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Homoclinic solutions for second order discrete *p*-Laplacian systems

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Abstract

Some new existence theorems for homoclinic solutions are obtained for a class of second-order discrete *p*-Laplacian systems by critical point theory, a homoclinic orbit is obtained as a limit of $2kT$-periodic solutions of a certain sequence of the second-order difference systems. A completely new and effective way is provided for dealing with the existence of solutions for discrete *p*-Laplacian systems, which is different from the previous study and generalize the results.

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1. Introduction

In this article, we shall be concerned with the existence of homoclinic orbits for the second-order discrete *p*-Laplacian systems:

$$\Delta(\varphi_p(\Delta u(n-1))) = \nabla F(n, u(n)) + f(n), \quad n \in \mathbb{Z},\ u \in \mathbb{R}^{\mathbb{N}}, \tag{1.1}$$

where $p > 1$, $\phi_p(s) = |s|^{p-2}s$ is the Laplacian operator, $\Delta u(n) = u(n+1) - u(n)$ is the forward difference operator, $F : \mathbb{Z} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a continuous function in the second variable and satisfies $F(n + T, u) = F(n, u)$ for a given positive integer T. As usual, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all natural numbers, integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, denote $\mathbb{Z}(a) = \{a, a+1, \ldots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \ldots b\}$ when $a \leq b$.

Differential equations occur widely in numerous settings and forms both in mathematics itself and in its application to statistics, computing, electrical circuit analysis, biology and other fields, so it is worthwhile to explore this topic. As is known to us, the development of the study of periodic solution and their connecting orbits of differential equations is relatively rapid. Many excellent results were obtained by variational methods [1-11]. It is well-known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably produce chaotic phenomenon.

On the other hand, we know that a differential equation model is often derived from a difference equation, and numerical solutions of a differential equation have to be obtained by discretizing the differential equation, therefore, the study of periodic solution and connecting orbits of difference equation is meaningful [12-24].

It is clear that system (1.1) is a discretization of the following second differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\dot{u}(t)|^{p-2}\dot{u}(t)) = \nabla F(t, u(t)) + f(t), \quad t \in \mathbb{R},\ u \in \mathbb{R}^{\mathbb{N}}. \tag{1.2}$$

Recently, the following second order self-adjoint difference equation

$$\Delta[p(n)\Delta u(n-1)] + q(n)u(n) = f(n, u(n)), \quad n \in \mathbb{Z},\ u \in \mathbb{R} \tag{1.3}$$

has been studied by using variational method. Yu and Guo established the existence of a periodic solution for Equation (1.3) by applying the critical point theory in [15]. Ma and Guo [20] obtained homoclinic orbits as the limit of the subharmonics for Equation (1.3) by applying the Mountain Pass theorem relying on Ekelands variational principle and the diagonal method, their results are based on scalar equation with $q(t) \neq 0$, if $q(t) = 0$, the traditional ways in [20] are inapplicable to our case.

Some special cases of (1.1) have been studied by many researchers via variational methods [15-17,22,23]. However, to our best knowledge, results on homoclinic solutions for system (1.1) have not been studied. Motivated by [9,10,20], the main purpose of this article is to give some sufficient conditions for the existence of homoclinic solutions to system (1.1).

Our main results are the following theorems.

Theorem 1.1 *Assume that F and f satisfy the following conditions:*

(H1) *$F(n, x)$ is T-periodic with respect to $n, T > 0$ and continuously differentiable in x;*

(H2) *There are constants $b_1 > 0$ and $\nu > 1$ such that for all $(n, x) \in \mathbb{Z} \times \mathbb{R}^{\mathbb{N}}$,*

$$F(n, x) \geq F(n, 0) + b_1|x|^{\nu};$$

(H3) *$f \neq 0$ is a bounded function such that $\sum_{n\in\mathbb{Z}} |f(n)|^{\nu/(\nu-1)} < \infty$.*

Then, system (1.1) possesses a homoclinic solution.

Theorem 1.2 *Assume that F and f satisfy the following conditions:*

(H4) *$F(n, x) = K(n, x) - W(n, x)$, where K, W is T-periodic with respect to $n, T > 0$, $K(n, x)$ and $W(n, x)$ are continuously differentiable in x;*

(H5) *There is a constant $\mu > p$ such that for every $n \in \mathbb{Z}$, $u \in \mathbb{R}^{\mathbb{N}}\backslash\{0\}$,*

$$0 < \mu W(n, x) \leq (\nabla W(n, x), x);$$

(H6) *$\nabla W(n,x) = o(|x|)$, as $|x| \to 0$ uniformly with respect to n;*

(H7) *There exist constants $b_2 > 0$ and $\gamma \in (1, p]$ such that for all $(n, u) \in \mathbb{Z} \times \mathbb{R}^{\mathbb{N}}$,*

$$K(n, 0) = 0, \quad K(n, x) \geq b_2|x|^{\gamma};$$

(H8) *There is a constant* $\varrho \in [p, \mu)$ such that

$$(x, \nabla K(n, x)) \leq \varrho K(n, x), \quad \forall (n, x) \in [0, T] \times \mathbb{R}^{\mathbb{N}};$$

(H9) *$f \neq 0$ is a bounded function such that*

$$\sum_{n\in\mathbb{Z}} |f(n)|^q < \frac{\left(\min\left\{\frac{\delta^{p-1}}{p}, b_2\delta^{\gamma-1} - M_1\delta^{\mu-1}\right\}\right)^q}{C^p},$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$M_1=\sup\{W(n,x)|n\in[0,T],x\in\mathbb{R}^{\mathbb{N}},|x|=1\},$$

C is given in (3.4) and $\delta\in(0,1]$ *such that*

$$b_2\delta^{\gamma-1}-M_1\delta^{\mu-1}=\max_{x\in[0,1]}\left(b_2x^{\gamma-1}-M_1x^{\mu-1}\right).$$

Then, system (1.1) *possesses a nontrivial homoclinic solution.*

Remark Obviously, condition (H9) holds naturally when $f = 0$. Moreover, if $b_2(\gamma - 1) \le M(\mu - 1)$, then

$$\delta=\left[\frac{b_2(\gamma-1)}{M(\mu-1)}\right]^{1/(\mu-\gamma)},$$

and so condition (H9) can be rewritten as

$$\sum_{n\in\mathbb{Z}}|f(n)|^q<\frac{\left(\min\left\{\frac{1}{p}\left[\frac{b_2(\gamma-1)}{M(\mu-1)}\right]^{(p-1)/(\mu-\gamma)},\frac{b_2(\mu-\gamma)}{\mu-1}\left[\frac{b_2(\gamma-1)}{M(\mu-1)}\right]^{(\gamma-1)/(\mu-\gamma)}\right\}\right)^q}{C^p},$$

if $b_2(\gamma - 1) > M(\mu - 1)$, then $\delta = 1$ and $b_2\delta^{(\gamma-1)} - M\delta^{(\mu-1)} = b_2 - M$, and so condition (H9) can be rewritten as

$$\sum_{n\in\mathbb{Z}}|f(n)|^q<C^{-p}(\min\{p^{-1},b_2-M\})^q. \tag{1.4}$$

2. Preliminaries

In this section, we recall some basic facts which will be used in the proofs of our main results. In order to apply the critical point theory, we make a variational structure.

