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Non-Archimedean Hyers-Ulam-Rassias stability of m-variable functional equation

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Abstract
The main goal of this paper is to study the Hyers-Ulam-Rassias stability of the following Euler-Lagrange type additive functional equation:

$$\sum_{j=1}^{m} f\left(-r_j x_j + \sum_{1\le i\le m, i\neq j} r_i x_i\right) + 2\sum_{i=1}^{m} r_i f(x_i) = mf\left(\sum_{i=1}^{m} r_i x_i\right),$$

where $r_1,\ldots,r_m \in \mathbb{R}$, $\sum_{i=k}^{m} r_k \neq 0$, and $r_i, r_j \neq 0$ for some $1 \le i < j \le m$, in non-Archimedean Banach spaces.
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1 Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r||s|$ and the triangle inequality is replaced by $|r+s| \le \max\{|r|,|s|\}$.

The field $\mathbb{K}$ is called a *valued field* if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are the examples of valuations.

Let us consider the valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by $|r+s| \le \max\{|r|,|s|\}$ for all $r,s \in \mathbb{K}$, then the function $|\cdot|$ is called a *non-Archimedean valuation* and the field is called a *non-Archimedean field*. Clearly, $|1| = |-1| = 1$ and $|n| \le 1$ for all integers $n \ge 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 1.1 Let X be a vector space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if the following conditions hold:

(a) $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$;
(b) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(c) the strong triangle inequality holds:

$$\|x + y\| \le \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2 Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

(a) A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is a *Cauchy sequence* iff the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero;
(b) The sequence $\{x_n\}$ is said to be *convergent* if, for any $\varepsilon > 0$, there is a positive integer N and $x \in X$ such that $\|x_n - x\| \le \varepsilon$, for all $n \ge N$. Then the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$;
(c) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

Example 1.1 Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the *p-adic number field*. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\ge n_x}^{\infty} a_k p^k$ where $|a_k| \le p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k\ge n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field.

Theorem 1.1 *Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that:*

(a) $d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \ge n_0$;
(b) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(c) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;
(d) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In this paper, we prove the generalized Hyers-Ulam stability of the following functional equation:

$$\sum_{j=1}^{m} f\left(-r_j x_j + \sum_{1\le i\le m, i\ne j} r_i x_i\right) + 2\sum_{i=1}^{m} r_i f(x_i) = mf\left(\sum_{i=1}^{m} r_i x_i\right), \tag{1.1}$$

where $r_1, \ldots, r_m \in \mathbb{R}$, $\sum_{k=1}^{m} r_k \ne 0$, and $r_i, r_j \ne 0$ for some $1 \le i < j \le m$, in non-Archimedean Banach spaces. A classical question in the theory of functional equations is the following: 'When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?'.

If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [34] in 1940.

In the next year D. H. Hyres [17], gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Th. M. Rassias [24] proved a generalization of Hyres' theorem for additive mappings.

The result of Th. M. Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations. In 1994, a generalization of Rassias' theorem was obtained by Găvruta [15] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x,y)$.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1–33]).

2 Non-Archimedean stability of the functional equation (1.1): a fixed point approach

In this section, using a fixed point alternative approach, we prove the generalized Hyers-Ulam stability of the functional equation (1.1) in non-Archimedean normed spaces. Throughout this section, let X be a non-Archimedean normed space and Y be a non-Archimedean Banach space. Also $|2| \neq 1$.

Lemma 2.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $r_1, \dots, r_n$ be real numbers with $\sum_{k=1}^n r_k \neq 0$ and $r_i, r_j \neq 0$ for some $1 \le i < j \le n$. Assume that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (1.1) for all $x_1, \dots, x_n \in \mathcal{X}$. Then the mapping f is Cauchy additive. Moreover, $f(r_k x) = r_k f(x)$ for all $x \in \mathcal{X}$ and all $1 \le k \le n$.*

Proof Since $\sum_{k=1}^n r_k \neq 0$, putting $x_1 = \cdots = x_n = 0$ in (1.1), we get $f(0) = 0$. Without loss of generality, we may assume that $r_1, r_2 \neq 0$. Letting $x_3 = \cdots = x_n = 0$ in (1.1), we get

$$f(-r_1x_1 + r_2x_2) + f(r_1x_1 - r_2x_2) + 2r_1 f(x_1) + 2r_2 f(x_2) = 2f(r_1x_1 + r_2x_2) \tag{2.1}$$

for all $x_1, x_2 \in \mathcal{X}$. Letting $x_2 = 0$ in (2.1), we get

$$2r_1 f(x_1) = f(r_1x_1) - f(-r_1x_1) \tag{2.2}$$

for all $x_1 \in \mathcal{X}$. Similarly, by putting $x_1 = 0$ in (2.1), we get

$$2r_2 f(x_2) = f(r_2x_2) - f(-r_2x_2) \tag{2.3}$$

for all $x_1 \in \mathcal{X}$. It follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned} &f(-r_1x_1 + r_2x_2) + f(r_1x_1 - r_2x_2) + f(r_1x_1) + f(r_2x_2) \\ &\quad - f(-r_1x_1) - f(-r_2x_2) = 2f(r_1x_1 + r_2x_2) \end{aligned} \tag{2.4}$$

for all $x_1, x_2 \in \mathcal{X}$. Replacing x_1 and x_2 by $\frac{x}{r_1}$ and $\frac{y}{r_2}$ in (2.4), we get

$$f(-x + y) + f(x - y) + f(x) + f(y) - f(-x) - f(-y) = 2f(x + y) \tag{2.5}$$

for all $x, y \in \mathcal{X}$. Letting $y = -x$ in (2.5), we get that $f(-2x) + f(2x) = 0$ for all $x \in \mathcal{X}$. So the mapping L is odd. Therefore, it follows from (2.5) that the mapping f is additive. Moreover, let $x \in \mathcal{X}$ and $1 \leq k \leq n$. Setting $x_k = x$ and $x_l = 0$ for all $1 \leq l \leq n$, $l \neq k$, in (1.1) and using the oddness of f, we get that $f(r_k x) = r_k f(x)$. □

Using the same method as in the proof of Lemma 2.1, we have an alternative result of Lemma 2.1 when $\sum_{k=1}^{n} r_k = 0$.

