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On the hyper exponent of convergence of zeros of $f^{(j)} - \varphi$ of higher order linear differential equations

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Abstract

In this paper, we deal with the relationship between the small function and the derivative of solutions of higher order linear differential equations

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0 \quad (k \geq 2),$$

where $A_j(z)$ ($j = 0, 1, \ldots, k-1$) are entire functions or meromorphic functions. The theorems of this paper improve the previous results given by Chen, Belaïdi, Liu.
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1 Introduction and main results

Complex oscillation theory of solutions of linear differential equations in the complex plane $\mathbb{C}$ was started by Bank and Laine [2, 3]. After their well-known work, many important results have been obtained on the complex oscillation theory of solutions of linear differential equations in $\mathbb{C}$, refer to [16, 17].

To state those correlated results, we require to give some explanation as follows.

We shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [11, 14, 16]). In addition, we will use the notation $\sigma(f)$ to denote the order of meromorphic function $f(z)$, $\lambda(f)$ to denote the exponent of convergence of the zero-sequence of $f(z)$ and $\overline{\lambda}(f)$ to denote exponent of convergence of distinct zero-sequence of meromorphic function $f(z)$, and $\tau(f)$ to denote the type of an entire function $f(z)$ with $0 < \sigma(f) = \sigma < +\infty$, which is defined to be (see [14])

$$\tau(f) = \limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\sigma}}.$$

We use $\sigma_2(f)$ to denote the hyper-order of $f(z)$, $\sigma_2(f)$ is defined to be (see [21])

$$\sigma_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r,f)}{\log r}.$$

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We use $\lambda_2(f)$ $(\overline{\lambda}_2(f))$ to denote the hyper-exponent of convergence of the zero-sequence (distinct zero-sequence) of meromorphic function $f(z)$, $\lambda_2(f)$, $\overline{\lambda}_2(f)$ are defined to be (see [11, 16])

$$\lambda_2(f) = \limsup_{r \to \infty} \frac{\log \log N(r, \frac{1}{f})}{\log r}, \quad \text{and} \quad \overline{\lambda}_2(f) = \limsup_{r \to \infty} \frac{\log \log \overline{N}(r, \frac{1}{f})}{\log r}.$$

Let $\varphi(z)$ be an entire function with $\sigma(\varphi) < \sigma(f)$ or $\sigma_2(\varphi) < \sigma_2(f)$, the hyper-exponent of convergence of zeros and distinct zeros of $f(z) - \varphi(z)$ are defined to be

$$\lambda_2(f - \varphi) = \limsup_{r \to \infty} \frac{\log \log N(r, \frac{1}{f-\varphi})}{\log r}, \qquad \overline{\lambda}_2(f - \varphi) = \limsup_{r \to \infty} \frac{\log \log N(r, \frac{1}{f-\varphi})}{\log r},$$

especially if $\varphi(z) = z$, we use $\lambda_2(f - z)$ and $\overline{\lambda}_2(f - z)$ to denote the hyper-exponent of convergence of fixed points and distinct fixed points of $f(z)$, respectively. We use $mE = \int_E dt$ to denote the linear measure of a set $E \subset (0, +\infty)$ and use $m_l E = \int_E \frac{dt}{t}$ to denote the logarithmic measure of a set $E \subset [1, +\infty)$. We denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o\big(T(r, f)\big),$$

as $r \to +\infty$, possibly outside of a set with finite measure. A meromorphic function $\psi(z)$ is called a small function with respect to f if $T(r, \psi) = S(r, f)$.

For the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1}$$

where $A(z)$ and $B(z)$ ($\not\equiv 0$) are entire functions, it is an interesting problem to investigate the complex oscillation of solutions of Equation (1). Many mathematicians obtained a lot of important and significant results (see [1, 6, 7, 10, 13]) by studying the above equation.

In 1996, Shon [15] investigated the hyper-order of the solutions of (1) and obtained the following result.

