Hartley and Lorenzo *Advances in Difference Equations* 2011, **2011**:59
http://www.advancesindifferenceequations.com/content/2011/1/59

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Order-distributions and the Laplace-domain logarithmic operator

Tom T Hartley[1*] and Carl F Lorenzo[2]

* Correspondence: thartley@uakron.edu
[1]Department of Electrical and Computer Engineering, University of Akron, Akron, OH 44325-3904, USA
Full list of author information is available at the end of the article

Abstract

This paper develops and exposes the strong relationships that exist between time-domain order-distributions and the Laplace-domain logarithmic operator. The paper presents the fundamental theory of the Laplace-domain logarithmic operator, and related operators. It is motivated by the appearance of logarithmic operators in a variety of fractional-order systems and order-distributions. Included is the development of a system theory for Laplace-domain logarithmic operator systems which includes time-domain representations, frequency domain representations, frequency response analysis, time response analysis, and stability theory. Approximation methods are included.

Keywords: Order-distribution, Laplace transform, Fractional-order systems, Fractional calculus

Introduction

The area of mathematics known as fractional calculus has been studied for over 300 years [1]. Fractional-order systems, or systems described using fractional derivatives and integrals, have been studied by many in the engineering area [2-9]. Additionally, very readable discussions, devoted to the mathematics of the subject, are presented by Oldham and Spanier [1], Miller and Ross [10], Oustaloup [11], and Podlubny [12]. It should be noted that there are a growing number of physical systems whose behavior can be compactly described using fractional-order system theory. Specific applications are viscoelastic materials [13-16], electrochemical processes [17,18], long lines [5], dielectric polarization [19], colored noise [20], soil mechanics [21], chaos [22], control systems [23], and optimal control [24]. Conferences in the area are held annually, and a particularly interesting publication containing many applications and numerical approximations is Le Mehaute et al. [25].

The concept of an order-distribution is well documented [26-31]. Essentially, an order-distribution is a parallel connection of fractional-order integrals and derivatives taken to the infinitesimal limit in delta-order. Order-distributions can arise by design and construction, or occur naturally. In Bagley [32], a thermo-rheological fluid is discussed. There it is shown that the order of the rheological fluid is roughly a linear function of temperature. Thus a spatial temperature distribution inside the material leads to a related spatial distribution of system orders in the rheological fluid, that is, the position-force dynamic response will be represented by a fractional-order derivative whose order varies with position or temperature inside the material. In Hartley and

Springer

© 2011 Hartley and Lorenzo; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Lorenzo [33], it is shown that various order-distributions can lead to a variety of transfer functions, many of which contain a ln(s) term, where s is the Laplace variable. Some of these results are reproduced in the tables at the end of this paper. In Adams et al. [34], it is shown that a conjugated-order derivative can lead directly to terms containing ln(s). In Adams et al. [35], it is also shown that the conjugated derivative is equivalent to the third generation CRONE control which has been applied extensively to control a variety of systems [25].

The purpose of this paper is to provide an understanding of the ln(s) operator, the Laplace-domain logarithmic operator, and determine what special properties are associated with it. The motivation is the frequent occurrence of the ln(s) operator in problems whose dynamics are expressed as fractional-order systems or order-distributions. This can be seen in Figure 1 which shows the transfer functions corresponding to several order-distributions, taken from Hartley and Lorenzo [33]. This figure demonstrates that the ln(s) operator appears frequently.

The next section will review the necessary results from fractional calculus and the theory of order-distributions. It will then be shown that the Laplace-domain logarithmic operator arises naturally as an order distribution, thereby providing a method for constructing a logarithmic operator either in the time or frequency domain. It is then shown that Laplace-domain logarithmic operators can be combined to form systems of logarithmic operators. Following this is the development of a system theory for Laplace-domain logarithmic operator systems which includes time-domain representations, frequency domain representations, frequency response analysis, time response analysis, and stability theory. Fractional-order approximations for logarithmic operators are then developed using finite differences. The paper concludes with some special order-distribution applications.

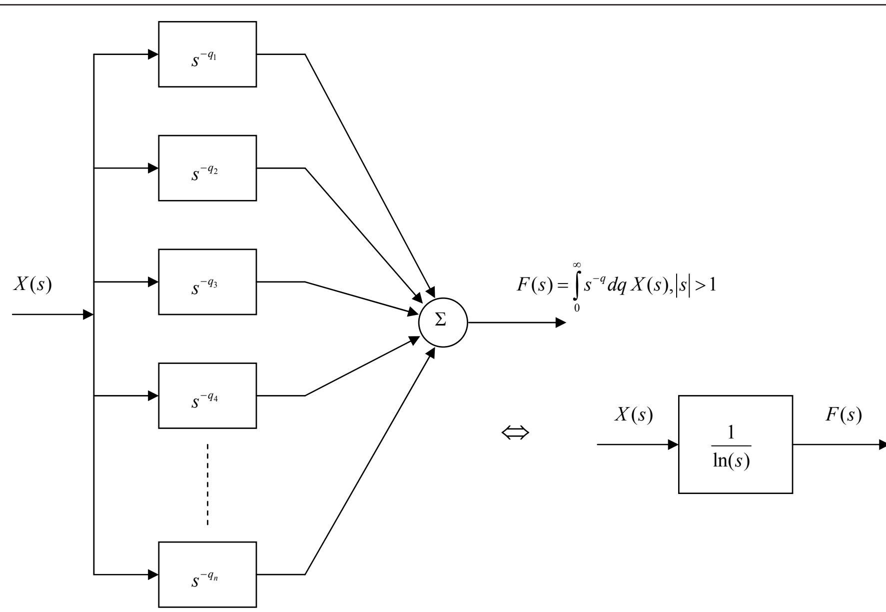


Figure 1 High frequency continuum order-distribution realization of the Laplace-domain reciprocal logarithmic operator.

Order-distributions

In Hartley and Lorenzo [30,33], the theory of order-distributions has been presented. It is based on the use of fractional-order differentiation and integration. The definition of the uninitialized *q*th-order Riemann-Liouville fractional integral is

$$ {}_0d_t^{-q}x(t) \equiv \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} x(\tau)d\tau, \quad q \geq 0 \tag{1}$$

The *p*th-order fractional derivative is defined as an integer derivative of a fractional integral

$$ {}_0d_t^{p}x(t) \equiv \frac{1}{\Gamma(1-p)} \frac{d}{dt} \int_0^t (t-\tau)^{-p} x(\tau)\mathrm{d}\tau 0, \quad < p < 1 \tag{2}$$