Lemma 2.2 *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $r_1, \dots, r_n$ be real numbers with $r_i, r_j \neq 0$ for some $1 \leq i < j \leq n$. Assume that a mapping $f : \mathcal{X} \to \mathcal{Y}$ with $f(0) = 0$ satisfying the functional equation (1.1) for all $x_1, \dots, x_n \in \mathcal{X}$. Then the mapping f is Cauchy additive. Moreover, $f(r_k x) = r_k f(x)$ for all $x \in \mathcal{X}$ and all $1 \leq k \leq n$.*

Remark 2.1 Throughout this paper, $r_1, \dots, r_m$ will be real numbers such that $r_i, r_j \neq 0$ for fixed $1 \leq i < j \leq m$ and $\varphi_{i,j}(x,y) := \varphi(0, \dots, 0, \underbrace{x}_{i\text{th}}, 0, \dots, 0, \underbrace{y}_{j\text{th}}, 0, \dots, 0)$ for all $x, y \in X$ and all $1 \leq i < j \leq m$.

Theorem 2.1 *Let $\varphi : X^m \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x_1}{2}, \dots, \frac{x_m}{2}\right) \leq \frac{L\varphi(x_1, \dots, x_m)}{|2|} \tag{2.6}$$

for all $x_1, \dots, x_m \in X$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying the following inequality:

$$\begin{aligned} &\left\| \sum_{j=1}^{m} f\left(-r_j x_j + \sum_{1 \leq i \leq m, i \neq j} r_i x_i\right) + 2\sum_{i=1}^{m} r_i f(x_i) - mf\left(\sum_{i=1}^{m} r_i x_i\right) \right\| \\ &\quad \leq \varphi(x_1, \dots, x_m) \end{aligned} \tag{2.7}$$

for all $x_1, \dots, x_m \in X$. Then there is a unique Euler-Lagrange type additive mapping $EL : X \to Y$ such that

$$\begin{aligned} \|f(x) - EL(x)\| \leq \frac{L}{|2| - |2|L} \max\Bigg\{ &\max\left\{ \varphi_{i,j}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2r_j}\right) \right\}, \\ &\frac{1}{|2|} \max\left\{ \varphi_{i,j}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \varphi_{i,j}\left(\frac{x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{r_j}\right) \right\} \Bigg\} \end{aligned} \tag{2.8}$$

for all $x \in X$.

Proof For each $1 \leq k \leq m$ with $k \neq i, j$, let $x_k = 0$ in (2.7). Then we get the following inequality:

$$\begin{aligned} &\|f(-r_i x_i + r_j x_j) + f(r_i x_i - r_j x_j) - 2f(r_i x_i + r_j x_j) + 2r_i f(x_i) + 2r_j f(x_j)\| \\ &\quad \leq \varphi_{i,j}(x_i, x_j) \end{aligned} \tag{2.9}$$

for all $x_i, x_j \in X$. Letting $x_i = 0$ in (2.9), we get

$$\|f(-r_j x_j) - f(r_j x_j) + 2r_j f(x_j)\| \leq \varphi_{i,j}(0, x_j) \tag{2.10}$$

for all $x_j \in X$. Similarly, letting $x_j = 0$ in (2.9), we get

$$\left\| f(-r_i x_i) - f(r_i x_i) + 2r_i f(x_i) \right\| \le \varphi_{i,j}(x_i, 0) \tag{2.11}$$

for all $x_i \in X$. It follows from (2.9), (2.10) and (2.11) that for all $x_i, x_j \in X$

$$\begin{aligned} &\| f(-r_i x_i + r_j x_j) + f(r_i x_i - r_j x_j) - 2f(r_i x_i + r_j x_j) + 2r_i f(x_i) + 2r_j f(x_j) \\ &\quad - \left(f(-r_i x_i) - f(r_i x_i) + 2r_i f(x_i) \right) - \left(f(-r_j x_j) - f(r_j x_j) + 2r_j f(x_j) \right) \| \\ &\le \max\left\{ \varphi_{i,j}(x_i, x_j), \varphi_{i,j}(0, x_j), \varphi_{i,j}(x_i, 0) \right\}. \end{aligned} \tag{2.12}$$

Replacing x_i and x_j by $\frac{x}{r_i}$ and $\frac{y}{r_j}$ in (2.12), we get that

$$\begin{aligned} &\left\| f(-x+y) + f(x-y) - 2f(x+y) + f(x) + f(y) - f(-x) - f(-y) \right\| \\ &\le \max\left\{ \varphi_{i,j}\left(\frac{x}{r_i}, \frac{y}{r_j} \right), \varphi_{i,j}\left(\frac{x}{r_i}, 0 \right), \varphi_{i,j}\left(0, \frac{y}{r_j} \right) \right\} \end{aligned} \tag{2.13}$$

for all $x, y \in X$. Putting $y = x$ in (2.13), we get

$$\left\| f(x) - f(-x) - f(2x) \right\| \le \frac{1}{|2|} \max\left\{ \varphi_{i,j}\left(\frac{x}{r_i}, \frac{x}{r_j} \right), \varphi_{i,j}\left(\frac{x}{r_i}, 0 \right), \varphi_{i,j}\left(0, \frac{x}{r_j} \right) \right\} \tag{2.14}$$

for all $x \in X$. Replacing x and y by $\frac{x}{2}$ and $-\frac{x}{2}$ in (2.13) respectively, we get

$$\left\| f(x) + f(-x) \right\| \le \max\left\{ \varphi_{i,j}\left(\frac{x}{2r_i}, -\frac{x}{2r_j} \right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0 \right), \varphi_{i,j}\left(0, -\frac{x}{2r_j} \right) \right\} \tag{2.15}$$