Let S be the vector space of all real sequences of the form

$$u=\{u(n)\}_{n\in\mathbb{Z}}=(\ldots,u(-n),u(-n+1),\ldots,u(-1),u(0),u(1),\ldots,u(n),\ldots),$$

namely

$$S=\{u=\{u(n)\}:u(n)\in\mathbb{R}^N,\quad n\in\mathbb{Z}\}.$$

For each $k\in\mathbb{N}$, let E_k denote the Banach space of $2kT$-periodic functions on $\mathbb{Z}$ with values in $\mathbb{R}^N$ under the norm

$$||u||_{E_k}:=\left[\sum_{n=-kT}^{kT-1}\left(|\Delta u(n-1)|^p+|u(n)|^p\right)\right]^{1/p}.$$

In order to receive a homoclinic solution of (1.1), we consider a sequence of systems:

$$\Delta(\varphi_p(\Delta u(n-1)))+\nabla F(n,u(n))=f_k(n),\quad n\in\mathbb{Z},\quad u\in\mathbb{R}^{\mathbb{N}}, \tag{2.1}$$

where $f_k:\mathbb{Z}\to\mathbb{R}^N$ is a $2kT$-periodic extension of restriction of f to the interval $[-kT, kT-1]$, $k\in\mathbb{N}$. Similar to [20], we will prove the existence of one homoclinic solution of (1.1) as the limit of the $2kT$-periodic solutions of (2.1).

For each $k \in \mathbb{N}$, let $l^p_{2kT}(\mathbb{Z}, \mathbb{R}^N)$ denote the Banach space of $2kT$-periodic functions on $\mathbb{Z}$ with values in $\mathbb{R}^N$ under the norm

$$||u||_{l^p_{2kT}} = \left(\sum_{n\in\mathbb{N}[-kT,\ kT-1]} |u(n)|^p \right)^{\frac{1}{p}}, \quad u \in l^p_{2kT}.$$

Moreover, l^∞_{2kT} denote the space of all bounded real functions on the interval $\mathbb{N}[-kT, kT - 1]$ endowed with the norm

$$||u||_{l^\infty_{2kT}} = \max_{n\in\mathbb{N}[-kT,\ kT-1]} \{|u(n)|\}, \quad u \in l^\infty_{2kT}.$$

Let

$$I_k(u) = \sum_{n=-kT}^{kT-1} \left[\frac{1}{p} |\Delta u(n-1)|^p + F(n, u(n)) + (f_k(n), u(n)) \right]. \tag{2.2}$$

Then $I_k \in C^1(E_k, \mathbb{R})$ and it is easy to check that

$$I'_k(u)v = \sum_{n=-kT}^{kT-1} [(|\Delta u(n-1)|^{p-2} \Delta u(n-1), \Delta v(n-1)) + (\nabla F(n, u(n)), v(n)) + (f_k(n), v_k(n))].$$

Furthermore, the critical points of I_k in E_k are classical $2kT$-periodic solutions of (2.1).

That is, the functional I_k is just the variational framework of (2.1).

In order to prove Theorem 1.2, we need the following preparations.

Let $\eta_k : E_k \to [0, +\infty)$ be such that

$$\eta k(u) = \left(\sum_{n=-kT}^{kT-1} [|\Delta u(n-1)|^p + pK(n, u)] \right)^{\frac{1}{p}}. \tag{2.3}$$

Then it follows from (2.2), (2.3), (H4) and (H8) that

$$I_k(u) = \frac{1}{p} \eta_k^p(u) + \sum_{n=-kT}^{kT-1} [-W(n, u(n)) + (f_k(n), u(n))], \tag{2.4}$$

and

$$I'_k(u)u \le \sum_{n=-kT}^{kT-1} [|\Delta u(n-1)|^p + \varrho K(n, u(n))] - \sum_{n=-kT}^{kT-1} (\nabla W(n, u(n)), u(n)) + \sum_{n=-kT}^{kT-1} (f_k(n), u(n)) \tag{2.5}$$

We will obtain the critical points of I by using the Mountain Pass Theorem. Since the minimax characterisation provides the critical value, it is important for what follows. Therefore, we state these theorems precisely.

Lemma 2.1 [7]*Let E be a real Banach space and* $I \in C^1(E, \mathbb{R})$ *satisfy (PS)-condition. Suppose that I satisfies the following conditions:*

(i) $I(0) = 0$;

(ii) *There exist constants* ρ, $\alpha > 0$ *such that* $I|_{\partial B_\rho(0)} \ge \alpha$;

(iii) *There exists* $e \in E\backslash\bar{B}_\rho(0)$ *such that* $I(e) < 0$.

Then I possesses a critical value $c \geq \alpha$ *given by*

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)),$$

where $B_\rho(0)$ *is an open ball in E of radius* ρ *centered at* 0, *and*

$$\Gamma = \{g \in C([0,1], E\} \quad : \quad g(0) = 0, g(1) = e\}.$$

Lemma 2.2 [4]*Let E be a Banach space,* $I : E \to \mathbb{R}$ *a functional bounded from below and differentiable on E. If I satisfies the (PS)-condition then I has a minimum on E.*

Lemma 2.3 [3]*For every* $n \in \mathbb{Z}$, *the following inequalities hold:*

$$W(n,u) \leq W\left(n, \frac{u}{|u|}\right)|u|^\mu, \quad \text{if} \quad 0 < |u| \leq 1, \tag{2.6}$$

$$W(n,u) \geq W\left(n, \frac{u}{|u|}\right)|u|^\mu, \quad \text{ifquad}|u| \geq 1. \tag{2.7}$$

Lemma 2.4 *Set* $m := \inf\{W(n, u) : n \in [0,T], |u| = 1\}$. *Then for every* $\zeta \in \mathbb{R}\backslash\{0\}$, $u \in E_k\backslash\{0\}$, *we have*

$$\sum_{n=-kT}^{kT-1} W(n, \zeta u(n)) \geq m|\zeta|^\mu \sum_{n=-kT}^{kT-1} |u(n)|^\mu - 2kTm. \tag{2.8}$$

Proof Fix $\zeta \in \mathbb{R}\backslash\{0\}$ and $u \in E_k\backslash\{0\}$.