Theorem A (see [15]) *Let $A(z)$ and $B(z)$ be entire functions such that $\sigma(A) < \sigma(B)$ or $\sigma(B) < \sigma(A) < \frac{1}{2}$, then every function $f \not\equiv 0$ of (1) satisfies $\sigma_2(f) \geq \max\{\sigma(A), \sigma(B)\}$.*

In 2006, Chen and Shon [8] investigated the zeros of the solution concerning small functions and fixed points of solutions of second order linear differential equations and obtained some results as follows.

Theorem B (see [8]) *Let $A_j(z) \not\equiv 0$ ($j = 1, 2$) be entire functions with $\sigma(A_j) < 1$, suppose that a, b are complex numbers and satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ ($0 < c < 1$). If $\varphi(z) \not\equiv 0$ is an entire function of finite order, then every non-trivial solution f of the equation*

$$f'' + A_1(z)e^{az}f' + A_2(z)e^{bz}f = 0$$

satisfies $\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty$.

Theorem C (see [8]) *Let $A_1(z) \not\equiv 0$, $\varphi(z) \not\equiv 0$, $Q(z)$ be entire functions with $\sigma(A_1) < 1$, $1 < \sigma(Q) < \infty$ and $\sigma(\varphi) < \infty$, then every non-trivial solution f of the equation*

$$f'' + A_1(z)e^{az}f' + Q(z)f = 0$$

satisfies $\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty$, where $a \neq 0$ is a complex number.

In the same year, Liu and Zhang [18] investigated the fixed points when the coefficients of the equations are meromorphic functions and obtained the following result.

Theorem D (see [18]) *Suppose that $k \geq 2$ and $A(z)$ is a transcendental meromorphic function satisfying $\delta(\infty, A) = \underline{\lim}_{r \to \infty} \frac{m(r,A)}{T(r,A)} = \delta > 0$, $\sigma(A) = \sigma < +\infty$. Then every meromorphic solution $f \not\equiv 0$ of the equation*

$$f^{(k)} + A(z)f = 0 \tag{2}$$

satisfies that f and $f', f'', \ldots, f^{(k)}$ all have infinitely many fixed points and $\overline{\lambda}(f^{(j)} - z) = \sigma$ ($j = 0, 1, \ldots, k$).

For Equation (2), Belaïdi [4], Belaïdi and El Farissi [5] investigated the fixed points and the relationship between small functions and differential polynomials of solutions of Equation (2) and obtained some results which improve Theorems D and C.

There naturally arises an interesting subject on the problems of fixed points of solutions of the differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0 \quad (k \geq 2), \tag{3}$$

where $A_j(z)$ ($j = 0, 1, \ldots, k - 1$) are entire functions.

In this paper, we will deal with the above equation and investigate the relationship between small functions and derivative of solutions of Equation (2) and obtain some theorems which improve the previous results given by Chen, Kwon and *etc*.

Theorem 1.1 *Let $A_j(z)$, $j = 0, 1, \ldots, k - 1$ be entire functions with finite order and satisfy one of the following conditions:*

(i) $\max\{\sigma(A_j) : j = 1, 2, \ldots, k - 1\} < \sigma(A_0) < \infty$;

(ii) $0 < \sigma(A_{k-1}) = \cdots = \sigma(A_1) = \sigma(A_0) < \infty$ and $\max\{\tau(A_j) : j = 1, 2, \ldots, k - 1\} = \tau_1 < \tau(A_0) = \tau$,

then for every solution $f \not\equiv 0$ of (3) and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$, we have

$$\overline{\lambda}_2(f - \varphi) = \overline{\lambda}_2(f' - \varphi) = \overline{\lambda}_2(f'' - \varphi) = \overline{\lambda}_2(f''' - \varphi) = \overline{\lambda}_2(f^{(i)} - \varphi) = \sigma_2(f) = \sigma(A_0) \quad (i \in \mathbb{N}).$$

Theorem 1.2 *Let $A_j(z)$, $j = 1, 2, \ldots, k - 1$ be polynomials, $A_0(z)$ be a transcendental entire function, then for every solution $f \not\equiv 0$ of (3) and for any entire function $\varphi(z)$ of finite order, we have*

(i) $\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = \infty$;

(ii) $\overline{\lambda}(f^{(i)} - \varphi) = \lambda(f^{(i)} - \varphi) = \sigma(f^{(i)} - \varphi) = \infty$ ($i \geq 1, i \in \mathbb{N}$).