If higher derivatives are desired ($p > 1$), multiple integer derivatives are taken of the appropriate fractional integral. The integer derivatives are taken as in the standard calculus. In what follows, it will be important to use the Laplace transform of the fractional integrals and derivatives. This Laplace transform is given in Equation 3, where it is assumed that any initialization is zero

$$ L_0 d_t^q x(t) = s^q X(s) \text{forall}\, q \tag{3}$$

By comparing the convolution operators with the Laplace transforms, a fundamentally useful Laplace transform pair is

$$ Lt^{q-1} = \frac{\Gamma(q)}{s^q}, \quad q > -1 \tag{4}$$

Here it can be seen that an operator such as that in Equation 3, can also be written as a Laplace-domain operator as

$$ F(s) = s^q X(s). $$

where for example, $f(t)$ could be force, and $x(t)$ could be displacement. Now if there exists a collection of these individual fractional-order operators driven by the same input, then their outputs can be combined

$$ F(s) = k_1 s^{q_1} X(s) + k_2 s^{q_2} X(s) + k_3 s^{q_3} X(s) + k_4 s^{q_4} X(s) + \cdots = \sum_{n=1}^{N} k_n s^{q_n} X(s) $$

where the k's are weightings on each fractional integral. Taking the summation to a continuum limit yields the definition of an order-distribution

$$ \left(\int_0^{q_{\max}} k(q) s^q X(s) dq \right) = \left(\int_0^{q_{\max}} k(q) s^q dq \right) X(s) = F(s), \tag{5}$$

where $q_{\max}$ is an upper limit on the differential order and should be finite for the integral to converge, and $k(q)$ must be such that the integral is convergent. Equation 5 has the uninitialized time domain representation

$$\int_0^{q\max} k(q)\,{}_0d_t^q x(t)dq = f(t) \tag{6}$$

Order-distributions can also be defined using integral operators instead of differential operators as

$$X(s) = \left(\int_0^\infty k(q)s^{-q}F(s)dq\right) = \left(\int_0^\infty k(q)s^{-q}dq\right)F(s), \tag{7}$$

The uninitialized time domain representation of Equation 7 is

$$\int_0^\infty k(q)\,{}_0d_t^q f(t)dq = x(t) \tag{8}$$

As a further generalization, in Equation 5, the lower limit of integration can be extended below zero to give

$$\left(\int_0^{q\max} k(q)s^q d(s)dq\right)X(s) = F(s) \tag{9}$$

where again, $k(q)$ must be chosen such that the integral converges. Even more generally, an order distribution can be written as

$$\left(\int_a^b k(q)s^{-q}dq\right)X(s) = -\left(\int_{-b}^{-a} k(-q)s^q dq\right)X(s) = F(s), a < b. \tag{10}$$

The Laplace-domain logarithmic operator

The logarithmic operator can now be defined using the order-distribution concept. In Equation 10, let $k(q)$ be unity over the region of integration, $a = 0$, and b equal to infinity. Then the order-distribution is

$$F(s) = \left(\int_0^\infty s^{-q}dq\right)X(s) = -\left(\int_{-\infty}^0 s^q dq\right)X(s). \tag{11}$$

Evaluating the first integral on the left gives

$$\begin{aligned} F(s) &= \left(\int_0^\infty s^{-q}dq\right)X(s) = \left(\int_0^\infty e^{-q\ln(s)}dq\right)X(s) = \left.\frac{e^{-q\ln(s)}}{-\ln(s)}X(s)\right|_0^\infty, |s| > 1, \\ &= \left[\frac{e^{-\infty\ln(s)}}{-\ln(s)} - \frac{e^{-0\ln(s)}}{-\ln(s)}\right]X(s), |s| > l \end{aligned}.$$

Thus

$$F(s) = \left(\int_0^\infty s^{-q}dq\right)X(s) = \left[\frac{1}{\ln(s)}\right]X(s), \quad |s| > 1 \tag{12}$$

With the constraint that $|s| > 1$ for the integral to converge, it is seen that this order-distribution is an exact representation of the Laplace domain logarithmic operator at high frequencies, $\omega > 1$, or small time. From Equation 12, it can be seen that the reciprocal Laplace-domain logarithmic operator can be represented by the sum of all fractional-order integrals at high frequencies, $\omega > 1$. This can be visualized as shown in Figure 2.

At high frequencies, $\omega > 1$, the time-domain operator corresponding to Equation 12 is

$$f(t) = \int_0^\infty \int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau\, dq, \tag{13}$$

so that

$$\left[\frac{1}{\ln(s)}\right] X(s) \Leftrightarrow \int_0^\infty \int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} x(\tau)\, d\tau\, dq.$$

These results are verified by the Laplace transform pair given in Roberts and Kaufman

$$\frac{1}{\ln(s)} \Leftrightarrow \int_0^\infty \frac{t^{q-1}}{\Gamma(q)} dq, \tag{14}$$

which is obtained from Equation 13 by letting $x(t) = \delta(t)$, a unit impulse. It is important to note that the time domain function on the right-hand side of Equation 14 is known as a Volterra function, and is defined for all positive time, not just at high frequencies (small time) [36].

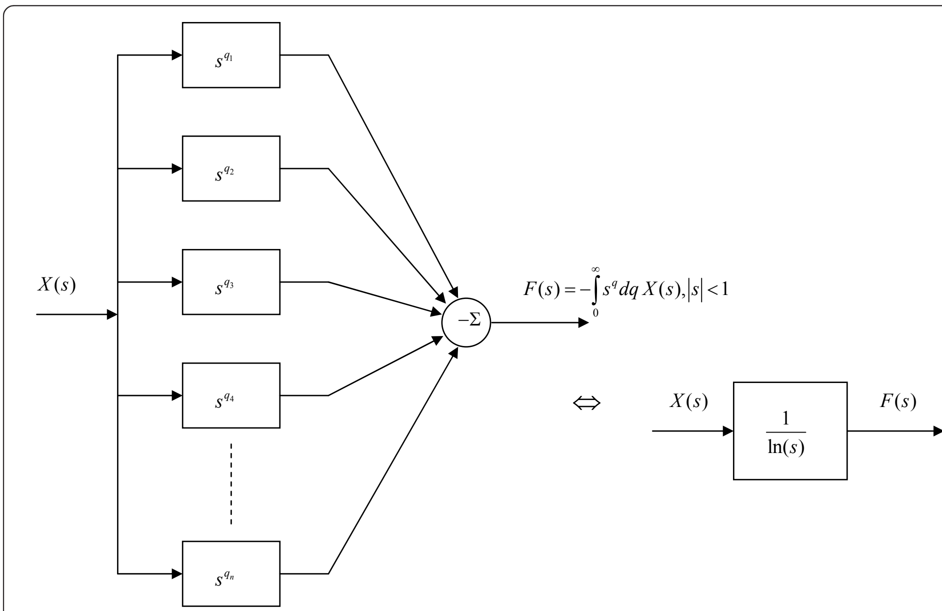


Figure 2 Low frequency continuum order-distribution realization of the Laplace-domain reciprocal logarithmic operator.

Referring back to Equation 10, again let $k(q)$ be unity over the region of integration, a = negative infinity, and b = 0. Then the order-distribution is

$$F(s) = \left(\int_{-\infty}^{0} s^{-q} dq \right) X(s) = -\left(\int_{0}^{+\infty} s^{q} dq \right) X(s). \tag{15}$$

Evaluating the integral on the right gives

$$\begin{aligned} F(s) &= -\left(\int_{0}^{\infty} s^{q} dq \right) X(s) = -\left(\int_{0}^{\infty} e^{q\ln(s)} dq \right) X(s) = -\frac{e^{q\ln(s)}}{\ln(s)} X(s) \Bigg|_{0}^{\infty}, \quad |s| < 1, \\ &= -\left[\frac{e^{\infty \ln(s)}}{\ln(s)} - \frac{e^{0\ln(s)}}{\ln(s)} \right] X(s), \quad |s| < l. \end{aligned}$$

Thus,

$$F(s) = -\left(\int_{0}^{\infty} s^{q} dq \right) X(s) = \left[\frac{1}{\ln(s)} \right] X(s), \ \ | \ s \ |< 1. \tag{16}$$

With the constraint that $|s| < 1$ for the integral convergence, it is seen that this order-distribution is an exact representation of the Laplace domain logarithmic operator at low frequencies, $\omega < 1$, or large time. From Equation 16, it can be seen that the reciprocal Laplace-domain logarithmic operator can be represented by the sum of all fractional-order derivatives at low frequencies (large time). This can be visualized as shown in Figure 3.