for all $x \in X$. It follows from (2.14) and (2.15) that

$$\begin{aligned} &\left\| f(2x) - 2f(x) \right\| \\ &= \left\| f(x) + f(-x) + f(x) - f(-x) - f(2x) \right\| \\ &\le \max\left\{ \max\left\{ \varphi_{i,j}\left(\frac{x}{2r_i}, -\frac{x}{2r_j} \right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0 \right), \varphi_{i,j}\left(0, -\frac{x}{2r_j} \right) \right\}, \right. \\ &\left. \frac{1}{|2|} \max\left\{ \varphi_{i,j}\left(\frac{x}{r_i}, \frac{x}{r_j} \right), \varphi_{i,j}\left(\frac{x}{r_i}, 0 \right), \varphi_{i,j}\left(0, \frac{x}{r_j} \right) \right\} \right\} \end{aligned} \tag{2.16}$$

for all $x \in X$. Replacing x by $\frac{x}{2}$ in (2.16), we obtain

$$\begin{aligned} &\left\| f(x) - 2f\left(\frac{x}{2} \right) \right\| \\ &\le \max\left\{ \max\left\{ \varphi_{i,j}\left(\frac{x}{4r_i}, -\frac{x}{4r_j} \right), \varphi_{i,j}\left(\frac{x}{4r_i}, 0 \right), \varphi_{i,j}\left(0, -\frac{x}{4r_j} \right) \right\}, \right. \\ &\left. \frac{1}{|2|} \max\left\{ \varphi_{i,j}\left(\frac{x}{2r_i}, \frac{x}{2r_j} \right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0 \right), \varphi_{i,j}\left(0, \frac{x}{2r_j} \right) \right\} \right\}. \end{aligned} \tag{2.17}$$

Consider the set $S := \{g : X \to Y; g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f,g) = \inf_{\mu\in\mathbb{R}^+}\Bigg\{\|g(x)-h(x)\| \le \mu \max\Bigg\{\max\Bigg\{\varphi_{i,j}\Big(\frac{x}{2r_i},-\frac{x}{2r_j}\Big), \varphi_{i,j}\Big(\frac{x}{2r_i},0\Big),\varphi_{i,j}\Big(0,-\frac{x}{2r_j}\Big)\Bigg\}, \frac{1}{|2|}\max\Bigg\{\varphi_{i,j}\Big(\frac{x}{r_i},\frac{x}{r_j}\Big),\varphi_{i,j}\Big(\frac{x}{r_i},0\Big),\varphi_{i,j}\Big(0,\frac{x}{r_j}\Big)\Bigg\}\Bigg\}, \forall x\in X\Bigg\},$$

where $\inf\emptyset = +\infty$. It is easy to show that (S,d) is complete (see [18], Lemma 2.1). Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\Big(\frac{x}{2}\Big)$$

for all $x \in X$. Let $g,h \in S$ be such that $d(g,h) = \epsilon$. Then

$$\|g(x)-h(x)\| \le \epsilon \max\Bigg\{\max\Bigg\{\varphi_{i,j}\Big(\frac{x}{2r_i},-\frac{x}{2r_j}\Big),\varphi_{i,j}\Big(\frac{x}{2r_i},0\Big),\varphi_{i,j}\Big(0,-\frac{x}{2r_j}\Big)\Bigg\}, \frac{1}{|2|}\max\Bigg\{\varphi_{i,j}\Big(\frac{x}{r_i},\frac{x}{r_j}\Big),\varphi_{i,j}\Big(\frac{x}{r_i},0\Big),\varphi_{i,j}\Big(0,\frac{x}{r_j}\Big)\Bigg\}\Bigg\}$$

for all $x \in X$, and so

$$\begin{aligned}\|Jg(x)-Jh(x)\| &= \Big\|2g\Big(\frac{x}{2}\Big)-2h\Big(\frac{x}{2}\Big)\Big\| \\ &\le |2|\epsilon \max\Bigg\{\max\Bigg\{\varphi_{i,j}\Big(\frac{x}{4r_i},-\frac{x}{4r_j}\Big),\varphi_{i,j}\Big(\frac{x}{4r_i},0\Big),\varphi_{i,j}\Big(0,-\frac{x}{4r_j}\Big)\Bigg\}, \\ &\quad \frac{1}{|2|}\max\Bigg\{\varphi_{i,j}\Big(\frac{x}{2r_i},\frac{x}{2r_j}\Big),\varphi_{i,j}\Big(\frac{x}{2r_i},0\Big),\varphi_{i,j}\Big(0,\frac{x}{2r_j}\Big)\Bigg\}\Bigg\} \\ &\le |2|\frac{L\epsilon}{|2|}\max\Bigg\{\max\Bigg\{\varphi_{i,j}\Big(\frac{x}{2r_i},-\frac{x}{2r_j}\Big),\varphi_{i,j}\Big(\frac{x}{2r_i},0\Big),\varphi_{i,j}\Big(0,-\frac{x}{2r_j}\Big)\Bigg\}, \\ &\quad \frac{1}{|2|}\max\Bigg\{\varphi_{i,j}\Big(\frac{x}{r_i},\frac{x}{r_j}\Big),\varphi_{i,j}\Big(\frac{x}{r_i},0\Big),\varphi_{i,j}\Big(0,\frac{x}{r_j}\Big)\Bigg\}\Bigg\}\end{aligned}$$

for all $x \in X$. Thus $d(g,h) = \epsilon$ implies that $d(Jg,Jh) \le L\epsilon$. This means that

$$d(Jg,Jh) \le Ld(g,h)$$

for all $g,h \in S$. It follows from (2.17) that

$$d(f,Jf) \le \frac{L}{|2|}.$$

By Theorem 1.1, there exists a mapping $EL : X \to Y$ satisfying the following:

(1) EL is a fixed point of J, that is,

$$EL\Big(\frac{x}{2}\Big) = \frac{1}{2}EL(x) \tag{2.18}$$

for all $x \in X$. The mapping EL is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g,h) < \infty\}.$$

This implies that EL is a unique mapping satisfying (2.18) such that there exists $\mu \in (0,\infty)$ satisfying

$$\begin{aligned}\|f(x) - EL(x)\| \le \mu \max\Big\{&\max\Big\{\varphi_{i,j}\Big(\frac{x}{2r_i}, -\frac{x}{2r_j}\Big), \varphi_{i,j}\Big(\frac{x}{2r_i}, 0\Big), \varphi_{i,j}\Big(0, -\frac{x}{2r_j}\Big)\Big\},\\ &\frac{1}{|2|}\max\Big\{\varphi_{i,j}\Big(\frac{x}{r_i}, \frac{x}{r_j}\Big), \varphi_{i,j}\Big(\frac{x}{r_i}, 0\Big), \varphi_{i,j}\Big(0, \frac{x}{r_j}\Big)\Big\}\Big\}\end{aligned} \tag{2.19}$$

for all $x \in X$.

