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Qualitative behavior of a rational difference equation $y_{n+1} = \frac{y_n + y_{n-1}}{p + y_n y_{n-1}}$

Xiao Qian* and Shi Qi-hong

* Correspondence: xiaoxiao_xq168@163.com
Department of Basic Courses, Hebei Finance University, Baoding 071000, China

Abstract

This article is concerned with the following rational difference equation $y_{n+1} = (y_n + y_{n-1})/(p + y_n y_{n-1})$ with the initial conditions; y_{-1}, y_0 are arbitrary positive real numbers, and p is positive constant. Locally asymptotical stability and global attractivity of the equilibrium point of the equation are investigated, and non-negative solution with prime period two cannot be found. Moreover, simulation is shown to support the results.

Keywords: Global stability attractivity, solution with prime period two, numerical simulation

Introduction

Difference equations are applied in the field of biology, engineer, physics, and so on [1]. The study of properties of rational difference equations has been an area of intense interest in the recent years [6,7]. There has been a lot of work deal with the qualitative behavior of rational difference equation. For example, Çinar [2] has got the solutions of the following difference equation:

$$x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}$$

Karatas et al. [3] gave that the solution of the difference equation:

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

In this article, we consider the qualitative behavior of rational difference equation:

$$y_{n+1} = \frac{y_n + y_{n-1}}{p + y_n y_{n-1}}, \quad n = 0, 1, \ldots, \tag{1}$$

with initial conditions $y_{-1}, y_0 \in (0, +\infty)$, $p \in R^+$.

Preliminaries and notation

Let us introduce some basic definitions and some theorems that we need in what follows.

Lemma 1. Let I be some interval of real numbers and

$$f : I^2 \to I$$

be a continuously differentiable function. Then, for every set of initial conditions, x_{-k}, x_{-k+1}, ..., $x_0 \in I$ the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 (Equilibrium point). A point $\bar{x} \in I$ is called an equilibrium point of Equation 2, if

$$\bar{x} = f(\bar{x}, \bar{x})$$

Definition 2 (Stability). (1) The equilibrium point $\bar{x}$ of Equation 2 is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$, such that for any initial data x_{-k}, x_{-k+1}, ..., $x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta,$$

we have $|x_n - \bar{x}| < \varepsilon$, for all $n \geq -k$.

(2) The equilibrium point $\bar{x}$ of Equation 2 is locally asymptotically stable if $\bar{x}$ is locally stable solution of Equation 2, and there exists $\gamma > 0$, such that for all x_{-k}, x_{-k+1}, ..., $x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(3) The equilibrium point $\bar{x}$ of Equation 2 is a global attractor if for all x_{-k}, x_{-k+1}, ..., $x_0 \in I$, we have $\lim_{n\to\infty} x_n = \bar{x}$..

(4) The equilibrium point $\bar{x}$ of Equation 2 is globally asymptotically stable if $\bar{x}$ is locally stable and $\bar{x}$ is also a global attractor of Equation 2.

(5) The equilibrium point $\bar{x}$ of Equation 2 is unstable if $\bar{x}$ is not locally stable.

Definition 3 The linearized equation of (2) about the equilibrium $\bar{x}$ is the linear difference equation:

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-i}} y_{n-i} \tag{3}$$

Lemma 2 [4]. Assume that $p_1, p_2 \in R$ and $k \in \{1, 2, \ldots\}$, then

$$|p_1| + |p_2| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} - p_1 x_n - p_2 x_{n-1} = 0, \quad n = 0, 1, \ldots \tag{4}$$

Moreover, suppose $p_2 > 0$, then, $|p_1| + |p_2| < 1$ is also a necessary condition for the asymptotic stability of Equation 4.

Lemma 3 [5]. Let $g:[p, q]^2 \to [p, q]$ be a continuous function, where p and q are real numbers with $p < q$ and consider the following equation:

$$x_{n+1} = g(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{5}$$

Suppose that g satisfies the following conditions:

(1) $g(x, y)$ is non-decreasing in $x \in [p, q]$ for each fixed $y \in [p, q]$, and $g(x, y)$ is non-increasing in $y \in [p, q]$ for each fixed $x \in [p, q]$.

(2) If (m, M) is a solution of system

$M = g(M, m)$ and $m = g(m, M)$,

then $M = m$.

Then, there exists exactly one equilibrium $\bar{x}$ of Equation 5, and every solution of Equation 5 converges to $\bar{x}$.