At low frequencies, $\omega < 1$, or large time, the integral over all the fractional derivatives must be used as in Equation 16. The time-domain operator corresponding to Equation 16 is then, with $q = p - u$,

$$f(t) = \int_{0}^{\infty} \frac{d^p}{dt^p} \int_{0}^{t} \frac{(t-\tau)^{u-1}}{\Gamma(u)} x(\tau) d\tau dq, \qquad p = 1, 2, 3, \ldots, \qquad p > q > p-1, \tag{17}$$

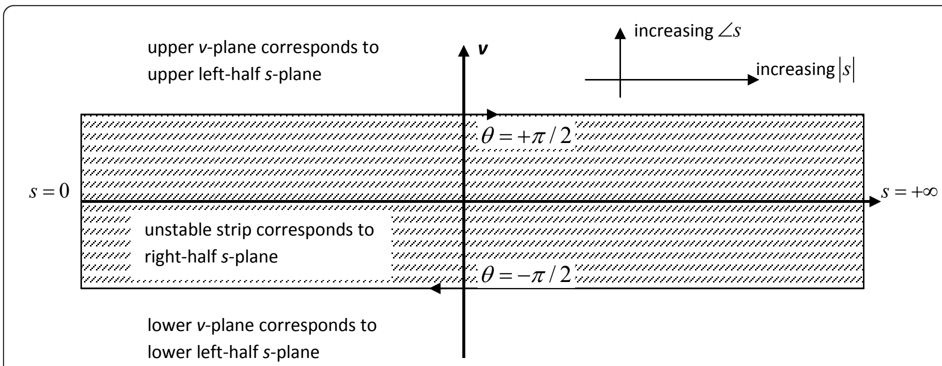


Figure 3 Stable and unstable regions of the v = ln(s) plane.

so that

$$\left[\frac{1}{\ln(s)}\right]X(s) \Leftrightarrow \int_0^\infty \frac{d^p}{dt^p}\int_0^t \frac{(t-\tau)^{u-1}}{\Gamma(u)}x(\tau)d\tau dq, \quad p=1,2,3,\ldots, \quad p>q>p-1, \; |s|<1. \tag{18}$$

Letting the input $x(t) = \delta(t)$, a unit impulse, this equation becomes

$$\left[\frac{1}{\ln(s)}\right]X(s) \Leftrightarrow \int_0^\infty \frac{d^p}{dt^p}\left(\frac{(t)^{u-1}}{\Gamma(u)}\right)dq, \quad p=1,2,3,\ldots, \quad p>q>p-1, \; |s|<1. \tag{19}$$

Performing the integral yields

$$\frac{1}{\ln(s)} \Leftrightarrow \int_0^\infty \frac{t^{-q-1}}{\Gamma(-q)}dq. \tag{20}$$

The properties of this integral require further study, although it appears to be convergent for large time due to the gamma function going to infinity when q passes through an integer and thus driving the integrand to zero there.

Higher powers of the Laplace-domain logarithmic operator

Higher powers of logarithmic operators can be generated using order distributions. In Equation 10, rather than letting $k(q)$ be unity over the region of integration, $a = 0$, and b equal to infinity, now set $k(q) = q$. Then, at high frequencies, the integral becomes

$$F(s) = \left(\int_0^\infty qs^{-q}dq\right)X(s) = \left(\int_0^\infty qe^{-q\ln(s)}dq\right)X(s).$$

Recognizing the rightmost term as the Laplace transform of q using $\ln(s)$ as the Laplace variable, gives

$$F(s) = \left(\int_0^\infty qe^{-q\ln(s)}dq\right)X(s) = \frac{1}{\ln^2(s)}X(s), \quad |s|>1,$$

the square of the logarithmic operator. Likewise, this process can be continued for other polynomial terms in q, to give

$$F(s) = \left(\int_0^\infty q^n s^{-q}dq\right)X(s) = \left(\int_0^\infty q^n e^{-q\ln(s)}dq\right)X(s) = \frac{n!}{\ln^{n+1}(s)}X(s), n=0,1,2,3,\ldots, \quad |s|>1$$

For non-integer values of n, this process gives

$$F(s) = \left(\int_0^\infty q^n s^{-q}dq\right)X(s) = \left(\int_0^\infty q^n e^{-q\ln(s)}dq\right)X(s) = \frac{\Gamma(n+1)}{\ln^{n+1}(s)}X(s), \quad |s|>1, \tag{21}$$

Referring back to Equation 10, rather than letting $k(q)$ be unity over the region of integration, a = negative infinity, and $b = 0$, now set $k(q) = q$. Thus, at low frequencies, the integral becomes

$$F(s) = \left(\int_{-\infty}^{0} q s^{-q} dq \right) X(s) = -\left(\int_{0}^{+\infty} q s^{q} dq \right) X(s) = -\left(\int_{0}^{\infty} q e^{q \ln(s)} dq \right) X(s). \tag{22}$$

Recognizing the rightmost term as the Laplace transform of q using $\ln(s)$ as the Laplace variable, gives

$$F(s) = -\left(\int_{0}^{\infty} q e^{q \ln(s)} dq \right) X(s) = \frac{1}{\ln^2(s)} X(s), \quad |s| < 1,$$

the square of the logarithmic operator. Likewise, this process can be continued for other polynomial terms in q, to give

$$F(s) = -\left(\int_{0}^{\infty} q^n s^q dq \right) X(s) = -\left(\int_{0}^{\infty} q^n e^{q \ln(s)} dq \right) X(s) = \frac{n!}{\ln^{n+1}(s)} X(s), \quad n = 0, 1, 2, 3, \ldots, \quad |s| < 1.$$

For non-integer values of n, this process gives

$$F(s) = -\left(\int_{0}^{\infty} q^n s^q dq \right) X(s) = -\left(\int_{0}^{\infty} q^n e^{q \ln(s)} dq \right) X(s) = \frac{\Gamma(n+1)}{\ln^{n+1}(s)} X(s), \quad |s| < 1, \tag{23}$$

Systems of Laplace-domain logarithmic operators

Using the definitions for higher powers of logarithmic operators, it is possible to create systems of Laplace-domain logarithmic operator equations. As an example, consider the high frequency realization

$$\begin{aligned} & a_2 \left(\int_{0}^{\infty} q^2 s^{-q} X(s) dq \right) + a_1 \left(\int_{0}^{\infty} q s^{-q} X(s) dq \right) + a_0 \left(\int_{0}^{\infty} s^{-q} X(s) dq \right) \\ & = b_2 \left(\int_{0}^{\infty} q^2 s^{-q} U(s) dq \right) + b_1 \left(\int_{0}^{\infty} q s^{-q} U(s) dq \right) + b_0 \left(\int_{0}^{\infty} s^{-q} U(s) dq \right), \quad |s| > 1. \end{aligned}$$

Simplifying this gives

$$\begin{aligned} & a_2 \left(\frac{2}{\ln^3(s)} X(s) \right) + a_1 \left(\frac{1}{\ln^2(s)} X(s) \right) + a_0 \left(\frac{1}{\ln(s)} X(s) \right) \\ & = b_2 \left(\frac{2}{\ln^3(s)} U(s) \right) + b_1 \left(\frac{1}{\ln^2(s)} U(s) \right) + b_0 \left(\frac{1}{\ln(s)} U(s) \right). \end{aligned}$$

or

$$2a_2 X(s) + a_1 \ln(s) X(s) + a_0 \ln^2(s) X(s) = 2b_2 U(s) + b_1 \ln(s) U(s) + b_0 \ln^2(s) U(s).$$

This results in the transfer function

$$\frac{X(s)}{U(s)} = \frac{b_0 \ln^2(s) + b_1 \ln(s) + 2b_2}{a_0 \ln^2(s) + a_1 \ln(s) + 2a_2}. \tag{24}$$

Properties of transfer functions of this type will be the subject of the remainder of the paper.