(2) $d(J^n f, EL) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big) = EL(x)$$

for all $x \in X$.

(3) $d(f, EL) \le \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, EL) \le \frac{L}{|2| - |2|L}.$$

This implies that the inequality (2.8) holds.

By (2.6) and (2.7), we obtain

$$\begin{aligned}&\Big\|\sum_{j=1}^m EL\Big(-r_j x_j + \sum_{1\le i\le m, i\ne j} r_i x_i\Big) + 2\sum_{i=1}^m r_i EL(x_i) - mEL\Big(\sum_{i=1}^m r_i x_i\Big)\Big\|\\ &\quad = \lim_{n\to\infty} |2|^n \Big\|\sum_{j=1}^m f\Big(\frac{-r_j x_j}{2^n} + \sum_{1\le i\le m, i\ne j} \frac{r_i x_i}{2^n}\Big) + 2\sum_{i=1}^m r_i f\Big(\frac{x_i}{2^n}\Big) - mf\Big(\sum_{i=1}^m \frac{r_i x_i}{2^n}\Big)\Big\|\\ &\quad \le \lim_{n\to\infty} |2|^n \varphi\Big(\frac{x_1}{2^n}, \ldots, \frac{x_m}{2^n}\Big)\\ &\quad \le \lim_{n\to\infty} |2|^n \cdot \frac{L^n}{|2|^n}\varphi(x_1, \ldots, x_m)\end{aligned}$$

for all $x_1, \ldots, x_m \in X$ and $n \in \mathbb{N}$. So EL satisfies (1.1). Thus, the mapping $EL : X \to Y$ is Euler-Lagrange type additive, as desired. □

Corollary 2.1 *Let $\theta \ge 0$ and r be a real number with $0 < r < 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying the inequality*

$$\begin{aligned}&\Big\|\sum_{j=1}^m f\Big(-r_j x_j + \sum_{1\le i\le m, i\ne j} r_i x_i\Big) + 2\sum_{i=1}^m r_i f(x_i) - mf\Big(\sum_{i=1}^m r_i x_i\Big)\Big\|\\ &\quad \le \theta\Big(\sum_{i=1}^m \|x_i\|^r\Big)\end{aligned} \tag{2.20}$$

for all $x_1, \ldots, x \in X$. *Then, the limit* $EL(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ *exists for all* $x \in X$ *and* $EL : X \to Y$ *is a unique Euler-Lagrange additive mapping such that*

$$\begin{aligned}\|f(x) - EL(x)\| &\le \frac{|2|}{|2|^{r+1} - |2|^2} \max\left\{\max\left\{\frac{|2|^r\theta\|x\|^r(|r_i|^r + |r_j|^r)}{|4|^r|r_ir_j|^r}, \frac{\theta\|x\|^r}{|2|^r|r_i|^r}, \frac{\theta\|x\|^r}{|2|^r|r_j|^r}\right\},\right.\\ &\quad \left.\frac{1}{|2|}\max\left\{\frac{\theta\|x\|^r(|r_i|^r + |r_j|^r)}{|r_ir_j|^r}, \frac{\theta\|x\|^r}{|r_i|^r}, \frac{\theta\|x\|^r}{|r_j|^r}\right\}\right\}\\ &\le \frac{\theta\|x\|^r(|r_i|^r + |r_j|^r)}{|r_ir_j|^r(|2|^{r+1} - |2|^2)}\end{aligned}$$

for all $x \in X$.

Proof The proof follows from Theorem 2.1 by taking $\varphi(x_1, \ldots, x_m) = \theta(\sum_{i=1}^m \|x_i\|^r)$ for all $x_1, \ldots, x_m \in X$. In fact, if we choose $L = |2|^{1-r}$, then we get the desired result. □

Theorem 2.2 *Let* $\varphi : X^m \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x_1, \ldots, x_m) \le |2|L\varphi\left(\frac{x_1}{2}, \ldots, \frac{x_m}{2}\right) \tag{2.21}$$

for all $x_1, \ldots, x_m \in X$. *Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *satisfying the inequality (2.7). Then, there is a unique Euler-Lagrange additive mapping* $EL : X \to Y$ *such that*

$$\begin{aligned}\|f(x) - EL(x)\| &\le \frac{1}{|2| - |2|L} \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2r_j}\right)\right\},\right.\\ &\quad \left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \varphi_{i,j}\left(\frac{x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{r_j}\right)\right\}\right\}.\end{aligned} \tag{2.22}$$

Proof By (2.16), we have

$$\begin{aligned}\left\|\frac{f(2x)}{2} - f(x)\right\| &\le \frac{1}{|2|} \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2r_j}\right)\right\},\right.\\ &\quad \left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \varphi_{i,j}\left(\frac{x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{r_j}\right)\right\}\right\}\end{aligned} \tag{2.23}$$

for all $x \in X$. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2}h(2x)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then

$$\begin{aligned}\|g(x) - h(x)\| &\le \epsilon \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2r_j}\right)\right\},\right.\\ &\quad \left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \varphi_{i,j}\left(\frac{x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{r_j}\right)\right\}\right\}\end{aligned}$$

for all $x \in X$, and so

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|\frac{g(2x)}{2} - \frac{h(2x)}{2}\right\| \\ &\le \frac{1}{|2|}\epsilon \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{r_i}, -\frac{x}{r_j}\right), \varphi_{i,j}\left(\frac{x}{r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{r_j}\right)\right\},\right. \\ &\quad \left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{2x}{r_i}, \frac{2x}{r_j}\right), \varphi_{i,j}\left(\frac{2x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{2x}{r_j}\right)\right\}\right\} \\ &\le |2|L\frac{\epsilon}{|2|}\max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2r_j}\right)\right\},\right. \\ &\quad \left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \varphi_{i,j}\left(\frac{x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{r_j}\right)\right\}\right\}\end{aligned}$$

for all $x \in X$. Thus $d(g,h) = \epsilon$ implies that $d(Jg, Jh) \le L\epsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$. It follows from (2.23) that

$$d(f, Jf) \le \frac{1}{|2|}.$$

By Theorem 1.1, there exists a mapping $EL : X \to Y$ satisfying the following:

(1) EL is a fixed point of J, that is,

$$EL(2x) = 2EL(x) \tag{2.24}$$

for all $x \in X$. The mapping EL is a unique fixed point of J in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}.$$

This implies that EL is a unique mapping satisfying (2.24) such that there exists $\mu \in (0, \infty)$ satisfying

$$\begin{aligned}\|g(x) - h(x)\| \le \mu \max&\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2r_j}\right)\right\},\right. \\ &\left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \varphi_{i,j}\left(\frac{x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{r_j}\right)\right\}\right\}\end{aligned}$$

for all $x \in X$.

