oleinik has shown that weak solutions satisfying Condition E are uniquely determined by their initial data. We shall discuss numerical approximations to weak solutions of (2.1) which are obtained by $(2K+1)$ -point explicit schemes in conservation form

$$u_j^{n+1} = u_j^n - \lambda \left(\bar{f}^n_{j+\frac{1}{2}} - \bar{f}^n_{j-\frac{1}{2}} \right), \tag{2.8}$$

where

$$\bar{f}^n_{j+\frac{1}{2}} = \bar{f}\left(u^n_{j-K+1}, ..., u^n_{j+K}\right). \tag{2.9}$$

where $u_j^n = u\left(j\Delta x, n\Delta t\right)$, and $\bar{f}$ is a numerical flux function. We require the numerical flux function to be consistent with the flux $f(u)$ in the following sense:

$$\bar{f}\left(u, ..., u\right) = f\left(u\right). \tag{2.10}$$

We note that $\bar{f}$ is a continuous function of each of its arguments. Let

$$\bar{f}_r = \frac{\partial \bar{f}}{\partial u_r}, \; r = -K+1, ..., K \tag{2.11}$$

$$\bar{f}_{-K} = 0, \tag{2.12}$$

$$\bar{f}_{k+1} = 0. \tag{2.13}$$

Equation 2.8 can be written as follows:

$$u_j^{n+1} = u_j^n - \lambda \left(\bar{f}^n_{j+\frac{1}{2}} - \bar{f}^n_{j-\frac{1}{2}} \right) \equiv G\left(u^n_{j-K}, ..., u^n_{j+K}\right). \tag{2.14}$$

It follows from (2.14) that

$$G\left(u_j^n, ..., u_j^n\right) = u_j^n - \lambda \left(f\left(u_j^n\right) - f\left(u_j^n\right)\right) = u_j^n. \tag{2.15}$$

Suppose that G is a smooth function of its all arguments, then

$$G_r = \frac{\partial G}{\partial u_r}, \tag{2.16}$$

$$G_{rs} = \frac{\partial^2 G}{\partial u_r \partial u_s}. \tag{2.17}$$

At last, one can derive the conservation form scheme approximation solutions of the viscous modified equation [9,10]

$$u_t + f(u)_x = \frac{1}{2}\Delta t \frac{\partial}{\partial x}\left[\beta\,(u,\lambda)\,u_x\right]. \tag{2.18}$$

where

$$\beta\,(u,\lambda) = \frac{1}{\lambda^2}\sum_{r=-K}^{K} r^2 G_r - \left(\frac{\partial f}{\partial u}\right)^2. \tag{2.19}$$

We claim that, except in a trivial case, $\beta(u, \lambda) \geq 0$ and $\beta(u, \lambda) \neq 0$; this shows that the scheme in conservative form is of first-order accuracy [9-11].

3. Mathematical preliminaries on Lie group analysis

All the problems to be addressed here can be described by a general system of non-linear differential equations of the nth order

$$\Delta_\nu\left(x, u^{(n)}\right) = 0, \tag{3.1}$$

where $\nu = 1,\ldots,l$ and $x = (x^1,\ldots,x^p) \in X$ are independent variables, $u = (u^1,\ldots,u^q) \in U$ are dependent variables, and $\Delta_\nu(x, u^{(n)}) = (\Delta_1(x, u^{(n)}),\ldots, \Delta_l(x, u^{(n)}))$ is a smoothing function that depends on x, u and derivatives of u up to order n with respect to $x^1,\ldots,x^p$. If we define a jet space $X \times U^{(n)}$ as a space whose coordinates are independent variables, dependent variables and derivatives of dependent variables up to order n then Δ is a smoothing mapping

$$\Delta : X \times U^{(n)} \to R^l. \tag{3.2}$$

Before studying the symmetries of difference schemes, let us briefly review the theory of symmetries for differential equations. For all details, proofs, and further information, we refer to the many excellent books on the subject, e.g., [12-14]. Here, we follow the style of [12], but the Lie symmetry description is made concise by emphasizing the significant points and results. In order to provide the reader with a relatively quick and painless introduction to Lie symmetry theory, some important concepts must be introduced.