Set

$$A_k = \{n \in [-kT, kT-1] : |\zeta u(n)| \leq 1\}, \quad B_k = \{n \in [-kT, kT-1] : |\zeta u(n)| \geq 1\}.$$

From (2.7), we have

$$\begin{aligned}\sum_{n=-kT}^{kT} W(n,\zeta u(n)) &\geq \sum_{n\in B_k} W(n,\zeta u(n)) \geq \sum_{n\in B_k} W\left(n, \frac{\zeta u(n)}{|\zeta u(n)|}\right)|\zeta u(n)|^\mu \\ &\geq m\sum_{n\in B_k}|\zeta u(n)|^\mu \\ &\geq m\sum_{n=-kT}^{kT-1}|\zeta u(n)|^\mu - m\sum_{n\in A_k}|\zeta u(n)|^\mu \\ &\geq m|\zeta|^\mu\sum_{n=-kT}^{kT-1}|u(n)|^\mu - 2kTm.\end{aligned}$$

3. Existence of subharmonic solutions

In this section, we prove the existence of subharmonic solutions. In order to establish the condition of existence of subharmonic solutions for (2.1), first, we will prove the following lemmas, based on which we can get results of Theorem 1.1 and Theorem 1.2.

Lemma 3.1 *Let* $a, b \in \mathbb{Z}$, $a, b \geq 0$ *and* $u \in E_k$. *Then for every* $n,t \in \mathbb{Z}$, *the following inequality holds:*

$$|u(n)| \le (a+b+1)^{-1/\nu}\left(\sum_{t=n-a}^{n+b}|u(t)|^{\nu}\right)^{1/\nu} + \frac{\max\{a+1,b\}}{(a+b+1)^{1/p}}\left(\sum_{t=n-a}^{n+b}|\Delta u(t-1)|^{p}\right)^{1/p}. \quad (3.1)$$

Proof Fix $n \in \mathbb{Z}$, for every $\tau \in \mathbb{Z}$,

$$|u(n)| \le |u(\tau)| + \left|\sum_{t=\tau+1}^{n}\Delta u(t-1)\right|, \quad (3.2)$$

then by (3.2) and Höder inequality, we obtain

$$\begin{aligned}(a+b+1)|u(n)| &\le \sum_{\tau=n-a}^{n+b}|u(\tau)| + \sum_{\tau=n-a}^{n+b}\sum_{t=\tau+1}^{n}|\Delta u(t-1)| \\ &\le \sum_{\tau=n-a}^{n+b}|u(\tau)| + \sum_{\tau=n-a}^{n}\sum_{t=n=a+1}^{n}|\Delta u(t-1)| + \sum_{\tau=n+1}^{n+b}\sum_{t=n+1}^{n+b}|\Delta u(t-1)| \\ &\le (a+b+1)^{(\nu-1)/\nu}\left(\sum_{t=n-a}^{n+b}|u(t)|^{\nu}\right)^{1/\nu} + \max\{a+1,b\}\sum_{t=n-a}^{n+b}|\Delta u(t-1)| \\ &\le (a+b+1)^{(\nu-1)/\nu}\left(\sum_{t=n-a}^{n+b}|u(t)|^{\nu}\right)^{1/\nu} \\ &\quad + \max\{a+1,b\}\sum_{t=n-a}^{n+b}(a+b+1)^{(p-1)/p}\left(\sum_{t=n-a}^{n+b}|\Delta u(t-1)|^{p}\right)^{1/p},\end{aligned}$$

which implies that (3.1) holds. The proof is complete.

Corollary 3.1 *Let $u \in E_k$. Then for every $n \in \mathbb{Z}$, the following inequality holds:*

$$||u(n)||_{l^{\infty}_{2kT}} \le T^{-1/\nu}\left(\sum_{n=-kT}^{kT-1}|u(n)|^{\nu}\right)^{1/\nu} + T^{(p-1)/p}\left(\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^{p}\right)^{1/p}, \quad (3.3)$$

Proof For $n \in [-kT, kT-1]$, we can choose $n^* \in [-kT, kT-1]$ such that $u(n^*) = \max_{n\in[-kT,\ kT-1]}|u(n)|$. Let $a \in [0,T)$ and $b = T - a - 1$ such that $-kT \le n^* - a \le n^* \le n^* + b \le kT - 1$. Then by (3.1), we have

$$\begin{aligned}|u(n^*)| &\le T^{-1/\nu}\left(\sum_{n=n^*-a}^{n^*+b}|u(n)|\nu\right)^{1/\nu} + T^{(p-1)/p}\left(\sum_{n=n^*-a}^{n^*+b}|\Delta u(n-1)|^{p}ds\right)^{1/p} \\ &\le T^{-1/\nu}\left(\sum_{n=-kT}^{kT-1}|u(n)|\nu\right)^{1/\nu} + T^{(p-1)/p}\left(\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|p\right)^{1/p},\end{aligned}$$

which implies that (3.3) holds. The proof is complete.

Corollary 3.2 *Let $u \in E_k$. Then for every $n \in \mathbb{Z}$, the following inequality holds:*

$$||u(n)||_{l^{\infty}_{2kT}} \le 2\max\{T^{(p-1)/p}, T^{-1}\}||u||_{E_k} \doteq C||u||_{E_k}. \quad (3.4)$$

Proof Let $\nu = p$ in (3.3), we have

$$\begin{aligned}||u(n)||^{p}_{l^{\infty}_{2kT}} &\le 2^{p}\left(T^{-1}\sum_{n=-kT}^{kT-1}|u(n)|^{p} + T^{p-1}\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^{p}\right) \\ &\le 2^{p}\max\{T^{p-1}, T^{-p}\}\left(\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^{p} + |u(n)|^{p}\right) \\ &= 2^{p}\max\{T^{p-1}, T^{-p}\}||u||^{p}_{E_k},\end{aligned}$$

which implies that (3.4) holds. The proof is complete.

For the sake of convenience, set $\Lambda = \min\left\{\frac{\delta^{p-1}}{p}, b_2\delta^{\gamma-1} - M_1\delta^{\mu-1}\right\}$. By (H9), we have

$$\sum_{n\in\mathbb{Z}} |f(n)|^q < \frac{\Lambda^q}{C^p}, \tag{3.5}$$

where C is given in (3.4).