(2) $d(J^n f, EL) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = EL(x)$$

for all $x \in X$.

(3) $d(f, EL) \leq \frac{d(f, Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality

$$d(f, EL) \leq \frac{1}{|2| - |2|L}.$$

This implies that the inequality (2.22) holds. The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.2 *Let $\theta \geq 0$ and r be a real number with $r > 1$. Let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (2.20). Then, the limit $EL(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in X$ and $EL : X \to Y$ is a unique cubic mapping such that*

$$\begin{aligned}\|f(x) - EL(x)\| \leq{}& \frac{1}{|2| - |2|^r} \max\left\{\max\left\{\frac{|2|^r\theta\|x\|^r(|r_i|^r + |r_j|^r)}{|4|^r|r_i r_j|^r}, \frac{\theta\|x\|^r}{|2|^r|r_i|^r}, \frac{\theta\|x\|^r}{|2|^r|r_j|^r}\right\},\right.\\ &\left.\frac{1}{|2|}\max\left\{\frac{\theta\|x\|^r(|r_i|^r + |r_j|^r)}{|r_i r_j|^r}, \frac{\theta\|x\|^r}{|r_i|^r}, \frac{\theta\|x\|^r}{|r_j|^r}\right\}\right\} \leq \frac{\theta\|x\|^r(|r_i|^r + |r_j|^r)}{|r_i r_j|^r(|2|^{r+1} - |2|^{r+2})}\end{aligned}$$

for all $x \in X$.

Proof The proof follows from Theorem 2.2 by taking $\varphi(x_1, \ldots, x_m) = \theta(\sum_{i=1}^m \|x_i\|^r)$ for all $x_1, \ldots, x_m \in X$. In fact, if we choose $L = |2|^{r-1}$, then we get the desired result. □

3 Non-Archimedean stability of the functional equation (1.1): a direct method

In this section, using a direct method, we prove the generalized Hyers-Ulam stability of the cubic functional equation (1.1) in non-Archimedean normed spaces. Throughout this section, we assume that G is an additive semigroup and X is a non-Archimedean Banach space.

Theorem 3.1 *Let $\varphi : G^m \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x_1}{2^n}, \ldots, \frac{x_m}{2^n}\right) = 0 \tag{3.1}$$

for all $x_1, \ldots, x_m \in G$ and let for each $x \in G$ the limit

$$\begin{aligned}\Theta(x) = \lim_{n\to\infty} \max\Bigg\{&|2|^k \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, -\frac{x}{2^{k+2}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, 0\right),\right.\right.\\ &\left.\varphi_{i,j}\left(0, -\frac{x}{2^{k+2}r_j}\right)\right\}, \frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, \frac{x}{2^{k+1}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, 0\right),\right.\\ &\left.\left.\varphi_{i,j}\left(0, \frac{x}{2^{k+1}r_j}\right)\right\}\right\} \Bigg| 0 \leq k < n\Bigg\}\end{aligned} \tag{3.2}$$

exist. Suppose that $f : G \to X$ is a mapping with $f(0) = 0$ satisfying the following inequality:

$$\begin{aligned}&\left\|\sum_{j=1}^m f\left(-r_j x_j + \sum_{1\leq i\leq m, i\neq j} r_i x_i\right) + 2\sum_{i=1}^m r_i f(x_i) - mf\left(\sum_{i=1}^m r_i x_i\right)\right\|\\ &\quad \leq \varphi(x_1, \ldots, x_m)\end{aligned} \tag{3.3}$$

for all $x_1, \dots, x_m \in X$. Then, the limit $EL(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for all $x \in G$ and defines an Euler-Lagrange type additive mapping $EL : G \to X$ such that

$$\|f(x) - EL(x)\| \le \Theta(x). \tag{3.4}$$

Moreover, if

$$\lim_{p\to\infty}\lim_{n\to\infty}\max\left\{|2|^k \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, -\frac{x}{2^{k+2}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, 0\right),\right.\right.\right.$$
$$\left.\varphi_{i,j}\left(0, -\frac{x}{2^{k+2}r_j}\right)\right\}, \frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, \frac{x}{2^{k+1}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, 0\right),\right.$$
$$\left.\left.\left.\varphi_{i,j}\left(0, \frac{x}{2^{k+1}r_j}\right)\right\}\right\}\,\Big|\, p \le k < n+p\right\}$$

then EL is the unique mapping satisfying (3.4).