The main tool used in Lie group theory and working with transformation groups is "infinitesimal transformation". In order to present this, we need first to develop the concept of a vector field on a manifold. We begin with a discussion of tangent vectors. Suppose C is a smooth curve on a manifold M, parameterized by

$$\phi : I \to M, \tag{3.3}$$

where I is a subinterval of R. In local co-ordinates $x = x^1,\ldots,x^p$, C is given by p smoothing functions

$$\phi\,(\varepsilon) = \left(\phi^1\,(\varepsilon)\,,\ldots,\phi^p\,(\varepsilon)\right), \tag{3.4}$$

of the real variable ε. At each point $x = \varphi(\varepsilon)$ of C the curve has a tangent vector, namely the derivative

$$\dot{\phi} = \frac{d\phi}{d\varepsilon} = \left(\frac{d\phi^1}{d\varepsilon},\ldots,\frac{d\phi^p}{d\varepsilon}\right). \tag{3.5}$$

In order to distinguish between tangent vectors and local coordinate expressions for a point on the manifold, we adopt the notation

$$V = \frac{d\varphi}{d\varepsilon} = \frac{d\varphi^1}{d\varepsilon} \cdot \frac{\partial}{\partial x^1} + \ldots + \frac{d\varphi^p}{d\varepsilon} \cdot \frac{\partial}{\partial x^p} \tag{3.6}$$

for the vector tangential to C at $x = \varphi(\varepsilon)$ The collection of all tangent vectors to all possible curves passing through a given point x in M is called the tangent space to M at x, and is denoted by TM. A vector field V on M assigns a tangent vector $V \in TM$ to each point $x \in M$, with V varying smoothly from point to point. In local coordinates, a vector field has the form

$$V = \xi^1(x) \cdot \frac{\partial}{\partial x^1} + \ldots + \xi^p(x) \cdot \frac{\partial}{\partial x^p}. \tag{3.7}$$

where each $\zeta^i(x)$ is a smoothing function of x.

If V is a vector field, we denote the parameterized maximal integral curve passing through x in M by $\Psi(\varepsilon, x)$ and call Ψ the flow generated by V. Thus for each x in M, and ε in some interval I_x containing 0, $\Psi(\varepsilon, x)$ is a point on the integral curve passing through x in M. The flow of a vector field has the basic properties:

$$\Psi(\delta, \Psi(\varepsilon, x)) = \Psi(\delta + \varepsilon, x), \tag{3.8}$$

for all $\delta, \varepsilon \in R$ such that both sides of equation are defined,

$$\Psi(0, x) = x, \tag{3.9}$$

and

$$\frac{d}{d\varepsilon}\Psi(\varepsilon, x) = V \tag{3.10}$$

for all ε where defined. We see that the flow generated by a vector field is the same as a local group action of the Lie group on the manifold M, often called a 'one parameter group of transformations'. The vector field V is called the infinitesimal generator of the action since by Taylor's theorem, in local coordinates

$$\Psi(\varepsilon, x) = x + \varepsilon\xi(x) + O\left(\varepsilon^2\right), \tag{3.11}$$

where $\zeta = (\zeta^1, \ldots, \zeta^p)$ are the coefficients of V. The orbits of the one-parameter group action are the maximal integral curves of the vector field V.

Definition 1: A symmetry group of Equation 3.1 is a one-parameter group of transformations G, acting on $X \times U$, such that if $u = f(x)$ is an arbitrary solution of (3.1) and $g_\varepsilon \in G$ then $g_\varepsilon \cdot f(x)$ is also a solution of (3.1).

The infinitesimal generator of a symmetry group is called an infinitesimal symmetry. Infinitesimal generators are used to formulate the conditions for a group G to make it a symmetry group. Working with infinitesimal generators is simple. First, we define a prolongation of a vector field. The symmetry group of a system of differential equations is the largest local group of transformations acting on the independent and dependent variables of the system such that it can transform one system solution to another. The main goal of Lie symmetry theory is to determine a useful, systematic, computational method that explicitly determines the symmetry group of any given system of differential equations. The search for the symmetry algebra L of a system of differential equations is best formulated in terms of vector fields acting on the space $X \times U$ of independent and dependent variables. The vector field tells us how the variables x, u transform. We also need to know how the derivatives, that is u_x, u_{xx},..., transform. This is given by the prolongation of the vector field V. Combining these, we have [[12], p. 110, Theorem 2.36].

Theorem 1

Let

$$V = \sum_{i=1}^{p} \xi^i (x, u)\, \partial_{x^i} + \sum_{a=1}^{q} \eta_a (x, u)\, \partial_{u^a}$$

be a vector defined on an open subset $M \subset X \times U$. The nth prolongation of the original vector filed is the vector field:

$$pr^{(n)} V = V + \sum_{a=1}^{q} \sum_{J} \eta_a^J \left(x, u^{(n)}\right) \frac{\partial}{\partial u_J^a}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$. The second summation here is over all (unordered) multi-indices $J = (j_1, j_2,...,j_k)$, with $1 \le j_k \le p$, $1 \le k \le n$,. The coefficient functions ϕ_a^J of $pr^{(n)}V$ are given by the following formula:

$$\eta_a^J \left(x, u^{(n)}\right) = D_J \left(\eta_a - \sum_{i=1}^{p} \xi^i u_i^a\right) + \sum_{i=1}^{p} \xi^i u_{J,i}^a$$

where $u_i^a = \dfrac{\partial u^a}{\partial x^i}$, and $u_{J,i}^a = \dfrac{\partial u_J^a}{\partial x^i}$, and D_J are the total derivative of η with respect to x^j.