The main results and their proofs

In this section, we investigate the local stability character of the equilibrium point of Equation 1. Equation 1 has an equilibrium point

$$\bar{x} = \begin{cases} 0, & p \geq 2 \\ 0, \sqrt{2-p} & p < 2 \end{cases}.$$

Let $f:(0, \infty)^2 \to (0, \infty)$ be a function defined by

$$f(u, v) = \frac{u+v}{p+uv} \tag{6}$$

Therefore, it follows that

$$f_u(u, v) = \frac{p - v^2}{(p+uv)^2}, \quad f_v(u, v) = \frac{p - u^2}{(p+uv)^2}.$$

Theorem 1. (1) Assume that $p > 2$, then the equilibrium point $\bar{x} = 0$ of Equation 1 is locally asymptotically stable.

(2) Assume that $0 < p < 2$, then the equilibrium point $\bar{x} = \sqrt{2-p}$ of Equation 1 is locally asymptotically stable, the equilibrium point $\bar{x} = 0$ is unstable.

Proof. (1) when $\bar{x} = 0$,

$$f_u(\bar{x}, \bar{x}) = \frac{1}{p}, \quad f_v(\bar{x}, \bar{x}) = \frac{1}{p}.$$

The linearized equation of (1) about $\bar{x} = 0$ is

$$y_{n+1} - \frac{1}{p} y_n - \frac{1}{p} y_{n-1} = 0. \tag{7}$$

It follows by Lemma 2, Equation 7 is asymptotically stable, if $p > 2$.

(2) when $\bar{x} = \sqrt{2-p}$,

$$f_u(\bar{x}, \bar{x}) = \frac{p-1}{2}, \quad f_v(\bar{x}, \bar{x}) = \frac{p-1}{2}.$$

The linearized equation of (1) about $\bar{x} = \sqrt{2-p}$ is

$$y_{n+1} - \frac{p-1}{2} y_n - \frac{p-1}{2} y_{n-1} = 0. \tag{8}$$

It follows by Lemma 2, Equation 8 is asymptotically stable, if

$$\left|\frac{p-1}{2}\right| + \left|\frac{p-1}{2}\right| < 1,$$

Therefore,

$$0 < p < 2.$$

Equilibrium point $\bar{x} = 0$ is unstable, it follows from Lemma 2. This completes the proof.

Theorem 2. Assume that $v_0^2 < p < u_0^2$, the equilibrium point $\bar{x} = 0$ and $\bar{x} = \sqrt{2 - p}$ of Equation 1 is a global attractor.

Proof. Let p, q be real numbers and assume that g:$[p, q]^2 \to [p, q]$ be a function defined by $g\,(u, v) = \dfrac{u + v}{p + uv}$, then we can easily see that the function $g(u, v)$ increasing in u and decreasing in v.

Suppose that (m, M) is a solution of system

$M = g(M, m)$ and $m = g(m, M)$.

Then, from Equation 1

$$M = \frac{M + m}{p + Mm}, \quad m = \frac{M + m}{p + Mm}.$$

Therefore,

$$pM + M^2 m = M + m, \tag{9}$$

$$pm + Mm^2 = M + m. \tag{10}$$

Subtracting Equation 10 from Equation 9 gives

$$\left(p + Mm\right)(M - m) = 0.$$

Since $p+Mm \neq 0$, it follows that

$$M = m.$$

Lemma 3 suggests that $\bar{x}$ is a global attractor of Equation 1 and then, the proof is completed.

Theorem 3. (1) has no non-negative solution with prime period two for all $p \in R^+$.

Proof. Assume for the sake of contradiction that there exist distinctive non-negative real numbers ϕ and ψ, such that

$$\ldots, \varphi, \psi, \varphi, \psi, \ldots$$

is a prime period-two solution of (1).

ϕ and ψ satisfy the system

$$\varphi\left(p + \varphi\psi\right) = \varphi + \psi, \tag{11}$$

$$\psi\left(p + \varphi\psi\right) = \psi + \varphi, \tag{12}$$

Subtracting Equation 11 from Equation 12 gives

$$(\varphi - \psi)\left(p + \varphi\psi\right) = 0,$$

so $\phi = \psi$, which contradicts the hypothesis $\phi \neq \psi$. The proof is complete.