Proof By (2.17), we know

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{4r_i}, -\frac{x}{4r_j}\right), \varphi_{i,j}\left(\frac{x}{4r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{4r_j}\right)\right\},\right.$$
$$\left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{2r_i}, \frac{x}{2r_j}\right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{2r_j}\right)\right\}\right\} \tag{3.5}$$

for all $x \in G$. Replacing x by $\frac{x}{2^n}$ in (3.5), we obtain

$$\left\|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right\|$$
$$\le |2|^n \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2^{n+2}r_i}, -\frac{x}{2^{n+2}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{n+2}r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2^{n+2}r_j}\right)\right\},\right.$$
$$\left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{2^{n+1}r_i}, \frac{x}{2^{n+1}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{n+1}r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{2^{n+1}r_j}\right)\right\}\right\}. \tag{3.6}$$

It follows from (3.1) and (3.6) that the sequence $\{2^n f(\frac{x}{2^n})\}_{n\ge1}$ is a Cauchy sequence. Since X is complete, so $\{2^n f(\frac{x}{2^n})\}_{n\ge1}$ is convergent. Set

$$EL(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right).$$

Using induction on n, one can show that

$$\left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\|$$
$$\le \max\left\{|2|^k \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, -\frac{x}{2^{k+2}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2^{k+2}r_j}\right)\right\},\right.\right.$$
$$\left.\left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, \frac{x}{2^{k+1}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{2^{k+1}r_j}\right)\right\}\right\}\,\Big|\, 0 \le k < n\right\} \tag{3.7}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (3.7) and using (3.2), one obtains (3.4). By (3.1) and (3.3), we get

$$\begin{aligned}
&\left\| \sum_{j=1}^{m} EL\left(-r_j x_j + \sum_{1\le i\le m, i\neq j} r_i x_i\right) + 2\sum_{i=1}^{m} r_i EL(x_i) - mEL\left(\sum_{i=1}^{m} r_i x_i\right) \right\| \\
&\quad = \lim_{n\to\infty} |2|^n \left\| \sum_{j=1}^{m} f\left(\frac{-r_j x_j}{2^n} + \sum_{1\le i\le m, i\neq j} \frac{r_i x_i}{2^n}\right) + 2\sum_{i=1}^{m} r_i f\left(\frac{x_i}{2^n}\right) - mf\left(\sum_{i=1}^{m} \frac{r_i x_i}{2^n}\right) \right\| \\
&\quad \le \lim_{n\to\infty} |2|^n \varphi\left(\frac{x_1}{2^n}, \dots, \frac{x_m}{2^n}\right) \\
&\quad = 0
\end{aligned}$$

for all $x_1, \dots, x_m \in X$. Therefore the function $EL : G \to X$ satisfies (1.1).

To prove the uniqueness property of EL, let $A : G \to X$ be another function satisfying (3.4). Then

$$\begin{aligned}
&\|EL(x) - A(x)\| \\
&\quad = \lim_{j\to\infty} |2|^j \left\| EL\left(\frac{x}{2^j}\right) - A\left(\frac{x}{2^j}\right) \right\| \\
&\quad \le \lim_{j\to\infty} |2|^j \max\left\{ \left\| EL\left(\frac{x}{2^j}\right) - f\left(\frac{x}{2^j}\right) \right\|, \left\| f\left(\frac{x}{2^j}\right) - A\left(\frac{x}{2^j}\right) \right\| \right\} \\
&\quad \le \lim_{j\to\infty} \lim_{n\to\infty} \max\left\{ |2|^k \max\left\{ \max\left\{ \varphi_{i,j}\left(\frac{x}{2^{k+2} r_i}, -\frac{x}{2^{k+2} r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+2} r_i}, 0\right), \right.\right.\right. \\
&\qquad \left. \varphi_{i,j}\left(0, -\frac{x}{2^{k+2} r_j}\right) \right\}, \frac{1}{|2|} \max\left\{ \varphi_{i,j}\left(\frac{x}{2^{k+1} r_i}, \frac{x}{2^{k+1} r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+1} r_i}, 0\right), \right. \\
&\qquad \left.\left.\left. \varphi_{i,j}\left(0, \frac{x}{2^{k+1} r_j}\right) \right\}\right\} \,\middle|\, j \le k < n + j \right\} \\
&\quad = 0
\end{aligned}$$

for all $x \in G$. Therefore $A = EL$, and the proof is complete. □

Corollary 3.1 *Let $\xi : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\xi\left(\frac{t}{|2|}\right) \le \xi\left(\frac{1}{|2|}\right)\xi(t) \quad (t \ge 0) \qquad \xi\left(\frac{1}{|2|}\right) < |2|^{-1}. \tag{3.8}$$

Let $\kappa > 0$ and $f : G \to X$ be a mapping with $f(0) = 0$ satisfying the following inequality:

$$\begin{aligned}
&\left\| \sum_{j=1}^{m} f\left(-r_j x_j + \sum_{1\le i\le m, i\neq j} r_i x_i\right) + 2\sum_{i=1}^{m} r_i f(x_i) - mf\left(\sum_{i=1}^{m} r_i x_i\right) \right\| \\
&\quad \le \kappa \left(\sum_{k=1}^{m} \xi(|x_k|)\right)
\end{aligned} \tag{3.9}$$

for all $x_1,\dots,x_m \in G$. *Then there exists a unique Euler-Lagrange type additive mapping* $EL : G \to X$ *such that*

$$\|f(x) - EL(x)\| \le \frac{\kappa}{|4|}\left\{\xi\left(\left|\frac{x}{r_i}\right|\right) + \xi\left(\left|\frac{x}{r_j}\right|\right)\right\}. \tag{3.10}$$

Proof Defining $\zeta : G^m \to [0,\infty)$ by $\varphi(x_1,\dots,x_m) := \kappa(\sum_{k=1}^m \xi(|x_k|))$, then we have

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x_1}{2^n},\dots,\frac{x_m}{2^n}\right) \le \lim_{n\to\infty}\left(|2|\xi\left(\frac{1}{|2|}\right)\right)^n \varphi(x_1,\dots,x_m) = 0$$

for all $x_1,\dots,x_m \in G$. On the other hand,

$$\begin{aligned}\Theta(x) &= \lim_{n\to\infty}\max\left\{|2|^k \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, -\frac{x}{2^{k+2}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, 0\right),\right.\right.\right.\\ &\qquad \left.\varphi_{i,j}\left(0, -\frac{x}{2^{k+2}r_j}\right)\right\}, \frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, \frac{x}{2^{n+1}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, 0\right),\right.\\ &\qquad \left.\left.\left.\varphi_{i,j}\left(0, \frac{x}{2^{k+1}r_j}\right)\right\}\right\}\middle| 0 \le k < n\right\}\\ &= \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{4r_i}, -\frac{x}{4r_j}\right), \varphi_{i,j}\left(\frac{x}{4r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{4r_j}\right)\right\},\right.\\ &\qquad \left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{2r_i}, \frac{x}{2r_j}\right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{2r_j}\right)\right\}\right\}\\ &= \frac{\kappa}{|4|}\left\{\xi\left(\left|\frac{x}{r_i}\right|\right) + \xi\left(\left|\frac{x}{r_j}\right|\right)\right\}\end{aligned}$$