In the following analysis, we only deal with one-dimensional scalar differential equations that are assumed to be differentiable up to the necessary order.

Consider the special case, where $p = 2$, $q = 1$ in the prolongation formula, so that we are looking at a partial differential equation involving the function $u = f(x, t)$. A general vector field on $X \times U \cong R^2 \times R$ then takes the form [[12], p. 114]

$$V = \xi (x, t, u) \frac{\partial}{\partial x} + \tau (x, t, u) \frac{\partial}{\partial t} + \eta (x, t, u) \frac{\partial}{\partial u} \tag{3.12}$$

The first prolongation of V is the vector field:

$$pr^{(1)} V = V + [\eta_x] \frac{\partial}{\partial u_x} + [\eta_t] \frac{\partial}{\partial u_t} \tag{3.13}$$

where

$$[\eta_x] = \eta_x + (\eta_u - \xi_x) u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t$$

and

$$[\eta_t] = \eta_t + (\eta_u - \tau_t) u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_x u_t$$

The subscripts on η, ζ, τ denote partial derivatives. Similarly,

$$pr^{(2)}V = pr^{(1)}V + [\eta_{xx}]\frac{\partial}{\partial u_{xx}} + [\eta_{xt}]\frac{\partial}{\partial u_{xt}} + [\eta_{tt}]\frac{\partial}{\partial u_{tt}} \tag{3.14}$$

where

$$[\eta_{xx}] = \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_xu_t - \xi_{uu}u_x^3$$
$$-\tau_{uu}u_x^2u_t + (\eta_u - 2\xi_x)u_{xx} - 2\tau_xu_{xt} - 3\xi_uu_{xx}u_x - \tau_uu_{xx}u_t - 2\tau_uu_{xt}u_t$$

$$\begin{aligned}[\eta_{xt}] = {} & \eta_{xt} + (\eta_{xu} - \tau_{tx})u_t + (\eta_{tu} - \xi_{tx})u_x - \tau_{xu}u_t^2 + (\eta_{uu} - \xi_{xu} - \tau_{ut})u_xu_t \\ & - \xi_{tu}u_x^2 - \tau_{uu}u_xu_t^2 - \xi_{uu}u_tu_x^2 - \tau_xu_{tt} + (\eta_u - \xi_x - \tau_t)u_{xt} - \xi_tu_{xx} - 2\tau_uu_tu_{xt} \\ & - 2\xi_uu_xu_{xt} - \tau_uu_xu_{tt} - \xi_uu_tu_{xx}\end{aligned}$$

$$[\eta_{tt}] = \eta_{tt} + (2\eta_{tu} - \tau_{tt})u_t - \xi_{tt}u_x + (\eta_{uu} - 2\tau_{tu})u_t^2 - 2\xi_{tu}u_xu_t - \tau_{uu}u_x^3$$
$$-\tau_{uu}u_t^2u_x + (\eta_u - 2\tau_t)u_{tt} - 2\xi_tu_{xt} - 3\tau_uu_{tt}u_t - \xi_uu_{tt}u_x - 2\xi_uu_{xt}u_t$$

From here on analysis of difference equations only concerns modified equations, which have third prolongation of the vector field. From work in CFD, we know that the right-hand side of the modified equation is written entirely in terms of x derivatives. So, investigation can be limited to the terms of the spatial derivatives in the following analysis. The coefficients of the various monomials in the third-order partial derivatives of u are given in the following:

$$pr^{(3)}V = pr^{(2)}V + [\eta_{xxx}]\frac{\partial}{\partial u_{xxx}} + [\eta_{xxt}]\frac{\partial}{\partial u_{xxt}} + [\eta_{xtt}]\frac{\partial}{\partial u_{xtt}} + [\eta_{ttt}]\frac{\partial}{\partial u_{ttt}} \tag{3.15}$$