Here and subsequently,

$$\mathbb{N}(k_0) \doteq \{k : k \in \mathbb{N}, k \geq k_0\}.$$

Lemma 3.2 *Assume that F and f satisfy (H1)-(H3). Then for every* $k \in \mathbb{N}$, *system (2.1) possesses a 2kT-periodic solution* $u_k \in E_k$ *such that*

$$\frac{1}{p}\sum_{n=-kT}^{kT-1} |\Delta u_k(n-1)|^p + b_1 \sum_{n=-kT}^{kT-1} |u_k|^\nu \leq M\left(\sum_{n=-kT}^{kT-1} |u_k|^\nu\right)^{1/\nu}, \tag{3.6}$$

where

$$M = \left(\sum_{n\in\mathbb{Z}} |f(n)|^{\nu/(\nu-1)}\right)^{(\nu-1)/\nu}. \tag{3.7}$$

Proof Set $C_0 = \sum_{n=0}^{T} F(n,0)$. By (H2), (H3), (2.2), and the Hӧder inequality, we have

$$\begin{aligned} I_k(u) &= \sum_{n=-kT}^{kT-1}\left[\frac{1}{p}|\Delta u(n-1)|^p + F(n,u(n)) + (f_k(n),u(n))\right] \\ &\geq \sum_{n=-kT}^{kT-1}\left[\frac{1}{p}|\Delta u(n-1)|^p + F(n,0) + b_1|u(n)|^\nu + (f_k(n),u(n))\right] \\ &= \frac{1}{p}\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + b_1\sum_{n=-kT}^{kT-1}|u(n)|^\nu + \sum_{n=-kT}^{kT-1}(f_k(n),u(n)) + 2kC_0 \\ &\geq \frac{1}{p}\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + b_1\sum_{n=-kT}^{kT-1}|u(n)|^\nu \\ &\quad - \left(\sum_{n=-kT}^{kT-1}|f_k(n)^{\nu/(\nu-1)}\right)\left(\sum_{n=-kT}^{kT-1}|u(n)|^\nu\right)^{1/\nu} + 2kC_0 \\ &\geq \frac{1}{p}\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + b_1\sum_{n=-kT}^{kT-1}|u(n)|^\nu \\ &\quad - M\left(\sum_{n=-kT}^{kT-1}|u(n)|^\nu\right)^{1/\nu} + 2kC_0. \end{aligned} \tag{3.8}$$

For any $x \in [0, +\infty)$, we have

$$\frac{b_1}{2}x^\nu - Mx \geq -\frac{b_1}{2}(\nu-1)\left(\frac{2M}{b_1\mu}\right)^{\nu/(\nu-1)} := -D.$$

It follows from (3.8) that

$$I_k(u) \geq \frac{1}{p}\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + \frac{b_1}{2}\sum_{n=-kT}^{kT-1}|u(n)|^\nu - D + 2kC_0.$$

Consequently, I_k is a functional bounded from below.

Set

$$\bar{u} = \frac{1}{2kT}\sum_{n=-kT}^{kT-1}u(n), \quad \text{and} \quad \tilde{u}(n) = u(n) = \bar{u}.$$

Then by Sobolev's inequality, we have

$$||\tilde{u}||_{l^\infty_{2kT}} \leq C_1||\Delta u(n-1)||_{l^p_{2kT}}, \quad \text{and} \quad ||\tilde{u}||_{l^p_{2kT}} \leq C_2||\Delta u(n-1)||_{l^p_{2kT}}. \tag{3.9}$$

In view of (3.9), it is easy to verify, for each $k \in \mathbb{N}$, that the following conditions are equivalent:

(i) $||u||_{E_k} \to \infty$;

(ii) $|\bar{u}|^p + \sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p \to \infty$;

(iii) $\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + \frac{b_1}{2}\sum_{n=-kT}^{kT-1}|u(n)|^\nu \to \infty$.

Hence, from (3.8), we obtain

$$I_k(u) \to +\infty \ \text{as} \ ||u||_{E_k} \to \infty.$$

Then, it is easy to verify that I_k satisfies (PS)-condition. Now by Lemma 2.2, we conclude that for every $k \in \mathbb{N}$ there exists $u_k \in E_k$ such that

$$I_k(u_k) = \inf_{u \in E_k} I_k(u).$$

Since

$$I_k(0) = \sum_{n=-kT}^{kT-1}F(n,0) = 2kC_0,$$

we have $I_k(u_k) \leq 2kC_0$. It follows from (3.8) that

$$\frac{1}{p}\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + b_1\sum_{n=-kT}^{kT-1}|u_k|^\nu \leq M\left(\sum_{n=-kT}^{kT-1}|u_k(n)|^p\right)^{1/p}.$$

This shows that (3.6) holds. The proof is complete.

Lemma 3.3 *Assume that all conditions of Theorem 1.2 are satisfied. Then for every k* $\in \mathbb{N}$ (k_0), *the system (2.1) possesses a* $2kT$*-periodic solution* $u_k \in E_k$.

Proof In our case it is clear that $I_k(0) = 0$. First, we show that I_k satisfies the (PS) condition. Assume that $\{u_j\}_{j\in\mathbb{N}}$ in E_k is a sequence such that $\{I_k(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I_k'(u_j) \to 0, \ j \to +\infty$. Then there exists a constant $C_k > 0$ such that

$$|I_k(u_j)| \leq C_k, \quad ||I_k'(u_j)||_{k^*} \leq C_k \tag{3.10}$$

for every $j \in \mathbb{N}$. We first prove that $\{u_j\}_{j\in\mathbb{N}}$ is bounded. By (2.3) and (H5), we have

$$\eta_k^p(u_j) \le pI_k(u_j) + \frac{p}{\mu}\sum_{t=-kT}^{kT-1}(\nabla W(n,u_j(n)),u_j(n)) - p\sum_{n=-kT}^{kT-1}(f_k(n),u(n)), \tag{3.11}$$

From (2.5), (3.5), (3.10) and (3.11), we have

$$\begin{aligned}\left(1-\frac{\varrho}{\mu}\right)\eta_k^p(u_j) &\le pI_k(u_j) - \frac{p}{\mu}I_k'(u_j)u_j - \left(p-\frac{p}{\mu}\right)\sum_{n=-kT}^{kT-1}(f_k(n),u(n))\\ &\le pC_k + \left[\frac{p}{\mu}\|I_k'(u_j)\|_{k*} + \left(p-\frac{p}{\mu}\right)\left(\sum_{n=-kT}^{kT-1}|f_k(n)|^q\right)^{1/q}\right]\|u_j\|_{E_k}\\ &\le pC_k + \left[\frac{pC_k}{\mu} + \frac{p(\mu-1)\Lambda}{C^{p-1}\mu}\right]\|u_j\|_{E_k}\\ &= pC_k + D_k\|u_j\|_{E_k}, \quad k\in\mathbb{N}(k_0),\end{aligned} \tag{3.12}$$

where

$$D_k = \frac{pC_k}{\mu} + \frac{p(\mu-1)\Lambda}{C^{p-1}\mu}.$$

Without loss of generality, we can assume that $||u_j||_{E_k} \neq 0$. Then from (2.3), (3.3), and (H7), we obtain for $j \in \mathbb{N}$,