Stability properties

The stability of systems composed only of Laplace-domain logarithmic operators must be studied in the complex ln(s)-plane. Generally, to study stability of an operator in a complex plane, which is a mapping of another complex variable, the boundary of stability in the original complex plane must be mapped through the operator into the new complex plane. For the ln(s) operator, let ν = ln(s), thus

$$\nu = \ln(s)\big|_{s=re^{j\theta}} = \ln(re^{j\theta}),$$

or

$$\nu = \ln(r) + j\theta + j2n\pi,$$

where n is generally all integers. Using only the primary strip, for n = 0, gives the plot of Figure 4. The stability boundary in the s-plane is the imaginary axis, or $\theta = \pm\pi/2$, and all r. Using the mapping, the positive imaginary s-axis maps into a line at $\nu = +j\pi/2$, which goes from minus infinity to plus infinity as r is varied from zero to plus infinity. Continuing around a contour with radius infinity in the left half of the s-plane, yields an image in the ν-plane moving downward out at plus infinity. Then moving back in the negative imaginary s-axis as r is varied from plus infinity to zero, gives a line in the ν-plane at $\nu = -j\pi/2$, which goes from plus infinity to minus infinity. Closing the contour in the s-plane by going around the origin on a semi-circle of radius zero, gives an upward vertical line at ν equal to minus infinity. As orientations are preserved through the mapping, the stable region always lies to the left of the contour. In the ν-plane, this is the region above the top horizontal line, and below the lower horizontal

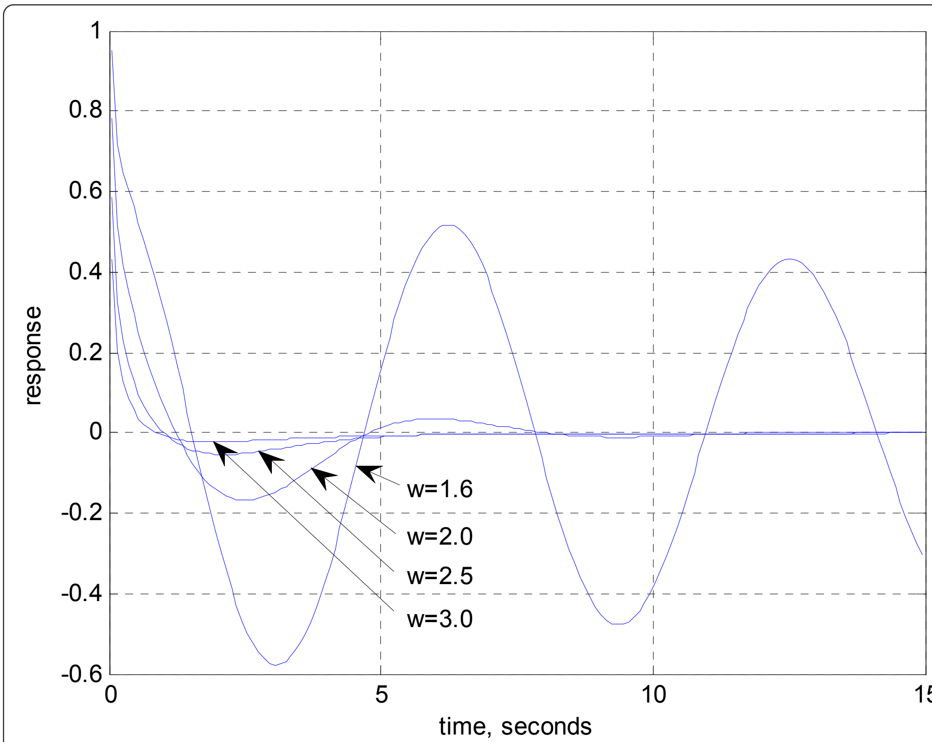


Figure 4 Time response associated with the example for various *w*.

line. Note that the origin of the v-plane corresponds to $s = 1$. Note that beyond $v = \pm j\pi$, the image in the s-plane moves inside the branch cut on the negative real s-axis.

Time-domain responses

Equation 24 can be rewritten using $v = \ln(s)$ as

$$\frac{X(v)}{U(v)} = \frac{b_0 v^2 + b_1 v + 2b_2}{a_0 v^2 + a_1 v + 2a_2}.$$

Letting $b_0 = 0$, $b_1 = 1$, $b_2 = 0$, $a_0 = 1$, $a_1 = 3$, $a_2 = 1$, results in

$$\frac{X(v)}{U(v)} = \frac{v}{v^2 + 3v + 2} = \frac{v}{v+1 \quad v+2}.$$

Let $u(t)$ be an impulse function, and write this equation as a partial fraction to give

$$X(v) = \frac{-1}{v+1} + \frac{2}{v+2}.$$

Now notice that the Laplace-domain logarithmic function has some interesting properties, particularly

$$\frac{1}{\ln(s)+c} = \frac{1}{\ln(s)+\ln(e^c)} = \frac{1}{\ln(s)+lna} = \frac{1}{\ln(as)} = \frac{1}{\ln(e^c s)}.$$

Using the scaling law $G(as) \Leftrightarrow \frac{1}{a} g\left(\frac{t}{a}\right)$, applied to Equation 14 gives the transform pair

$$\frac{1}{\ln(as)} \Leftrightarrow \frac{1}{a}\int_0^\infty \frac{\left(\frac{t}{a}\right)^{q-1}}{\Gamma(q)} dq, \tag{25}$$

or letting $a = e^c$ gives

$$\frac{1}{\ln(s)+c} = \frac{1}{\ln(e^c s)} \Leftrightarrow \frac{1}{e^c}\int_0^\infty \frac{\left(\frac{t}{e^c}\right)^{q-1}}{\Gamma(q)} dq. \tag{26}$$

Thus, the time response for this system becomes

$$X(s) = \frac{-1}{\ln(s)+1} + \frac{2}{\ln(s)+2} \quad \Leftrightarrow \quad x(t) = \frac{1}{e^2}\int_0^\infty \frac{t/e^{2^{q-1}}}{\Gamma(q)} dq - \frac{1}{e}\int_0^\infty \frac{t/e^{q-1}}{\Gamma(q)} dq.$$

For this system, the v-plane poles are at $v = -1,-2$, or $s = e^v = e^{-1}, e^{-2}$, which implies an unstable time response.