where,

$$\begin{aligned}[\eta_{xxx}] = {} & \eta_{xxx} + (3\eta_{xxu} - \xi_{xxx})u_x - \tau_{xxx}u_t + 3(\eta_{xuu} - \xi_{xxu})u_x{}^2 - 3\tau_{xxu}u_xu_t \\ & + (\eta_{uuu} - 3\xi_{xuu})(u_x)^3 + 3(\eta_{xu} - \xi_{xx})u_{xx} - 3\tau_{xx}u_{xt} - 3\tau_{xuu}(u_x)^2u_t \\ & + 3(\eta_{uu} - 3\xi_{xu})u_xu_{xx} - 3\tau_{xu}u_tu_{xx} - 6\tau_{xu}u_{xt}u_x - 3\tau_xu_{xxt} + (\eta_u - 3\xi_x)u_{xxx} \\ & - \xi_{xxx}(u_x)^4 - 6\xi_{uu}(u_x)^2u_{xx} - 3\tau_{uu}(u_x)^2u_{xt} - \tau_{uuu}(u_x)^3u_t - 3\xi_u(u_{xx})^2 \\ & - 3\tau_uu_{xxt}u_x - 3\tau_uu_{xt}u_{xx} - 3\tau_{uu}u_{xx}u_xu_t - 4\xi_uu_{xxx}u_x - \tau_uu_{xxx}u_t\end{aligned}$$

Suppose we are given an nth order system of differential equations, or, equivalently, a subvariety of the jet space $M^{(n)} \subset X \times U^{(n)}$. A symmetry group of this system is a local transformation G acting on $M \subset X \times U$. which transforms solutions of the system to other solutions. We can reduce the important infinitesimals condition for a group G to be a symmetry group of a given system of differential equations. The following theorem [[12], p. 104, Theorem 2.31] provides the infinitesimal conditions for a group G to be a symmetry group.

Theorem 2

Suppose

$$\Delta_\nu\left(x, u^{(n)}\right) = 0, \nu = 1, 2, ..., l$$

is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$pr^{(n)}V \circ \left[\Delta_\nu\left(x, u^{(n)}\right)\right] = 0, \nu = 1, 2, ..., l$$

whenever

$$\Delta_\nu\left(x, u^{(n)}\right) = 0,$$

for every infinitesimal generator V of G, then G is a symmetry group of the system.

In the following sections, this theorem is used to deduce explicitly different infinitesimal conditions for specific problems. It must be remembered, however, that, in all cases, though only the scalar differential problem is being discussed, Δ_ν is still used to denote different differential equations.

4. Galilean group and its prolongation

It is well known that as for Navier-Stokes equations, the intrinsic symmetries, except for the scaling symmetries, are just macroscopic consequences of the basic symmetries of Newton's equations governing microscopic molecular motion (in classical approximation). Any physical difference scheme should inherit the elementary symmetries (at least for Galilean symmetry) from the Navier-Stokes equations. This means that Galilean invariance has been an important issue in CFD. Furthermore, we stress that Galilean invariance is a basic requirement that is demanded for any physical difference scheme.

We have the Galilean transformation

$$\begin{cases} x' = x + t\varepsilon \\ t' = t \\ u' = u + \varepsilon \end{cases} \tag{4.1}$$

Thus, the vector of the Galilean transformation is

$$V = t\partial_x + \partial_u \tag{4.2}$$

According to Theorem 1, we have

$$pr^{(1)}V = V - \rho_x \frac{\partial}{\partial \rho_t} - u_x \frac{\partial}{\partial u_t} - p_x \frac{\partial}{\partial p_t} \tag{4.3}$$

$$pr^{(2)}V = pr^{(1)}V - \rho_{xx} \frac{\partial}{\partial \rho_{xt}} - u_{xx} \frac{\partial}{\partial u_{xt}} - p_{xx} \frac{\partial}{\partial p_{xt}} \tag{4.4}$$

5. Galilean invariance of first-order conservative form scheme

The main prototype equation here is the modified equation. Equation 2.18 can be recast into

$$\Delta_1 \equiv u_t + uu_x - \frac{1}{2}\Delta t\beta\left(u, \lambda\right)u_{xx} - \frac{1}{2}\Delta t\beta_u u_x u_x = 0. \tag{5.1}$$

Based on the prolongation formula presented in Section 4, the Galilean invariance condition reads

$$\Delta_1 = 0. \tag{5.2}$$

$$pr^{(2)}V \circ \Delta_1 = 0. \tag{5.3}$$

Before beginning the group analysis, some detailed but mechanical calculations must be performed:

$$d_1 = \partial_u \circ \Delta_1 = u_x - \frac{1}{2}\Delta t \beta_u u_{xx} - \frac{1}{2}\Delta t \beta_{uu} u_x u_x. \tag{5.4}$$

$$d_2 = \partial_{u_t} \circ \Delta_1 = 1. \tag{5.5}$$

With these formulas, it is clear from Equation 5.3 that the invariance condition reduces into

$$\beta_u u_{xx} + \beta_{uu} u_x u_x = 0. \tag{5.6}$$

Hence, we have

$$u_x u_x = -\frac{\beta_u}{\beta_{uu}} u_{xx}. \tag{5.7}$$

we can then write the model equation as

$$u_t + uu_x - \frac{1}{2}\Delta t \left[\beta - \frac{\beta_u \beta_u}{\beta_{uu}}\right] u_{xx} = 0. \tag{5.8}$$

with

$$\frac{1}{2}\Delta t \left[\beta - \frac{\beta_u \beta_u}{\beta_{uu}}\right] = \nu_1. \tag{5.9}$$

This manipulation yields the Burgers equation as following

$$u_t + uu_x = \nu_1 u_{xx}. \tag{5.10}$$

where ν_1 = constant.