$$\begin{aligned}\eta_k^p(u_j) &= \left(\sum_{n=-kT}^{kT-1}[|\Delta u_j(n-1)|^p + pK(n,u_j)]\right)\\ &\ge \left(\sum_{n=-kT}^{kT-1}[|\Delta u_j(n-1)|^p + pb_2|u_j(n)|\gamma]\right)\\ &\ge \left(\sum_{n=-kT}^{kT-1}\left[|\Delta u_j(n-1)|^p + pb_2(C||u_j(n)||_{E_k})^{\gamma-p}\sum_{n=-kT}^{kT-1}|u_j(n)|^p\right]\right)\\ &\ge \min\{1, pb_2(C||u_j(n)||_{E_k})^{\gamma-p}\}\left(\sum_{n=-kT}^{kT-1}|\Delta u_j(n-1)|^p + \sum_{t=-kT}^{kT-1}|u_j(n)|p\right)\\ &= \min\{1, pb_2(C||u_j(n)||_{E_k})^{\gamma-p}\}||u_j||_{E_k}^p\\ &= \min\{||u_j||_{E_k}^p, pb_2C^{\gamma-p}||u_j(n)||_{E_k}^\gamma\}\end{aligned} \tag{3.13}$$

Combining (3.12) with (3.13), we have

$$\min\left\{\| u_j \|_{E_k}^p, pb_2C^{\gamma-p} \| u_j(n) \|_{E_k}^\gamma\right\} \le \frac{\mu}{\mu-\varrho}(pC_k + D_k \| u_j\|_{E_k}) \tag{3.14}$$

It follows from (3.14) that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E_k, it is easy to prove that $\{u_j\}_{j\in\mathbb{N}}$ has a convergent subsequence in E_k. Hence, I_k satisfies the Palais-Smale condition.

We now show that there exist constants ρ, $\alpha > 0$ independent of k such that I_k satisfies assumption (ii) of Lemma 2.1 with these constants. If $\| u\|_{E_k} = \delta/C := |\rho$, then it follows from (3.4) that $|u(n)| \le \delta \le 1$ for $n \in [-kT, kT - 1]$ and $k \in \mathbb{N}(k_0)$. By Lemma 2.3 and (H9), we have

$$\begin{aligned}\sum_{n=-kT}^{kT-1} W(n,u(n)) &= \sum_{n\in[-kT,kT-1]|u(n)\neq 0} W(n,u(n)) \\ &\le \sum_{n\in[-kT,kT-1]|u(n)\neq 0} W\left(n,\frac{u(n)}{|u(n)|}\right)|u(n)|^{\mu} \\ &\le M_1\sum_{n=-kT}^{kT-1}|u(n)|^{\mu} \\ &\le M_1\delta^{\mu-\gamma}\sum_{n=-kT}^{kT-1}|u(n)|^{\gamma}, k\in\mathbb{N}(k_0),\end{aligned} \tag{3.15}$$

and

$$\sum_{n=-kT}^{kT-1}|u(n)|^p \le \delta^{p-\gamma}\sum_{n=-kT}^{kT-1}|u(n)|^{\gamma}, k\in\mathbb{N}(k_0). \tag{3.16}$$

Set

$$\alpha = \frac{\delta}{C}\left[\frac{1}{C^{p-1}}\min\left\{\frac{\delta^{p-1}}{p}, b_2\delta^{\gamma-1} - M_1\delta^{\mu-1}\right\} - \sum_{n\in\mathbb{Z}}|f(n)|^q\right]. \tag{3.17}$$

Hence, from (2.1), (3.4) and (3.15)-(3.17), we have

$$\begin{aligned} I_k(u) &= \sum_{n=-kT}^{kT-1}\left[\frac{1}{p}|\Delta u(n-1)|^p + K(n,u(n)) - W(n,u(n)) + (f_k(n),u(n))\right] \\ &\ge \frac{1}{p}\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + b_2\sum_{n=-kT}^{kT-1}|u(n)|^{\gamma} - \sum_{n=-kT}^{kT-1}W(n,u(n)) + \sum_{n=-kT}^{kT-1}(f_k(n),u(n)) \\ &\ge \frac{1}{p}\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + (b_2 - M_1\delta^{\mu-\gamma})\sum_{n=-kT}^{kT-1}|u(n)|^{\gamma} \\ &\quad - \left(\sum_{n=-kT}^{kT-1}|f_k(n)|^q\right)^{1/q}\left(\sum_{n=-kT}^{kT-1}|u(n)|^p\right)^{1/p} \\ &\ge \frac{1}{p}\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + (b_2 - M_1\delta^{\mu-\gamma})\sum_{n=-kT}^{kT-1}|u(n)|^{\gamma} \\ &\quad - \left(\sum_{n\in\mathbb{Z}}|f_k(n)|^q\right)^{1/q}\left(\sum_{n=-kT}^{kT-1}|u(n)|^p\right)^{1/p} \\ &\ge \min\left\{\frac{1}{p}, b_2\delta^{\gamma-p} - M_1\delta^{\mu-p}\right\}\left(\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + \sum_{n=-kT}^{kT-1}|u(n)|^p\right) \\ &\quad - \left(\sum_{n\in\mathbb{Z}}|f_k(n)|^q\right)^{1/q}\left(\sum_{n=-kT}^{kT-1}|\Delta u(n-1)|^p + \sum_{n=-kT}^{kT-1}|u(n)|^p\right)^{1/p} \\ &= \min\left\{\frac{1}{p}, b_2\delta^{\gamma-p} - M_1\delta^{\mu-p}\right\}\|u\|_{E_k}^p - \|u\|_{E_k}\left(\sum_{n\in\mathbb{Z}}|f_k(n)|^q\right)^{1/q} \\ &= \frac{\delta}{C}\left[\frac{1}{C^{p-1}}\min\left\{\frac{\delta^{p-1}}{p}, b_2\delta^{\gamma-1} - M_1\delta^{\mu-1}\right\} - \left(\sum_{n\in\mathbb{Z}}|f_k(n)|^q\right)^{1/q}\right] \\ &= \alpha, k\in\mathbb{N}(k_0). \end{aligned} \tag{3.18}$$

(3.18) shows that $\|u\|_{E_k} = \rho$ implies that $I_k(u) \ge \alpha$ for $k \in \mathbb{N}(k_0)$.