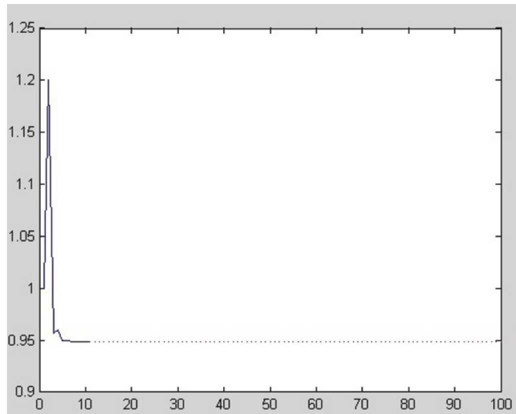

Figure 1 Plot of $x(n+1) = (x(n)+x(n-1))/(1.1+x(n)*x(n-1))$. This figure shows the solution of $y_{n+1} = \frac{y_n + y_{n-1}}{1.1 + y_n y_{n-1}}$, where $x_0 = 1$, $x_1 = 1.2$

Numerical simulation

In this section, we give some numerical simulations to support our theoretical analysis. For example, we consider the equation:

$$y_{n+1} = \frac{y_n + y_{n-1}}{1.1 + y_n y_{n-1}} \tag{13}$$

$$y_{n+1} = \frac{y_n + y_{n-1}}{1.5 + y_n y_{n-1}} \tag{14}$$

$$y_{n+1} = \frac{y_n + y_{n-1}}{5 + y_n y_{n-1}} \tag{15}$$

We can present the numerical solutions of Equations 13-15 which are shown, respectively in Figures 1, 2 and 3. Figure 1 shows the equilibrium point $\bar{x} = \sqrt{2 - 1.1}$ of Equation 13 is locally asymptotically stable with initial data $x_0 = 1$, $x_1 = 1.2$. Figure 2 shows the equilibrium point $\bar{x} = \sqrt{2 - 1.5}$ of Equation 14 is locally asymptotically

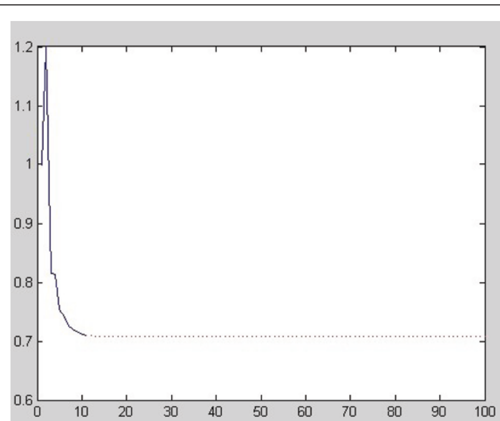

Figure 2 Plot of $x(n+1) = (x(n)+x(n-1))/(1.5+x(n)*x(n-1))$. This figure shows the solution of $y_{n+1} = \frac{y_n + y_{n-1}}{1.5 + y_n y_{n-1}}$, where $x_0 = 1$, $x_1 = 1.2$

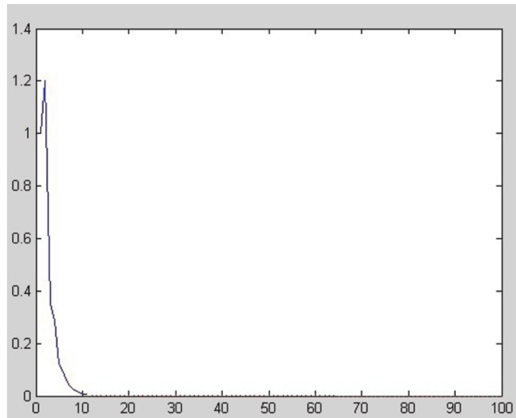

Figure 3 Plot of Plot of *x*(*n* + 1) = (*x*(*n*) + *x*(*n*-1))/(5 + *x*(*n*)*x(*n* - 1)). This figure shows the solution of $y_{n+1} = \dfrac{y_n + y_{n-1}}{5 + y_n y_{n-1}}$, where $x_0 = 1$, $x_1 = 1.2$

stable with initial data $x_0 = 1$, $x_1 = 1.2$. Figure 3 shows the equilibrium point $\bar{x} = 0$ of Equation 15 is locally asymptotically stable with initial data $x_0 = 1$, $x_1 = 1.2$.

Authors' contributions

Xiao Qian carried out the theoretical proof and drafted the manuscript. Shi Qi-hong participated in the design and coordination. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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