Based on the analysis of Equation 5.9, one have

$$\beta = \beta_0 \exp\ (\alpha u) + \frac{2\nu_1}{\Delta t}. \tag{5.11}$$

where β_0, α are some parameters.

Here, it is useful to list some well-known first-order conservative schemes to show their unified character.

5.1. Lax-Friedrichs scheme

$$\bar{f}^n_{j+\frac{1}{2}} = \frac{1}{2}\left[f\left(u^n_{j+1}\right) + f\left(u^n_j\right) - \frac{1}{\lambda}\left(u^n_{j+1} - u^n_j\right)\right]. \tag{5.12}$$

$$G_1 = \frac{1}{2}\left[1 - \lambda\frac{\partial f}{\partial u}\right], \tag{5.13}$$

$$G_{-1} = \frac{1}{2}\left[1 + \lambda\frac{\partial f}{\partial u}\right]. \tag{5.14}$$

$$\begin{aligned}\beta(u,\lambda) &= \frac{1}{\lambda^2} - \left(\frac{\partial f}{\partial u}\right)^2 \\ &= \frac{1}{\lambda^2} - u^2.\end{aligned} \tag{5.15}$$

5.2. 3-point monotonicity scheme (Godunov, 1959)

$$u^{n+1}_j = \sum_{r=-K}^{K} C_l u^n_{j+r}, \tag{5.16}$$

$$G_r = C_r \tag{5.17}$$

$$\begin{aligned}\beta(u,\lambda) &= \frac{1}{\lambda^2}\sum_{r=-K}^{K} r^2 G_r - \left(\frac{\partial f}{\partial u}\right)^2 \\ &= \frac{1}{\lambda^2}\sum_{r=-K}^{K} r^2 C_r - u^2.\end{aligned} \tag{5.18}$$

5.3. General 3-point conservation scheme

$$\bar{f}\left(u_j, u_{j+1}\right) = \frac{1}{2}\left[f\left(u_j\right) + f\left(u_{j+1}\right) - \frac{1}{\lambda}Q\left(\lambda\bar{a}_{j+\frac{1}{2}}\right)\Delta_{j+\frac{1}{2}}u\right], \tag{5.19}$$

where

$$\bar{a}_{j+\frac{1}{2}} = \frac{f\left(u_{j+1}\right) - f\left(u_j\right)}{\Delta_{j+\frac{1}{2}}u}, \text{ when } \Delta_{j+\frac{1}{2}}u \neq 0,$$

$$\bar{a}_{j+\frac{1}{2}} = a\left(u_j\right), \text{ when } \Delta_{j+\frac{1}{2}}u = 0,$$

Here $Q(x)$ is some function, which is often referred to as the coefficient of numerical viscosity.

Harten's lemma. Let $Q(x)$ in (5.19) satisfy the inequalities

$$|x| \leq Q(x) \leq 1 \text{ for } 0 \leq |x| \leq \mu \leq 1;$$

then finite-difference scheme is TVNI under the CFL-like restriction $\lambda \max_j |\bar{a}_{j+\frac{1}{2}}| \leq \mu$.

The coefficient of numerical viscosity could be expressed in terms of the β as follows

$$\beta(u,\lambda) = \frac{1}{2}\left[Q(u) - u^2\right]. \tag{5.20}$$

Therefore, one can have

$$Q(u) = u^2 + 2\beta_0 \exp(\alpha u) + \frac{4\nu_1}{\Delta t}. \tag{5.21}$$

If we choose

$$\beta_0 << 1 \tag{5.22}$$

Then Equation 5.21 is consistence with the results of Harten's. In summary, we obtain

Theorem 3

If we let the coefficients in (2.18) satisfy the equality

$$\beta = \beta_0 \exp(\alpha u) + \frac{2\nu_1}{\Delta t}.$$

where β_0, α is a dimensionless constant, then the first-order conservative finite different scheme satisfies Galilean invariant condition.