Finally, it remains to show that I_k satisfies assumption (iii) of Lemma 2.1. Set

$$a_1 = \max\{K(n,x) | n \in [0,T], x \in \mathbb{R}^N, |x| = 1\},$$

and

$$a_2 = \max\{K(n,x) | n \in [0,T], x \in \mathbb{R}^N, |x| \leq 1\},$$

Then by (H8) and $0 < a_1 \leq a_2 < \infty$,

$$K(n,x) \leq a_1 |x|^{\varrho} + a_2, \quad \text{for } (n,x) \in \mathbb{Z} \times \mathbb{R}^N. \tag{3.19}$$

By (2.2), (3.19) and Lemma 2.4, we have for every $\zeta \in \mathbb{R} \setminus \{0\}$ and $u \in E_k \setminus \{0\}$

$$\begin{aligned} I_k(\zeta u) &= \sum_{n=-kT}^{kT-1} \left[\frac{1}{p} |\Delta u(n-1)|^p + K(n, \zeta u(n)) - W(n, \zeta u(n)) + \zeta (f_k(n), u(n)) \right] \\ &\leq \frac{|\zeta|^p}{p} \sum_{n=-kT}^{kT-1} |\Delta u(n-1)|^p + a_1 |\zeta|^{\varrho} \sum_{n=-kT}^{kT-1} |u(n)|^{\varrho} - m |\zeta|^{\mu} \sum_{n=-kT}^{kT-1} |u(n)|^{\mu} \\ &\quad + |\zeta| \left(\sum_{n \in \mathbb{Z}} |f_k(n)|^q \right)^{1/q} \| u \|_{E_k} + 2kT(m + a_2), \quad k \in \mathbb{N}(k_0). \end{aligned} \tag{3.20}$$

Take $Q \in E_{k_0}$ such that $Q(\pm k_0 T) = 0$ and $Q \neq 0$. Since $p \leq \varrho < \mu$ and $m > 0$, (3.20) implies that there exists $\xi > 0$ such that $\| \xi Q \|_{E_{k_0}} > \rho$ and $I_{k_0}(\xi Q) < 0$. Set $e_{k_0}(n) = \xi Q(n)$ and

$$e_k(n) = \begin{cases} e_{k_0}(n), & \text{for } |n| \leq k_0 T, \\ 0, & \text{for } k_0 T < |n| \geq kT. \end{cases} \tag{3.21}$$

Then $e_k \in E_k$, $\|e_k\|_{E_k} = \left\| e_{k_0} \right\|_{E_{k_0}} > \rho$ and $I_k(e_k) = I_{k_0}(e_{k_0}) < 0$ for $k \in \mathbb{N}(k_0)$. By Lemma 2.1, I_k possesses a critical value $c_k \geq \alpha$ given by

$$c_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} I_k(g(s)), \quad k \in \mathbb{N}(k_0).$$

where

$$\Gamma_k = \{ g \in C([0,1], E_k) : g(0) = 0, g(1) = e_k \}, \quad k \in \mathbb{N}(k_0).$$

Hence, for $k \in \mathbb{N}(k_0)$, there exists $u_k \in E_k$ such that

$$I_k(u_k) = c_k, \quad \text{and } I_k'(u_k) = 0.$$

Then function u_k is a desired classical $2kT$-periodic solution of (1.1) for $k \in \mathbb{N}(k_0)$. Since $c_k > 0$, u_k is a nontrivial solution even if $f_k(n) = 0$. The proof is complete.

4. Existence of homoclinic solutions

Lemma 4.1 *Let $u_k \in E_k$ be the solution of system (2.1) that satisfies (3.6) for $k \in \mathbb{N}$. Then there exists a positive constant d_1 independent of k such that*

$$\| u_k \|_{l^\infty_{2kT}} \leq d_1, \quad k \in \mathbb{N}.$$

Proof By (3.6), we have

$$b_1 \sum_{n=-kT}^{kT-1} |u_k(n)|^\nu \leq M \left(\sum_{n=-kT}^{kT-1} |u_k(n)|^\nu \right)^{1/\nu},$$

which implies that

$$\sum_{n=-kT}^{kT-1} |u_k(n)|^\nu \leq \left(\frac{M}{b_1} \right)^{\nu/(\nu-1)}. \tag{4.1}$$

From (3.6), we obtain

$$\sum_{n=-kT}^{kT-1} |\Delta u_k(n-1)|^p \leq pM \left(\frac{M}{b_1} \right)^{1/(\nu-1)}. \tag{4.2}$$

It follows from (3.3), (4.1) and (4.2) that

$$\begin{aligned} \| u_k \|_{l^\infty_{2kT}} &\leq T^{-1/\nu} \left(\sum_{n=-kT}^{kT-1} |u(n)|^\nu \right)^{1/\nu} + T^{(p-1)/p} \left(\sum_{n=-kT}^{kT-1} |\Delta u(n-1)|^p \right)^{1/p} \\ &\leq T^{-1/\nu} \left(\frac{M}{b_1} \right)^{1/(\nu-1)} + T^{(p-1)/p} (pM)^{1/p} \left(\frac{M}{b_1} \right)^{1/p(\nu-1)} \\ &:= d_1. \end{aligned}$$

Lemma 4.2 *Let $u_k \in E_k$ be the solution of system (1.1) which satisfies Lemma 3.3 for $k \in \mathbb{N}(k_0)$. Then there exists a positive constant d_2 independent of k such that*

$$\| u_k \|_{l^\infty_{2kT}} \leq d_2. \tag{4.3}$$

Proof For $k \in \mathbb{N}(k_0)$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$ where e_k is defined by (3.21). Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_{k_0}(g_{k_0}(s))$ for all $k \in \mathbb{N}(k_0)$ and $s \in [0,1]$, Therefore,

$$c_k \leq \max_{s \in [0,1]} I_{k_0}(g_{k_0}(s)) \equiv d_0, \quad k \in \mathbb{N}(k_0).$$

where d_0 is independent of k.

As $I_k'(u_k) = 0$, we get from (2.2), (2.5) and (H5)

$$\begin{aligned} c_k &= I_k(u_k) - \frac{1}{\varrho} I_k'(u_k) u_k \\ &\geq \left(\frac{\mu}{\varrho} - 1 \right) \sum_{n=-kT}^{kT-1} W(n, u(n)) + \frac{\varrho - 1}{\varrho} \sum_{n=-kT}^{kT-1} (f_k(n), u_k(n)), \end{aligned}$$

and hence

$$\sum_{n=-kT}^{kT-1} W(n, u_k(n)) \leq \frac{1}{\mu - \varrho} \left[\varrho c_k - (\varrho - 1) \sum_{n=-kT}^{kT-1} (f_k(n), u_k(n)) \right].$$

Combining the above with (2.4), we have

$$\begin{aligned} \eta_k^p(u_k) &= pI_k(u_k) + p\sum_{n=-kT}^{kT-1} W(n, u_k(n)) - p\sum_{n=-kT}^{kT-1} (f_k(n), u_k(n)) \\ &\le \frac{p\mu d_0}{\mu-\varrho} + \frac{p(\mu-1)}{\mu-\varrho}\left(\sum_{n=-kT}^{kT-1} |f_k(n)|^q\right)^{1/q} \| u_k\|_{E_k} \\ &\le \frac{p\mu d_0}{\mu-\varrho} + \frac{p(\mu-1)\Lambda}{C^{p-1}(\mu-\varrho)} \| u_k\|_{E_k}, \quad k \in \mathbb{N}(k_0). \end{aligned} \tag{4.4}$$