Now in Equation 24, letting $b_0 = 0$, $b_1 = 0$, $b_2 = 0.5$, $a_0 = 1$, $a_1 = 0$, $a_2 = 2$, results in

$$\frac{X(v)}{U(v)} = \frac{1}{v^2+4} = \frac{1}{(v+j2)(v--j2)}.$$

Let $u(t)$ be an impulse function, and write this equation as a partial fraction to give

$$X(v) = \frac{j0.25}{v + j2} + \frac{-j0.25}{v - j2} \ or \ X(s) = \frac{j0.25}{\ln(s) + j2} + \frac{-j0.25}{\ln(s) - j2}.$$

Using the transform pair from Equation 26 for each term yields

$$x(t) = +j0.25\frac{1}{e^{+j2}}\int_0^\infty \frac{\left(\frac{t}{e^{+j2}}\right)^{q-1}}{\Gamma(q)}dq - j0.25\frac{1}{e^{-j2}}\int_0^\infty \frac{\left(\frac{t}{e^{-j2}}\right)^{q-1}}{\Gamma(q)}dq,$$

a real function of time. For this system, the v-plane poles are at $v = +j2$, $-j2$, $s = e^v = e^j2$, e^{-j2}, which implies a stable and damped-oscillatory time response.

This time-response can be seen in Figure 5. The time-response can also be found for the more general transfer function

$$X(v) = \frac{1}{\ln^2(s) + w^2} = \frac{1}{v^2 + w^2} = \frac{1}{(v + jwv + jw)} \Leftrightarrow$$

$$X(s) = +\frac{j}{2w}\frac{1}{e^{+jw}}\int_0^\infty \frac{\left(\frac{t}{e^{+jw}}\right)^{q-1}}{\Gamma(q)}dq - \frac{j}{2w}\frac{1}{e^{-jw}}\int_0^\infty \frac{\left(\frac{t}{e^{-jw}}\right)^{q-1}}{\Gamma(q)}dq$$

Time-response plots for this system are also shown in Figure 5 with $w = 1.6$, 2.5, and 3.0 in addition to $w = 2.0$. Note that the initial value of these functions is infinity, and that the response becomes unstable for $w < \frac{\pi}{2}$.

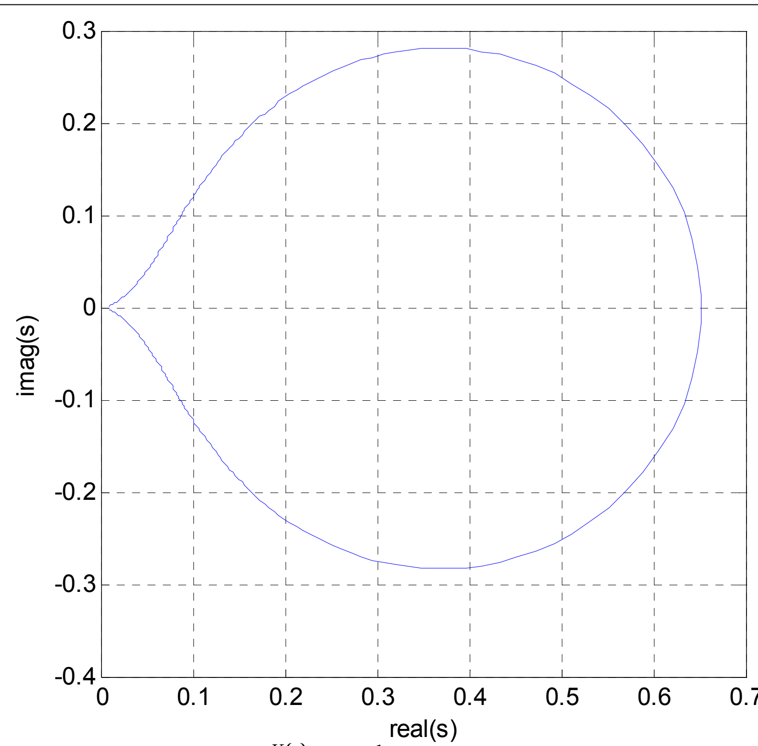


Figure 5 Nyquist plot of the system $\frac{X(s)}{U(s)} = \frac{1}{\ln^2(s) + 4}$.

Frequency responses of systems with Laplace-domain logarithmic operators

As shown earlier, the mapping from the s-plane to the v-plane is

$$v = \ln(s)\big|_{s=re^{j\theta}} = \ln(re^{j\theta}) \text{ } or \text{ } v = \ln(r) + j\theta + J2n\pi.$$

or

$$v = \ln(r) + j\theta + j2n\pi.$$

Again staying on the primary strip, with $n = 0$, the frequency response can be found to be

$$v = \ln(s)\big|_{s=j\omega=\omega e^{j\pi/2}} = \ln(\omega) + j\pi/2.$$

For the stable and damped-oscillatory example of the last section

$$\left.\frac{X(s)}{U(s)}\right|_{s=j\omega} = \left.\frac{1}{\ln^2(s)+4}\right|_{s=j\omega} = \frac{1}{\left[\ln(\omega) + j\frac{\pi}{2}\right]^2 + 4}$$
$$= \frac{1}{\ln^2(\omega) - \frac{\pi^2}{4} + j\pi\ln(\omega) + 4}$$

This frequency response is plotted in Figures 6 and 7, where a resonance can be seen as expected. It is interesting to notice that the low and high frequency phase shift is zero degrees for this system, yet there is a resonance, and a -100° phase shift through the resonant frequency.

An approximation to the logarithm

Equation 11 provides an interesting insight to an approximation for the logarithm, using discrete steps in q in Figure 2. Approximating the integral in Equation 11 using a sum of rectangles gives

$$\frac{1}{\ln(s)} = \left(\int_0^\infty s^{-q}\mathrm{d}q\right) = \lim_{Q\to 0} Qs^0 + Qs^{-Q} + Qs^{-2Q} + Qs^{-3Q} + \cdots = \lim_{Q\to 0}\sum_{n=0}^{\infty} Qs^{-nQ}, \quad \text{—}s\text{—} > 1. \tag{27}$$

with Q as the step size in order. It should be noticed that the sum on the right has the closed form representation

$$\sum_{n=0}^{\infty} Qs^{-nQ} = \frac{Q}{1-s^{-Q}}, |s| > 1, \tag{28}$$

This now provides a definition and an approximation, for the logarithmic operator at high frequencies

$$\frac{1}{\ln(s)} \equiv \lim_{Q\to 0}\frac{Q}{1-s^{-Q}} = \lim_{Q\to 0}\frac{Qs^Q}{s^Q-1}, \quad |s| > 1, \tag{29}$$

Likewise

$$\ln(s) \equiv \lim_{Q\to 0}\frac{1-s^{-Q}}{Q} = \lim_{Q\to 0}\frac{s^Q-1}{Qs^Q}, \quad |s| > 1, \tag{30}$$