6. Galilean invariance of second-order conservative scheme

The same manipulation could be conducted for the case of the second-order conservative scheme. The main prototype equation here is [15]

$$\Delta_1 \equiv u_t + uu_x = \frac{1}{6}\Delta t^3 \frac{\partial}{\partial x}\left[\gamma(u,\lambda)u_{xx} + \delta(u,\lambda)u_x u_x\right]. \tag{6.1}$$

where

$$\gamma(u,\lambda) = \frac{1}{\lambda^3}\sum_{r=-K}^{K} r^3 G_r + \left(\frac{\partial f}{\partial u}\right)^3, \tag{6.2}$$

$$\delta(u,\lambda) = \frac{1}{\lambda^3}\sum_{r,s=-K}^{K} \frac{1}{4}(r+s)\left[2rs - (r-s)^2 G_{rs}\right] + 3\left(\frac{\partial f}{\partial u}\right)^2\left(\frac{\partial^2 f}{\partial u^2}\right). \tag{6.3}$$

According to Theorem 2, one have

$$\Delta_2 \equiv u_t + uu_x - \frac{1}{6}\Delta t^3\left[\gamma u_{xxx} + (\gamma_u + 2\delta)u_x u_{xx} + \delta_u (u_x)^3\right]. \tag{6.4}$$

Before beginning the group analysis, some detailed but mechanical calculations must be performed:

$$d_1 = \partial_u \circ \Delta_2 = u_x - \frac{1}{6}\Delta t^3\left[\gamma_u u_{xxx} + (\gamma_{uu} + 2\delta_u)u_x u_{xx} + \delta_{uu}(u_x)^3\right]. \tag{6.5}$$

$$d_2 = \partial_{u_t} \circ \Delta_2 = 1. \tag{6.6}$$

$$pr^{(3)}V = t\partial_x + \partial_u - u_x\partial_{u_t}. \tag{6.7}$$

The corresponding Galilean invariant condition reads:

$$\Delta_2 = 0. \tag{6.8}$$

$$pr^{(3)}V \circ \Delta_2 = 0. \tag{6.9}$$

The substitution leads

$$u_x - \frac{1}{6}\Delta t^3\left[\gamma_u u_{xxx} + (\gamma_{uu} + 2\delta_u)\, u_x u_{xx} + \delta_{uu}(u_x)^3\right] - u_x = 0 \tag{6.10}$$

Hence, we have

$$\gamma_u u_{xxx} + (\gamma_{uu} + 2\delta_u)\, u_x u_{xx} + \delta_{uu}(u_x)^3 = 0. \tag{6.11}$$

It is clear that

$$u_x u_{xx} = -\frac{\delta_{uu}}{\gamma_{uu} + 2\delta_u} \cdot (u_x)^3 - \frac{\gamma_u}{\gamma_{uu} + 2\delta_u} \cdot u_{xxx}. \tag{6.12}$$

After some manipulation, one could obtain the model equation as follows

$$\begin{aligned}\Delta_2 \equiv & u_t + uu_x - \frac{1}{6}\Delta x^3\left(\gamma - \frac{\gamma_u + 2\delta}{\gamma_{uu} + 2\delta_u} \cdot \gamma_u\right).u_{xxx} \\ & - \frac{1}{6}\Delta x^3\left(\delta_u - \frac{\gamma_u + 2\delta}{\gamma_{uu} + 2\delta_u} \cdot \delta_{uu}\right)(u_x)^3.\end{aligned} \tag{6.13}$$

In order to obtain the non-oscillation solution of shock, we could let the term of $(u_x)^3$ to be zero, then we have

$$\delta_u - \frac{\gamma_u + 2\delta}{\gamma_{uu} + 2\delta_u} \cdot \delta_{uu} = 0. \tag{6.14}$$

This equation can be rewritten as below

$$\frac{\delta_{uu}}{\delta_u} = \frac{\gamma_{uu} + 2\delta_u}{\gamma_u + 2\delta} \tag{6.15}$$

$$\frac{d}{du}\left[\ln \delta_u\right] = \frac{d}{du}\left[\ln\ (\gamma_u + 2\delta)\right]. \tag{6.16}$$

$$\delta_u = \gamma_u + 2\delta. \tag{6.17}$$

or

$$\gamma_u = \delta_u + 2\delta. \tag{6.18}$$

The coefficient of the term of u_{xxx} could be rewritten as

$$\begin{aligned}&\gamma - \frac{\gamma_u + 2\delta}{\gamma_{uu} + 2\delta_u} \cdot \gamma_u \\ &= \gamma - \frac{\delta_u}{\delta_{uu}} \cdot \gamma_u \\ &= \gamma - \frac{\delta_u}{\delta_{uu}} \cdot (\delta_u - 2\delta)\end{aligned} \qquad (6.19)$$

If we set

$$\delta = au^n + b \qquad (6.20)$$

where a, b, m are some parameters. It is easy to show that

$$\gamma = \frac{an}{n-1}u^n - \frac{2a}{n-1}u^{n+1} - \frac{2b}{n-1}u + \gamma_0. \qquad (6.21)$$

where γ_0 = constant. In summary, we obtain

Theorem 4

If we let the coefficients in (6.1) satisfy the equality

$$\delta(u, \lambda) = au^n + b,$$

$$\gamma(u, \lambda) = \frac{an}{n-1}u^n - \frac{2a}{n-1}u^{n+1} - \frac{2b}{n-1}u + \gamma_0.$$

where a, b, n, γ_0 is a dimensionless constant, then the second-order conservative finite different scheme satisfies Galilean invariant condition.