Since $u_k \neq 0$, similar to the proof of (3.13), we have

$$\eta_k^p(u_k) \ge \min\{\| u_k \|_{E_k}^p, pbC^{\gamma-p} \| u_k \|_{E_k}^\gamma\}, \quad k \in \mathbb{N}(k_0). \tag{4.5}$$

From (4.4) and (4.5), we obtain

$$\min\{\| u_k \|_{E_k}^p, pbC^{\gamma-p} \| u_k \|_{E_k}^\gamma\} \le \frac{p\mu d_0}{\mu-\varrho} + \frac{p(\mu-1)\Lambda}{C^{p-1}(\mu-\varrho)} \| u_k\|_{E_k}, \quad k \in \mathbb{N}(k_0). \tag{4.6}$$

Since all coefficients of (4.6) are independent of k, we see that there is $d_2 > 0$ independent of k such that

$$\| u_k\|_{E_k} \le d_2, \quad k \in \mathbb{N}(k_0), \tag{4.7}$$

which, together with (3.4), implies that (4.3) holds. The proof is complete.

5. Proofs of theorems

Proof of Theorem 1.1 The proof is similar to that of [20], but for the sake of completeness, we give the details.

We will show that $\{u_k\}_{k\in \mathbb{N}}$ possesses a convergent subsequence $\{u_{k_m}\}$ in $E_{loc}(\mathbb{Z},\mathbb{R})$ and a nontrivial homoclinic orbit u_∞ emanating from 0 such that $u_{k_m} \to u_\infty$ as $k_m \to \infty$.

Since $u_k = \{u_k(t)\}$ is well defined on $\mathbb{N}[-kT, kT-1]$ and $||u_k||_k \le d$ for all $k \in \mathbb{N}$, we have the following consequences.

First, let $u_k = \{u_k(t)\}$ be well defined on $\mathbb{N}[-T,T-1]$. It is obvious that $\{u_k\}$ is isomorphic to $\mathbb{R}^{2T}$. Thus, there exists a subsequence $\left\{u_{k_m}^1\right\}$ and $u^1 \in E^1$ of $\{u_k\}_{k\in \mathbb{N}\setminus\{1\}}$ such that

$$\| u_{k_m}^1 - u^1\|_1 \to 0.$$

Second, let $\left\{u_{k_m}^1\right\}$ be restricted to $\mathbb{N}[-2T, 2T-1]$. Clearly, $\left\{u_{k_m}^1\right\}$ is isomorphic to $\mathbb{R}^{4T}$. Thus there exists a further subsequence $\left\{u_{k_m}^2\right\}$ of $\left\{u_{k_m}^1\right\}$ satisfying $u_2 \notin \left\{u_{k_m}^2\right\}$ and $u^2 \in E_2$ such that

$$\| u_{k_m}^2 - u^2\|_2 \to 0 \quad k_m \to \infty.$$

Repeat this procedure for all $k \in \mathbb{N}$. We obtain sequence $\left\{u_{k_m}^r\right\} \subset \left\{u_{k_m}^{r-1}\right\}, u_p \notin \left\{u_{k_m}^r\right\}$ and there exists $u_r \in E_r$ such that

$$\| u_{k_m}^r - u^r \|_r \to 0, \quad k_m \to \infty, \quad r = 1, 2, \dots$$

Moreover, we have

$$\| u^{r+1} - u^r \|_r \leq \| u_{k_m}^{r+1} - u^{r+1} \|_r + \| u_{k_m}^{r+1} - u^r \|_r \to 0,$$

which leads to

$$u^{r+1}(s) = u_r(s), s \in \mathbb{N}[-rT, rT - 1].$$

So, for the sequence $\{u^r\}$, we have $u^r \to u_\infty$, $r \to \infty$, where $u_\infty(s) = u^r(s)$ for $s \in \mathbb{N}[-rT, rT - 1]$ and $r \in \mathbb{N}$. Then take a diagonal sequence $\{u_{k_m}\} : u_{k_1}^1, u_{k_2}^2, \dots u_{k_m}^m, \dots,$ since $\{u_{k_m}^m\}$ is a sequence of $\{u_{k_m}^r\}$ for any $r \geq 1$, it follows that

$$\| u_{k_m}^m - u_\infty \| = \| u_{k_m}^m - u^m \|_m \to 0, \ m \in \mathbb{N}.$$

It shows that

$$u_{k_m} \to u_\infty \text{ as } k_m \to \infty, \text{ in } E_{\text{loc}}(\mathbb{Z}, \mathbb{R}),$$

where $u_\infty \in E_\infty(\mathbb{Z}, \mathbb{R}),\ E_\infty(\mathbb{Z}, \mathbb{R}) = \left\{u \in S \mid \| u\|_\infty = \Sigma_{m=-\infty}^{+\infty}(|\Delta u(n-1)|^p + |u(n)|^p) < \infty\right\}$.
By series convergence theorem, u_∞ satisfy

$$u_\infty(n) \to 0, \ \Delta u_\infty(n - 1) \to 0,$$

and

$$\sum_{n=-rT}^{rT-1} \{[|\Delta u_{k_m}^m(n-1)|^p + |u_{k_m}^m(n)|^p] < \infty\} = \| u_{k_m}^m \|,$$

as $|n| \to \infty$.
Letting $n \to \infty$, $\forall\ r \geq 1$, we have

$$\sum_{n=-rT}^{rT-1} \left[\frac{1}{p}|\Delta u_{k_m}^m(n-1)|^p + F(n, u_{k_m}^m(n)) + (f_k(n), u_{k_m}^m(n))\right] \leq d_1,$$

as $m \geq r$, $k_m \geq r$, where d_1 is independent of k, $\{k_m\} \subset \{k\}$ are chosen as above, we have

$$\sum_{n=-kT}^{kT-1} \left[\frac{1}{p}|\Delta u_\infty(n-1)|^p + F(n, u_\infty(n)) + (f(n), u_\infty(n))\right] \leq d_1.$$

Letting $p \to \infty$, by the continuity of $F(t,u)$ and $I_k^{'}$, which leads to

$$I_\infty(u_\infty) = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{p}|\Delta u_\infty(n-1)|^p + F(n, u_\infty(n)) + (f(n), u_\infty(n))\right] \leq d_1, \forall u \in E_\infty,$$

and

$$I_\infty^{'}(u_\infty) = 0.$$

Clearly, u_∞ is a solution of (1.1).