for all $x \in G$, exists. Also

$$\begin{aligned}&\lim_{p\to\infty}\lim_{n\to\infty}\max\left\{|2|^k \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, -\frac{x}{2^{k+2}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+2}r_i}, 0\right),\right.\right.\right.\\ &\qquad \left.\varphi_{i,j}\left(0, -\frac{x}{2^{k+2}r_j}\right)\right\}, \frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, \frac{x}{2^{n+1}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{k+1}r_i}, 0\right),\right.\\ &\qquad \left.\left.\left.\varphi_{i,j}\left(0, \frac{x}{2^{k+1}r_j}\right)\right\}\right\}\middle| p \le k < n+p\right\}\\ &= \lim_{p\to\infty}|2|^p \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2^{p+2}r_i}, -\frac{x}{2^{p+2}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{p+2}r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2^{p+2}r_j}\right)\right\},\right.\\ &\qquad \left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{2^{p+1}r_i}, \frac{x}{2^{p+1}r_j}\right), \varphi_{i,j}\left(\frac{x}{2^{p+1}r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{2^{p+1}r_j}\right)\right\}\right\}\\ &= 0.\end{aligned}$$

Applying Theorem 3.1, we get the desired result. □

Theorem 3.2 *Let* $\varphi : G^m \to [0,+\infty)$ *be a function such that*

$$\lim_{n\to\infty}\frac{\varphi(2^n x_1,\dots,2^n x_m)}{|2|^n} = 0 \tag{3.11}$$

for all $x_1, \ldots, x_m \in G$ and let for each $x \in G$ the limit

$$\begin{aligned}\Theta(x) = \lim_{n\to\infty} \max\bigg\{ &\frac{1}{|2|^k} \max\bigg\{\max\bigg\{\varphi_{i,j}\bigg(\frac{2^{k-1}x}{r_i}, -\frac{2^{k-1}x}{r_j}\bigg), \\ &\varphi_{i,j}\bigg(\frac{2^{k-1}x}{r_i}, 0\bigg), \varphi_{i,j}\bigg(0, -\frac{2^{k-1}x}{r_j}\bigg)\bigg\}, \\ &\frac{1}{|2|}\max\bigg\{\varphi_{i,j}\bigg(\frac{2^k x}{r_i}, \frac{2^k x}{r_j}\bigg), \varphi_{i,j}\bigg(\frac{2^k x}{r_i}, 0\bigg), \varphi_{i,j}\bigg(0, \frac{2^k x}{r_j}\bigg)\bigg\}\bigg\}\bigg| 0 \le k < n\bigg\} \qquad (3.12)\end{aligned}$$

exist. Suppose that $f: G \to X$ is a mapping with $f(0) = 0$ satisfying (3.3). Then, the limit $EL(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in G$ and defines an Euler-Lagrange type additive mapping $EL: G \to X$, such that

$$\big\|f(x) - EL(x)\big\| \le \frac{1}{|2|}\Theta(x). \qquad (3.13)$$

Moreover, if

$$\begin{aligned}\lim_{p\to\infty}\lim_{n\to\infty} \max\bigg\{ &\frac{1}{|2|^k} \max\bigg\{\max\bigg\{\varphi_{i,j}\bigg(\frac{2^{k-1}x}{r_i}, -\frac{2^{k-1}x}{r_j}\bigg), \\ &\varphi_{i,j}\bigg(\frac{2^{k-1}x}{r_i}, 0\bigg), \varphi_{i,j}\bigg(0, -\frac{2^{k-1}x}{r_j}\bigg)\bigg\}, \\ &\frac{1}{|2|}\max\bigg\{\varphi_{i,j}\bigg(\frac{2^k x}{r_i}, \frac{2^k x}{r_j}\bigg), \varphi_{i,j}\bigg(\frac{2^k x}{r_i}, 0\bigg), \varphi_{i,j}\bigg(0, \frac{2^k x}{r_j}\bigg)\bigg\}\bigg\}\bigg| p \le k < n+p\bigg\} = 0\end{aligned}$$

then EL is the unique Euler-Lagrange type additive mapping satisfying (3.13).

Proof It follows from (2.16) that

$$\begin{aligned}&\bigg\|\frac{f(2x)}{2} - f(x)\bigg\| \\ &\quad\le \frac{1}{|2|}\max\bigg\{\max\bigg\{\varphi_{i,j}\bigg(\frac{x}{2r_i}, -\frac{x}{2r_j}\bigg), \varphi_{i,j}\bigg(\frac{x}{2r_i}, 0\bigg), \varphi_{i,j}\bigg(0, -\frac{x}{2r_j}\bigg)\bigg\}, \\ &\qquad \frac{1}{|2|}\max\bigg\{\varphi_{i,j}\bigg(\frac{x}{r_i}, \frac{x}{r_j}\bigg), \varphi_{i,j}\bigg(\frac{x}{r_i}, 0\bigg), \varphi_{i,j}\bigg(0, \frac{x}{r_j}\bigg)\bigg\}\bigg\} \qquad (3.14)\end{aligned}$$

for all $x \in G$. Replacing x by $2^n x$ in (3.14), we obtain

$$\begin{aligned}&\bigg\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\bigg\| \\ &\quad\le \frac{1}{|2|^{n+1}}\max\bigg\{\max\bigg\{\varphi_{i,j}\bigg(\frac{2^{n-1}x}{r_i}, -\frac{2^{n-1}x}{r_j}\bigg), \varphi_{i,j}\bigg(\frac{2^{n-1}x}{r_i}, 0\bigg), \varphi_{i,j}\bigg(0, -\frac{2^{n-1}x}{r_j}\bigg)\bigg\}, \\ &\qquad \frac{1}{|2|}\max\bigg\{\varphi_{i,j}\bigg(\frac{2^n x}{r_i}, \frac{2^n x}{r_j}\bigg), \varphi_{i,j}\bigg(\frac{2^n x}{r_i}, 0\bigg), \varphi_{i,j}\bigg(0, \frac{2^n x}{r_j}\bigg)\bigg\}\bigg\}. \qquad (3.15)\end{aligned}$$