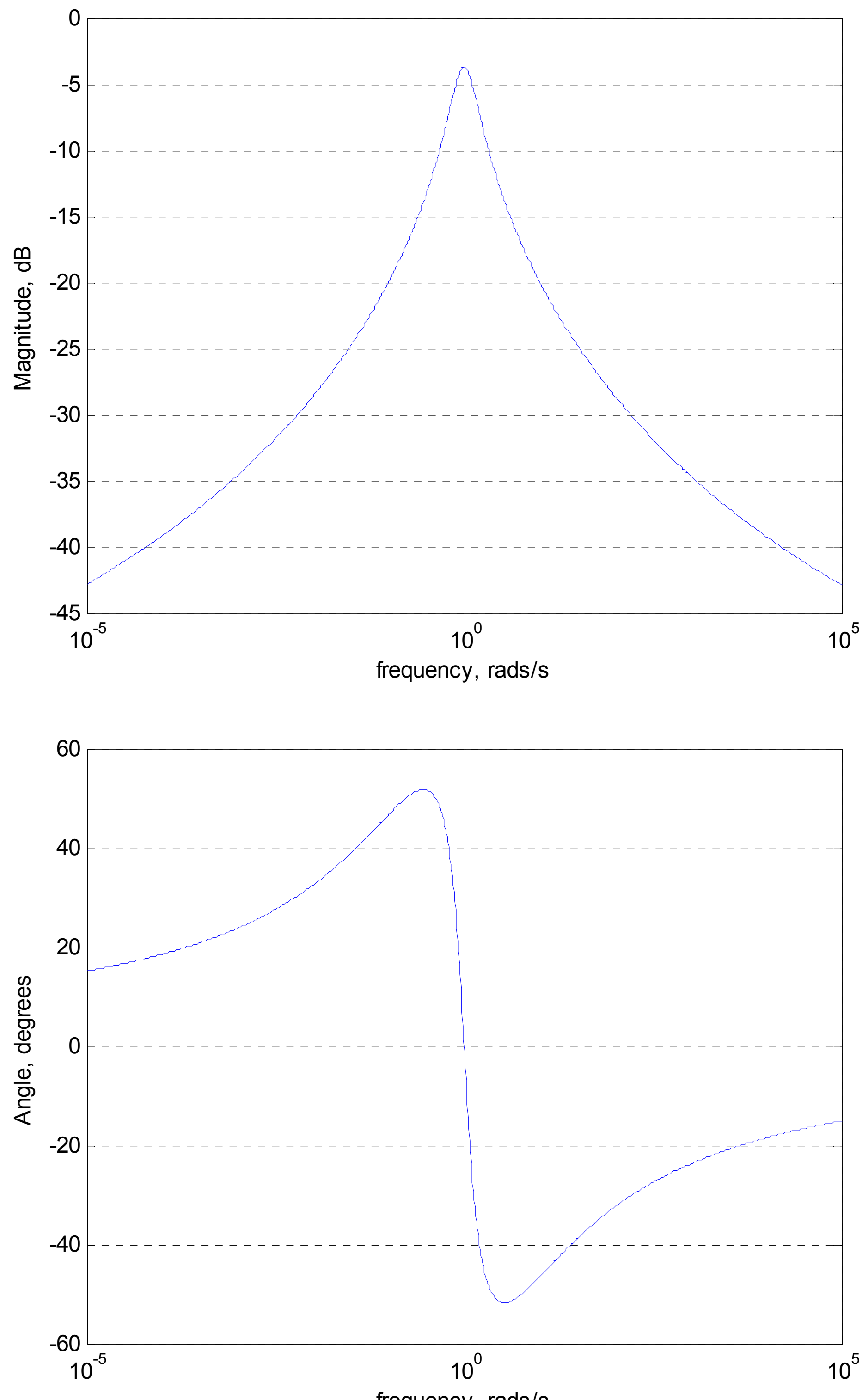
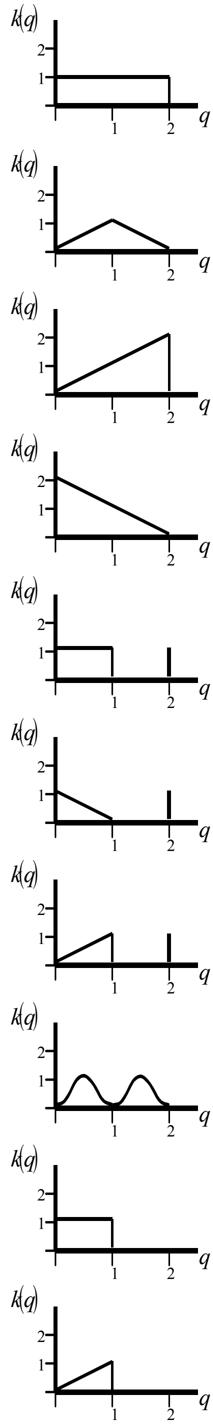


Figure 6 Bode plot of the system $\frac{X(s)}{U(s)} = \frac{1}{\ln^2(s) + 4}$.

Order-distribution



Laplace Transfer Function

$$P(s) = \frac{s^2 - 1}{\ln(s)} \quad (1)$$

$$P(s) = \frac{1 - 2s + s^2}{(\ln(s))^2} \quad (2)$$

$$P(s) = \frac{1 - (1 - 2\ln(s))s^2}{(\ln(s))^2} \quad (3)$$

$$P(s) = \frac{s^2 - 1 - 2\ln(s)}{(\ln(s))^2} \quad (4)$$

$$P(s) = \frac{s - 1 + s^2 \ln(s)}{\ln(s)} \quad (5)$$

$$P(s) = \frac{s^2(\ln(s))^2 - \ln(s) + s - 1}{(\ln(s))^2} \quad (6)$$

$$P(s) = \frac{1 - s + s\ln(s) + s^2(\ln(s))^2}{(\ln(s))^2} \quad (7)$$

$$P(s) = \frac{4\pi^2(s^2 - 1)}{2\ln(s)\left((\ln(s))^2 + 4\pi^2\right)} \quad (8)$$

$$P(s) = \frac{s - 1}{\ln(s)} \quad (9)$$

$$P(s) = \frac{1 - s(1 - \ln(s))}{(\ln(s))^2} \quad (10)$$

Figure 7 Order-distributions for orders between 0 and 2, and their transfer functions, using $\left(\int_0^\infty k(q)e^{q\ln(s)}\mathrm{d}q\right)X(s) = P(s)X(s) = F(s)$.

This definition can be found in Spanier and Oldham [37].

A similar discussion can be given for a low frequency approximation.

Approximating the integral in Equation 15 using a sum of rectangles gives

$$\frac{1}{\ln(s)} = -\left(\int_0^{\infty} s^q \mathrm{d}q\right) = -\lim_{Q\to 0} Qs^0 + Qs^Q + Qs^{2Q} + Qs^{3Q} + \cdots = -\lim_{Q\to 0} \sum_{n=0}^{\infty} Qs^{nQ}, \quad |\, s \,| < 1, \tag{31}$$

with Q as the step size in order. It should be noticed that the sum on the right has the closed form representation

$$-\sum_{n=0}^{\infty} Qs^{nQ} = -\frac{Q}{1-s^Q} = \frac{Q}{s^Q - 1}, \quad |\, s \,| < 1, \tag{32}$$

This now provides a definition and an approximation, for the logarithmic operator at low frequencies

$$\frac{1}{\ln(s)} \equiv \lim_{Q\to 0} \frac{Q}{s^Q - 1}, \quad |\, s \,| < 1, \tag{33}$$

These approximations were found to agree at high and low frequencies as predicted for $1/\ln(s)$.

Laplace-domain logarithmic operator representation of ODE's

Equation 11 can be rewritten to demonstrate that any ODE or FODE can result from using incomplete logarithmic operators.