To complete the proof of Theorem 1.2, it remains to prove that $u_\infty \not\equiv 0$.. By the above argument, we obtain

$$\Delta(\varphi_p(\Delta u_\infty(n-1))) = \nabla F(n, u_\infty(n)) + f(n), \tag{5.1}$$

By (H5) and (H7), it is easy to see that

$$\nabla F(n,0) = -\nabla K(n,0) + \nabla W(n,0) = 0$$

This shows that $u = 0$ is not a solution of (1.1) with $f \neq 0$ and so $u_\infty \neq 0$.

6. Examples

In this section, we give some examples to illustrate our results.

Example 6.1 Consider the second order discrete p-Laplacian systems:

$$\Delta(\varphi_p(\Delta u(n-1))) = \nabla F(n, u(n)) + f(n), \quad n \in \mathbb{Z},\ u \in \mathbb{R}^{\mathbb{N}}, \tag{6.1}$$

where

$$F(n,x) = \sin^2\frac{\pi}{T}n + |x| + b_1|x|^2,\ f(n) = \frac{1}{\sqrt{|n|}},\ \nu = \frac{4}{3}.$$

Then it is easy to verify that all conditions of Theorem 1.1 are satisfied. By Theorem 1.1, the system (6.1) has a nontrivial homoclinic solution.

Example 6.2 Consider the second order discrete systems:

$$\Delta^2 u(n-1) = \nabla F(n, u(n)) + f(n), \quad n \in \mathbb{Z},\ u \in \mathbb{R}^{\mathbb{N}}, \tag{6.2}$$

where

$$p = 2,\ K(n,x) = |x|^2 + 3|x|^{\frac{5}{2}},\ W(n,x) = \frac{1}{6}\left(1 + \sin\frac{n\pi}{2}\right)|x|^4,\ f(n) = \frac{a}{1+|n|}.$$

It is easy to verify that conditions (H4)-(H8) are satisfied with $\gamma = 2, \varrho = \frac{5}{2}, \mu = 4, T = 4$ and $b_2 = 1$.

Noting that

$$M = \sup\left\{\frac{1}{6}\left(1 + \sin\frac{n\pi}{2}\right)|x|^4 | n \in \{0,1,2,3\}, x \in \mathbb{R}^N, |x| = 1\right\} = \frac{1}{3}.$$

Therefore, $b_2(\gamma - 1) = M(\mu - 1) = 1$. Since

$$\sum_{n\in\mathbb{Z}} |f(n)|^2 = a^2 \sum_{n\in\mathbb{Z}} \frac{1}{(1+|n|)^2} = a^2\left(\frac{\pi^2}{3} - 1\right).$$

so (1.4) holds, i.e., condition (H9) holds if $0 < a^2 < \frac{3}{32(\pi^2-3)}$. In view of Theorem 1.2, the system (6.2) possesses a nontrivial homoclinic solution.

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Authors' contributions

All authors carried out the proof and authors conceived of the study. All authors read and approved the final manuscript.

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References

1. Ambrosetti, A, Rabinowitz, PH: Dual variational methods in critical point theory and applications. J Funct Anal. **14**(4), 349–381 (1973)
2. Chen, P, Tang, XH: New existence of homoclinic orbits for a second-order Hamiltonian system. Comput Math Appl. **62**, 131–141 (2011)
3. Izydorek, M, Janczewska, J: Homoclinic solutions for a class of the second order Hamiltonian systems. J Differ Equ. **219**, 375–389 (2005)
4. Mawhin, J, Willem, M: Critical Point Theory and Hamiltonian Systems. Springer, New York (1989)
5. Omana, W, Willem, M: Homoclinic orbits for a class of Hamiltonian systems. Differ Integr Equ. **5**(5), 1115–1120 (1992)
6. Peil, T, Peterson, A: Asymptotic behavior of solutions of a two-term difference equation. Rocky Mountain J Math. **24**, 233–251 (1994)
7. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications in Differential Equations. CBMS Reg Conf Series **65** (1986). Amer. Math. Soc., Providence
8. Rodriguez, J, Etheridge, DL: Periodic solutions of nonlinear second order difference equations. Adv Differ Equ. **2005**, 173–192 (2005)
9. Tang, XH, Lin, XY, Xiao, L: Homoclinic solutions for a class of second order discrete Hamiltonian systems. J Differ Equ Appl. **11**, 1257–1273 (2010)
10. Tang, XH, Xiao, L: Homoclinic solutions for nonautonomous second-order Hamilto-nian systems with a coercive potential. J Math Anal Appl. **351**, 586–594 (2009)
11. Xue, YF, Tang, CL: Existence of a periodic solution for subquadratic second-order discrete Hamiltonian system. Nonlinear Anal. **67**, 2072–2080 (2007)
12. Chen, P, Tang, XH: Existence of homoclinic solutions for the second-order p-Laplacian systems. Taiwanese Journal of Mathematics. **15**, 2123–2143 (2011)
13. Chen, P, Tang, XH: Existence of infinitely many homoclinic orbits for fourth-order difference systems containing both advance and retardation. Appl Math Comput. **217**, 4408–4415 (2011)
14. Agarwal, RP, Perera, K, O'Regan, D: Multiple positive solutions of singular discrete p-Laplacian problems via variational methods. Adv Differ Equ. **2005**(2), 93–99 (2005)
15. Guo, ZM, Yu, JS: The existence of periodic and subharmonic solutions for second order superlinear difference equations. Sci China Ser A. **46**, 506–513 (2003)
16. Guo, ZM, Yu, JS: Periodic and subharmonic solutions for superquadratic discrete Hamiltonian systems. Nonlinear Anal. **55**, 969–983 (2003)
17. Guo, ZM, Yu, JS: The existence of periodic and subharmonic solutions of subquadratic second order difference equations. J London Math Soc. **68**, 419–430 (2003)
18. Kocic, VL, Ladas, G: Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer, Dordrecht (1993)
19. Liang, HH, Weng, PX: Existence and multiple solutions for a second order difference boundary value problem via critical point theory. J Math Anal Appl. **326**, 511–520 (2007)
20. Ma, M, Guo, ZM: Homoclinic orbits and subharmonics for nonlinear second order difference equations. Nonlinear Anal. **67**, 1737–1745 (2007)
21. Ma, M, Guo, ZM: Homoclinic orbits for second order self-adjoint difference equations. J Math Anal Appl. **323**(1), 513–521 (2006)
22. Yu, JS, Guo, ZM, Zou, X: Positive periodic solutions of second order self-adjoint difference equations. J London Math Soc. **71**(2), 146–160 (2005)
23. Chen, P, Tang, XH: Existence and multiplicity of homoclinic orbits for 2nth-order nonlinear difference equations containing both many advances and retardations. J Math Anal Appl. **381**, 485–505 (2011)
24. Deng, X, Liu, X, Shi, H, Zhou, T: Homoclinic orbits for second-order nonlinear p-Laplacian difference equations. J Cont Math Anal. **46**(3), 172–182 (2011)

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