Here, we could give the details of the corresponding analysis by using the Lax-Wendroff scheme. It is well known that the Lax-Wendroff difference approximation to (2.1) is defined by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{f(u_{j+1}^n) - f(u_{j-1}^n)}{2\Delta x} = \frac{1}{2}\Delta t \frac{a^2_{j+\frac{1}{2}}(u_{j+1}^n - u_j^n) - a^2_{j-\frac{1}{2}}(u_j^n - u_{j-1}^n)}{\Delta x^2} \qquad (6.22)$$

where

$$u_j^n = u(n\Delta t, j\Delta x)$$

$$a(u) = \frac{\partial f(u)}{\partial u}$$

$$a_{j+\frac{1}{2}} = a(u^n_{j+\frac{1}{2}})$$

$$u^n_{j+\frac{1}{2}} = \frac{1}{2}(u_{j+1}^n + u_j^n)$$

It is well known that the Lax-Wendroff scheme was designed to have the following desirable computational features [16-18]: conservation of form; to have a three-point scheme; second-order accuracy on smooth solutions. Numerical spikes and downstream oscillations are generated in the vicinity of the shock.

The numerical flux for Lax-Wendroff scheme can be written as

$$\bar{f}^n_{j+\frac{1}{2}} = \frac{1}{2}\left[f\left(u^n_{j+1}\right) + f\left(u^n_j\right) - \frac{\Delta t}{\Delta x} a^2_{j+\frac{1}{2}}\left(u^n_{j+1} - u^n_j\right)\right]. \tag{6.23}$$

Using the general method presented in Section 2, one can obtain

$$G_1 = -\frac{1}{2}\lambda\frac{\partial f}{\partial u} + \frac{1}{2}\lambda^2\left(\frac{\partial f}{\partial u}\right)^2 \tag{6.24}$$

$$G_{-1} = \frac{1}{2}\lambda\frac{\partial f}{\partial u} + \frac{1}{2}\lambda^2\left(\frac{\partial f}{\partial u}\right)^2 \tag{6.25}$$

$$G_{11} = -\frac{\lambda}{2}\left(1 - 2\lambda\frac{\partial f}{\partial u}\right)\frac{\partial^2 f}{\partial u^2}, \tag{6.26}$$

$$G_{00} = -2\lambda^2\frac{\partial f}{\partial u}\frac{\partial^2 f}{\partial u^2}, \tag{6.27}$$

$$G_{-1,-1} = \frac{\lambda}{2}\left(1 + 2\lambda\frac{\partial f}{\partial u}\right)\frac{\partial^2 f}{\partial u^2}. \tag{6.28}$$

and for the other case of the index r, s

$$G_{rs} = 0. \tag{6.29}$$

$$\gamma^{\mathrm{LW}}(u, \lambda) = -\frac{1}{\lambda^2}\left(1 - \lambda^2\left(\frac{\partial f}{\partial u}\right)^2\right)\frac{\partial f}{\partial u}, \tag{6.30}$$

$$\delta^{\mathrm{LW}}(u, \lambda) = -\frac{1}{\lambda^2}\left(1 - 3\lambda^2\left(\frac{\partial f}{\partial u}\right)^2\right)\frac{\partial^2 f}{\partial u^2}, \tag{6.31}$$

$$\gamma^{\mathrm{LW}}(u, \lambda) = -\frac{1}{\lambda^2}u + u^3, \tag{6.32}$$

$$\delta^{\mathrm{LW}}(u, \lambda) = -\frac{1}{\lambda^2} + 3u^2. \tag{6.33}$$

Using the results of Theorem 4, we have

$$\delta^G(u, \lambda) = au^n + b = \delta^{LW}. \tag{6.34}$$

This leads the following corresponding relation for Lax-Wendroff scheme

$$n = 2, \tag{6.35}$$

$$a = 3,\ b = -\frac{1}{\lambda^2}. \tag{6.36}$$

$$\begin{aligned}\gamma^G(u,\lambda) &= \frac{an}{n-1}u^n - \frac{2a}{n-1}u^{n+1} - \frac{2b}{n-1}u + \gamma_0 \\ &= 6u^2 - 6u^3 + \frac{2}{\lambda^2}u + \gamma_0.\end{aligned} \tag{6.37}$$

The comparison of Equations 6.32 and 6.37 help us to draw some conclusion as follows: based on this analysis, we have known that the well-known Lax-Wendroff scheme can recover the Galilean symmetry approximately.