It follows from (3.11) and (3.15) that the sequence $\{\frac{f(2^n x)}{2^n}\}_{n\geq 1}$ is convergent. Set

$$EL(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}.$$

On the other hand, it follows from (3.15) that

$$\begin{aligned}
&\left\| \frac{f(2^p x)}{2^p} - \frac{f(2^q x)}{2^q} \right\| \\
&= \left\| \sum_{k=p}^{q-1} \frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k} \right\| \\
&\leq \max\left\{ \left\| \frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k} \right\|; p \leq k \leq q-1 \right\} \\
&\leq \max\left\{ \frac{1}{|2|^{k+1}} \max\left\{ \max\left\{ \varphi_{i,j}\left(\frac{2^{k-1}x}{r_i}, -\frac{2^{k-1}x}{r_j}\right), \varphi_{i,j}\left(\frac{2^{k-1}x}{r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{2^{k-1}x}{r_j}\right) \right\}, \right.\right. \\
&\quad \left.\left. \frac{1}{|2|} \max\left\{ \varphi_{i,j}\left(\frac{2^k x}{r_i}, \frac{2^k x}{r_j}\right), \varphi_{i,j}\left(\frac{2^k x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{2^k x}{r_j}\right) \right\} \right\} \middle| p \leq k < q \right\}
\end{aligned}$$

for all $x \in G$ and all nonnegative integers p, q with $q > p \geq 0$. Letting $p = 0$ and passing the limit $q \to \infty$ in the last inequality and using (3.12), we obtain (3.13). The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.2 *Let* $\xi : [0,\infty) \to [0,\infty)$ *be a function satisfying*

$$\xi(|2t|) \leq \xi(|2|)\xi(t) \quad (t \geq 0) \qquad \xi(|2|) < |2|. \tag{3.16}$$

Let $\kappa > 0$ *and* $f : G \to X$ *be a mapping with* $f(0) = 0$ *satisfying the following inequality (3.9). Then there exists a unique Euler-Lagrange type additive mapping* $EL : G \to X$ *such that*

$$\begin{aligned}
\|f(x) - EL(x)\| &\leq \frac{\kappa}{|2|} \max\left\{ \xi\left(\left|\frac{x}{2r_i}\right|\right) + \xi\left(\left|\frac{x}{2r_j}\right|\right), \frac{1}{|2|}\xi\left(\left|\frac{x}{r_i}\right|\right) + \xi\left(\left|\frac{x}{r_j}\right|\right) \right\} \\
&= \frac{\kappa}{|2|}\left[\xi\left(\left|\frac{x}{2r_i}\right|\right) + \xi\left(\left|\frac{x}{2r_j}\right|\right) \right].
\end{aligned} \tag{3.17}$$

Proof Defining $\zeta : G^m \to [0,\infty)$ by $\varphi(x_1,\ldots,x_m) := \kappa(\sum_{k=1}^m \xi(|x_k|))$, then, we have

$$\lim_{n\to\infty} \frac{\varphi(2^n x_1,\ldots,2^n x_m)}{|2|^n} \leq \lim_{n\to\infty} \left(\frac{\xi(|2|)}{|2|}\right)^n \varphi(x_1,\ldots,x_m) = 0$$

for all $x_1,\ldots,x_m \in G$. On the other hand,

$$\begin{aligned}
\Theta(x) = \lim_{n\to\infty} \max&\left\{ \frac{1}{|2|^k} \max\left\{ \max\left\{ \varphi_{i,j}\left(\frac{2^{k-1}x}{r_i}, -\frac{2^{k-1}x}{r_j}\right), \right.\right.\right. \\
&\left. \varphi_{i,j}\left(\frac{2^{k-1}x}{r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{2^{k-1}x}{r_j}\right) \right\}, \\
&\left.\left. \frac{1}{|2|} \max\left\{ \varphi_{i,j}\left(\frac{2^k x}{r_i}, \frac{2^k x}{r_j}\right), \varphi_{i,j}\left(\frac{2^k x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{2^k x}{r_j}\right) \right\} \right\} \middle| 0 \leq k < n \right\}
\end{aligned}$$

$$= \max\left\{\max\left\{\varphi_{i,j}\left(\frac{x}{2r_i}, -\frac{x}{2r_j}\right), \varphi_{i,j}\left(\frac{x}{2r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{x}{2r_j}\right)\right\},\right.$$
$$\left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{x}{r_i}, \frac{x}{r_j}\right), \varphi_{i,j}\left(\frac{x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{x}{r_j}\right)\right\}\right\}$$

for all $x \in G$, exists. Also

$$\lim_{p\to\infty}\lim_{n\to\infty}\max\left\{\frac{1}{|2|^k}\max\left\{\max\left\{\varphi_{i,j}\left(\frac{2^{k-1}x}{r_i}, -\frac{2^{k-1}x}{r_j}\right),\right.\right.\right.$$
$$\left.\varphi_{i,j}\left(\frac{2^{k-1}x}{r_i}, 0\right), \varphi_{i,j}\left(0, -\frac{2^{k-1}x}{r_j}\right)\right\},$$
$$\left.\left.\frac{1}{|2|}\max\left\{\varphi_{i,j}\left(\frac{2^k x}{r_i}, \frac{2^k x}{r_j}\right), \varphi_{i,j}\left(\frac{2^k x}{r_i}, 0\right), \varphi_{i,j}\left(0, \frac{2^k x}{r_j}\right)\right\}\right\}\middle| p \le k < n+p\right\} = 0.$$

Applying Theorem 3.2, we get the desired result. □

Remark 3.1 We remark that if $\xi(|2|) = 0$, then $\xi = 0$ identically, and so f is itself additive. Thus, for the nontrivial ξ, we observe that $\xi(|2|) \neq 0$ and

$$1 \le \xi\left(|1|\right) \le \xi\left(|2|\right)\xi\left(\frac{1}{|2|}\right) \le |2|\xi\left(\frac{1}{|2|}\right)$$

implies that $\frac{1}{|2|} \le \xi(\frac{1}{|2|})$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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