An incomplete Laplace-domain logarithmic operator can be defined as

$$F(s) = \left(\int_{-\infty}^{a} s^q dq\right) X(s), \tag{34}$$

where now the right-most integral is preferred. Evaluating the integral gives

$$\begin{aligned} F(s) &= \left(\int_{-\infty}^{a} s^q dq\right) X(s) = \left(\int_{-\infty}^{a} \mathrm{e}^{q\ln(s)} dq\right) X(s) = \left.\frac{\mathrm{e}^{q\ln(s)}}{\ln(s)} X(s)\right|_{-\infty}^{a} \\ &= \left[\frac{\mathrm{e}^{a\ln(s)}}{\ln(s)} - \frac{\mathrm{e}^{-\infty\ln(s)}}{\ln(s)}\right] X(s) \\ &= \frac{s^a}{\ln(s)} X(s), \ |s| > 1 \end{aligned} \tag{35}$$

Notice that this equation has mixed terms, containing both an s and a $\ln(s)$, a result that is generally easy to obtain using order-distributions. A similar equation can be found for small s by reversing the limits of integration.

A two-sided incomplete Laplace-domain logarithmic operator can also be defined as

$$F(s) = \left(\int_{b}^{a} s^q dq\right) X(s) \tag{36}$$

This expression can be evaluated as

$$\begin{aligned} F(s) &= \left(\int_b^a s^q dq\right) X(s) = \left(\int_b^a e^{q\ln(s)} dq\right) X(s) = \left.\frac{e^{q\ln(s)}}{\ln(s)} X(s)\right|_b^a \\ &= \left[\frac{e^{a\ln(s)}}{\ln(s)} - \frac{e^{b\ln(s)}}{\ln(s)}\right] X(s) \\ &= \frac{s^a - s^b}{\ln(s)} X(s). \end{aligned} \tag{37}$$

Transform Pairs

$$\left\{\left(\int_0^\infty s^{-q} dq\right), |s| > 1\right\} = \frac{1}{\ln(s)} \Leftrightarrow \int_0^\infty \frac{t^{q-1}}{\Gamma(q)} dq$$

$$\left\{\left(-\int_0^\infty s^{q} dq\right), |s| < 1\right\} = \frac{1}{\ln(s)} \Leftrightarrow \int_0^\infty \frac{t^{-q-1}}{\Gamma(-q)} dq$$

$$\frac{1}{\ln(as)} \Leftrightarrow \frac{1}{a}\int_0^\infty \frac{\left(\frac{t}{a}\right)^{q-1}}{\Gamma(q)} dq$$

$$\frac{1}{\ln(s)+c} = \frac{1}{\ln(e^c s)} \Leftrightarrow \frac{1}{e^c}\int_0^\infty \frac{\left(\frac{t}{e^c}\right)^{q-1}}{\Gamma(q)} dq$$

$$\left\{\left(\int_0^\infty q s^{-q} dq\right), |s| > 1\right\} = \frac{1}{\ln^2(s)} \Leftrightarrow \int_0^\infty \frac{q\, t^{q-1}}{\Gamma(q)} dq$$

$$\left\{\left(-\int_0^\infty q s^{q} dq\right), |s| < 1\right\} = \frac{1}{\ln^2(s)} \Leftrightarrow \int_0^\infty \frac{q\, t^{q-1}}{\Gamma(q)} dq$$

$$\left\{\left(\int_0^\infty q^n s^{-q} dq\right), |s| > 1\right\} = \frac{\Gamma(n+1)}{\ln^{n+1}(s)} \Leftrightarrow \int_0^\infty \frac{q^n\, t^{q-1}}{\Gamma(q)} dq$$

$$\left\{\left(-\int_0^\infty q^n s^{q} dq\right), |s| < 1\right\} = \frac{\Gamma(n+1)}{\ln^{n+1}(s)} \Leftrightarrow \int_0^\infty \frac{q^n\, t^{q-1}}{\Gamma(q)} dq$$

Figure 8 Transform pairs for logarithmic and related systems.

It is interesting to observe that a rational or fractional-order transfer function can arise from incomplete logarithmic operators. Consider

$$\begin{aligned} &a_2\left(\int_{-\infty}^{q_2} s^q X(s)dq\right) + a_1\left(\int_{-\infty}^{q_1} s^q X(s)dq\right) + a_0\left(\int_{-\infty}^{q_0} s^q X(s)dq\right) \\ &= b_1\left(\int_{-\infty}^{r_1} s^q U(s)dq\right) + b_0\left(\int_{-\infty}^{r_0} s^{-q} U(s)dq\right), \quad |s| > 1. \end{aligned} \tag{38}$$

Using Equation 35, this equation reduces to

$$\begin{aligned} &a_2\frac{s^{q_2}}{\ln(s)}X(s) + a_1\frac{s^{q_1}}{\ln(s)}X(s) + a_0\frac{s^{q_0}}{\ln(s)}X(s) \\ &= b_1\frac{s^{r_1}}{\ln(s)}U(s) + b_0\frac{s^{r_0}}{\ln(s)}U(s), \quad |s| > 1, \end{aligned} \tag{39}$$

Laplace-domain Operation	Time-domain Operation
$\left\{F(s) = \left(\int_0^\infty s^{-q}dq\right)X(s), \lvert s\rvert > 1\right\} \Leftrightarrow$	$f(t) = \int_0^\infty\int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)}\,x(\tau)\,d\tau\,dq$
$F(s) = \left[\frac{1}{\ln(s)}\right]X(s) \Leftrightarrow$	$f(t) = \int_0^\infty\int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)}\,x(\tau)\,d\tau\,dq$
$\left\{F(s) = -\left(\int_0^\infty s^{q}dq\right)X(s), \lvert s\rvert < 1\right\} \Leftrightarrow$	$\int_0^\infty \frac{d^p}{dt^p}\int_0^t \frac{(t-\tau)^{u-1}}{\Gamma(u)}\,x(\tau)\,d\tau\,dq,$ $p = 1, 2, 3, \ldots,\ p > q > p-1$
$F(s) = \left[\frac{1}{\ln(s)}\right]X(s) \Leftrightarrow$	$\int_0^\infty \frac{d^p}{dt^p}\int_0^t \frac{(t-\tau)^{u-1}}{\Gamma(u)}\,x(\tau)\,d\tau\,dq,$ $p = 1, 2, 3, \ldots,\ p > q > p-1, \lvert s\rvert < 1$
$\left\{F(s) = \left(\int_0^\infty q s^{-q}dq\right)X(s), \lvert s\rvert > 1\right\} \Leftrightarrow$	$f(t) = \int_0^\infty\int_0^t \frac{q(t-\tau)^{q-1}}{\Gamma(q)}\,x(\tau)\,d\tau\,dq$
$F(s) = \frac{1}{\ln^2(s)}X(s) \Leftrightarrow$	$f(t) = \int_0^\infty\int_0^t \frac{q(t-\tau)^{q-1}}{\Gamma(q)}\,x(\tau)\,d\tau\,dq$
$\left\{F(s) = \left(\int_0^\infty q^n s^{-q}dq\right)X(s), \lvert s\rvert > 1\right\} \Leftrightarrow$	$f(t) = \int_0^\infty\int_0^t \frac{q^n(t-\tau)^{q-1}}{\Gamma(q)}\,x(\tau)\,d\tau\,dq$
$F(s) = \frac{\Gamma(n+1)}{\ln^{n+1}(s)}X(s) \Leftrightarrow$	$f(t) = \int_0^\infty\int_0^t \frac{q^n(t-\tau)^{q-1}}{\Gamma(q)}\,x(\tau)\,d\tau\,dq$

Figure 9 Time-domain operations for Laplace-domain logarithmic operations.

or equivalently

$$\frac{X(s)}{U(s)} = \frac{b_1 s^{r_1} + b_0 s^{r_0}}{a_2 s^{q_2} + a_1 s^{q_1} + a_0 s^{q_0}}. \tag{40}$$

Thus it can be seen that in some cases, systems of order-distributions can surprisingly be represented by standard fractional-order systems.