7. Conclusions

It is known that the numerical solutions calculated by finite difference schemes are always associated with numerical dissipation and dispersion. Such errors can lead to undesirable numerical effects, especially for shock capturing. A full understanding of the nature of this odd numerical phenomenon is still lacking. Regardless of definition, spurious oscillations and overshoots are the most common symptoms of numerical stability. In one natural interpretation, these numerical phenomena are due to nonlinear stability, which links some symmetry breaking. This article uses a Lie symmetry analysis method to investigate Galilean invariance properties of several difference schemes for nonlinear equations. Two theorems have been obtained, which have demonstrated that the properties of Galilean invariance from a modification equation, can serve as the positive constrain condition for general conservative finite difference schemes.

It should be pointed out that the conclusions presented in this article have preliminary character and demand further study.

Acknowledgements
The author appreciates the numerous corrections and very useful comments from the referees on the manuscript. The study was supported by the National Natural Science Foundation of China (Grant Nos. 90816013, 10572083), and also supported by the State Key Laboratory for Turbulence and Complex System.

Competing interests
The authors declare that they have no competing interests.

Received: 4 November 2010 Accepted: 7 November 2011 Published: 7 November 2011

References

1. Harlow, FH: The particle-in-cell computing method for fluid dynamics. Methods Comput Phys. **3**, 319 (1964)
2. Yanenko, NN, Shokin, YI: On the group classification of difference schemes for systems of equations in gas dynamics. Proceedings of the second International Conference on Numerical Methods in Fluid Dynamics. (1970)
3. Winternitz, P: Symmetries of discrete system. Lecture presented at the CIMPA Winter School on Discrete Integrable System. (2003)
4. Ran, Z: Lie symmetries preservation and shock capturing methods. SIAM J Numer Anal. **46**(1), 325–343 (2007)
5. Ran, Z: Symmetry and non-physical oscillation in shock capturing. Chin Quart Mech **26**, 4 (2005). (In Chinese)
6. Harten, A: High resolution schemes for hyperbolic conservation laws. J Comput Phys. **49**, 357–393 (1983). doi:10.1016/0021-9991(83)90136-5
7. Harten, A, Lax, PD: A random choice finite difference scheme for hyperbolic conservation laws. SIAM J Numer Anal. **18**, 289–315 (1981). doi:10.1137/0718021
8. Oleinik, OA: Discontinuous solutions of non-linear differential equations. Usp Mat Nauk **12**((3(75))):3–73 (1957). (in Russian)
9. Harten, A, Lax, PD, Leer, BV: On upstream difference and Godunov-type schemes for hyperbolic conservation laws. SIAM Rev. **25**(1), 35–61 (1983). doi:10.1137/1025002
10. Harten, A, Hyman, JM, Lax, PD: On finite difference approximations and entropy conditions for shocks. Commun. Pure Appl Math. **29**, 297–322 (1976). doi:10.1002/cpa.3160290305
11. Osher, S, Chakravarthy, S: High resolution schemes and the entropy condition. SIAM J Numer Anal. **21**(5), 955–984 (1984). doi:10.1137/0721060
12. Olver, PJ: Applications of Lie Group to Differential Equations. GTM 107. Springer-Verlag, New York (1986)

13. Bluman, GW, Kumei, S: Symmetries and Differential Equations. In Applied Mathematical Sciences, vol. 81, pp. 194–195. Springer-Verlag, World Publishing Corp (1991)
14. Marsden, JE, Ratiu, TS: Introduction to Mechanics and Symmetry: A Basic Exposition of Classical Mechanical Systems. In Texts Appl Math, vol. 17,Springer Verlag, New York (1994)
15. Shui, HS: Finite Difference Methods for One Dimensional Fluid Dynamics. National Defence Industrial Press. Beijing (1998) (in Chinese)
16. Lax, PD, Wendroff, B: Systems of conservation laws. Commun Pure Appl Math. **23**, 217–237 (1960)
17. Majda, A, Osher, S: A systematic approach for scalar conservation laws. Numer Math. **30**, 429–452 (1978). doi:10.1007/BF01398510
18. Rubin, EL, Burstein, SZ: Difference methods for inviscid and viscous equations of compressible gas. J Comput Phys. **l2**, 178–196 (1976)

doi:10.1186/1687-1847-2011-53
Cite this article as: Ran: Galilean invariance and the conservative difference schemes for scalar laws. *Advances in Difference Equations* 2011 **2011**:53.