Discussion

This paper develops and exposes the strong relationships that exist between time-domain order-distributions and the Laplace-domain logarithmic operator. This paper has presented a theory of Laplace-domain logarithmic operators. The motivation is the appearance of logarithmic operators in a variety of fractional-order systems and order-distributions. A system theory for Laplace-domain logarithmic operator systems has been developed which includes time-domain representations, frequency domain representations, frequency response analysis, time response analysis, and stability theory. Approximation methods are also included. Mixed systems with s and $\ln(s)$ require further study. These considerations have provided several Laplace transform pairs that are expected to be useful for applications in science and engineering where variations of properties are involved. These pairs are shown in Figures 8 and 9. More research is required to understand the behavior of systems containing both the Laplace variable, s, and the Laplace-domain logarithmic operator, $\ln(s)$.

Acknowledgements
The authors gratefully acknowledge the continued support of NASA Glenn Research Center and the Electrical and Computer Engineering Department of the University of Akron. The authors also want to express their great appreciation for the valuable comments by the reviewers.

Author details
[1]Department of Electrical and Computer Engineering, University of Akron, Akron, OH 44325-3904, USA [2]NASA Glenn Research Center, Cleveland, OH 44135, USA

Authors' contributions
TH and CL worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 15 December 2010 Accepted: 30 November 2011 Published: 30 November 2011

References

1. Oldham, KB, Spanier, J: The Fractional Calculus. Academic Press, San Diego (1974)
2. Bush, V: Operational Circuit Analysis. Wiley, New York (1929)
3. Carslaw, HS, Jeager, JC: Operational Methods in Applied Mathematics. Oxford University Press, Oxford, 2 (1948)
4. Goldman, S: Transformation Calculus and Electrical Transients. Prentice-Hall, New York (1949)
5. Heaviside, O: Electromagnetic Theory. Chelsea Edition 1971, New YorkII (1922)
6. Holbrook, JG: Laplace Transforms for Electronic Engineers. Pergamon Press, New York, 2 (1966)
7. Mikusinski, J: Operational Calculus. Pergamon Press, New York (1959)
8. Scott, EJ: Transform Calculus with an Introduction to Complex Variables. Harper, New York (1955)
9. Starkey, BJ: Laplace Transforms for Electrical Engineers. Iliffe, London (1954)
10. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
11. Oustaloup, A: La derivation non entiere. Hermes, Paris (1995)
12. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999) ISBN 0-12-558840-2
13. Bagley, RL, Calico, RA: Fractional order state equations for the control of viscoelastic structures. J Guid Cont Dyn. **14**(2):304–311 (1991). doi:10.2514/3.20641
14. Koeller, RC: Application of fractional calculus to the theory of viscoelasticity. J Appl Mech. **51**, 299–307 (1984). doi:10.1115/1.3167616

15. Koeller, RC: Polynomial operators, Stieltjes convolution, and fractional calculus in hereditary mechanics. Acta Mech. **58**, 251–264 (1986). doi:10.1007/BF01176603
16. Skaar, SB, Michel, AN, Miller, RK: Stability of viscoelastic control systems. IEEE Trans Auto Cont. **33**(4):348–357 (1988). doi:10.1109/9.192189
17. Ichise, M, Nagayanagi, Y, Kojima, T: An analog simulation of non-integer order transfer functions for analysis of electrode processes. J Electroanal Chem Interfacial Electrochem. **33**, 253–265 (1971)
18. Sun, HH, Onaral, B, Tsao, Y: Application of positive reality principle to metal electrode linear polarization phenomena. IEEE Trans Biomed Eng. **31**(10):664–674 (1984)
19. Sun, HH, Abdelwahab, AA, Onaral, B: Linear approximation of transfer function with a pole of fractional order. IEEE Trans Auto Control. **29**(5):441–444 (1984). doi:10.1109/TAC.1984.1103551
20. Mandelbrot, B: Some noises with 1/f spectrum, a bridge between direct current and white noise. IEEE Trans Inf Theory. **13**(2):289–298 (1967)
21. Robotnov, YN: Elements of Hereditary Solid Mechanics (In English). MIR Publishers, Moscow (1980)
22. Hartley, TT, Lorenzo, CF, Qammar, HK: Chaos in a fractional order chua system. IEEE Trans Circ Syst I. **42**(8):485–490 (1995). doi:10.1109/81.404062
23. Oustaloup, A, Mathieu, B: La commande CRONE. Hermes, Paris (1999)
24. Agrawal, OP, Defterli, O, Baleanu, D: Fractional optimal control problems with several state and control variables. J Vibrat Control. **16**(13):1967–1976 (2010). doi:10.1177/1077546309353361
25. Le Mehaute, A, Tenreiro Machado, JA, Trigeassou, JC, Sabatier, J: Fractional differentiation and its applications. Proceedings of IFAC-FDA'04. (2005)
26. Bagley, RL, Torvik, PJ: On the existence of the order domain and the solution of distributed order differential equations. Parts I and II. Int J Appl Math **7**(8):865–882 (2000). 965-987, respectively
27. Caputo, M: Distributed order differential equations modeling dielectric induction and diffusion. Fract Calculus Appl Anal. **4**, 421–442 (2001)
28. Diethelm, K, Ford, NJ: Numerical solution methods for distributed order differential equations. Fract Calculus Appl Anal. **4**, 531–542 (2001)
29. Hartley, TT, Lorenzo, CF: Dynamics and control of initialized fractional-order systems. Nonlinear Dyn. **29**(1-4):201–233 (2002)
30. Hartley, TT, Lorenzo, CF: Fractional system identification: an approach using continuous order-distributions. NASA/TM-1999-209640. (1999)
31. Lorenzo, CF, Hartley, TT: Variable order and distributed order fractional operators. Nonlinear Dyn. **29**(1-4):57–98 (2002)
32. Bagley, RL: The thermorheologically complex material. Int J Eng Sci. **29**(7):797–806 (1991). doi:10.1016/0020-7225(91)90002-K
33. Hartley, TT, Lorenzo, CF: Fractional system identification based continuous order-distributions. Signal Process. **83**, 2287–2300 (2003). doi:10.1016/S0165-1684(03)00182-8
34. Adams, JL, Hartley, TT, Lorenzo, CF: Identification of complex order-distributions. J Vibrat Control. **14**(9-10):1375–1388 (2008). doi:10.1177/1077546307087443
35. Adams, JL, Hartley, TT, Lorenzo, CF: Complex order-distributions using conjugated-order differintegrals. In: Agrawal, J, Tenreiro, OP, Machado, JA (eds.) Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering Sabatier. pp. 347–360. Springer, Berlin (2007)
36. Erdelyi, A., *et al*: Higher Transcendental Functions. Dover**3** (2007)
37. Spanier, J, Oldham, K: An Atlas of Functions. Hemisphere Publishing, New York (1987)

doi:10.1186/1687-1847-2011-59
Cite this article as: Hartley and Lorenzo: **Order-distributions and the Laplace-domain logarithmic operator.** *Advances in Difference Equations* 2011 **2011**:59.

Submit your manuscript to a SpringerOpen journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com