Theorem 1.3 *Let $A_j(z)$, $j = 0, 1, \ldots, k - 1$ be meromorphic functions satisfying $\max\{\sigma(A_j) : j = 1, 2, \ldots, k - 1\} < \sigma(A_0)$ and $\delta(\infty, A_0) > 0$. Then for every meromorphic solution $f \not\equiv 0$ of (3) and for any meromorphic function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$, we have $\overline{\lambda}_2(f^{(i)} - \varphi) = \lambda_2(f^{(i)} - \varphi) \geq \sigma(A_0)$ ($i = 0, 1, \ldots$), where $f^{(0)} = f$.*

Remark 1.1 The following example shows that Theorem 1.3 is not valid when $A_j(z)$, $j = 0, 1, \ldots, k - 1$ do not satisfy the condition $\max\{\sigma(A_j) : j = 1, 2, \ldots, k - 1\} < \sigma(A_0)$.

Example 1.1 For the equation

$$f'' + \frac{e^{2z} + e^z - 1}{1 - e^z} f' + \frac{-e^{2z}}{1 - e^z} f = 0, \tag{$\sharp$}$$

we can easily get that Equation ($\sharp$) has a solution $f(z) = e^{e^z} + e^z$. And the functions $\frac{e^{2z}+e^z-1}{1-e^z}$, $\frac{-e^{2z}}{1-e^z}$ are meromorphic and satisfy $\delta(\infty, \frac{-e^{2z}}{1-e^z}) = \frac{1}{2}$. Take $\varphi(z) = e^z$, then $\sigma_2(\varphi) < \sigma(\frac{-e^{2z}}{1-e^z})$. Thus, we can get that $\overline{\lambda}_2(f' - \varphi) = \overline{\lambda}_2(e^{e^z} e^z) = 0 \neq 1 = \sigma(\frac{-e^{2z}}{1-e^z})$.

From Theorems 1.1-1.3, if $\varphi(z) = z$, we can get the following corollaries easily.

Corollary 1.1 *Under the assumptions of Theorem 1.1, if $\varphi(z) = z$, for every solution $f \not\equiv 0$ of (3), we have*

$$\overline{\lambda}_2(f - z) = \overline{\lambda}_2(f' - z) = \overline{\lambda}_2(f'' - z) = \overline{\lambda}_2(f^{(i)} - z) = \sigma_2(f) = \sigma(A_0) \quad (i \in \mathbb{N}).$$

Corollary 1.2 *Under the assumptions of Theorem 1.2, if $\varphi(z) = z$, for every solution $f \not\equiv 0$ of (3), we have*
  (i) $\overline{\lambda}(f - z) = \lambda(f - z) = \sigma(f) = \infty$;
  (ii) $\overline{\lambda}(f^{(i)} - z) = \lambda(f^{(i)} - z) = \sigma(f^{(i)} - z) = \infty$ ($i \geq 1, i \in \mathbb{N}$).

Corollary 1.3 *Under the assumptions of Theorem 1.3, if $\varphi(z) = z$, for every meromorphic solution $f \not\equiv 0$ of (3), we have $\overline{\lambda}_2(f^{(i)} - z) = \lambda_2(f^{(i)} - z) \geq \sigma_2(A_0)$ ($i = 0, 1, \ldots$), where $f^{(0)} = f$.*

Remark 1.2 In Theorem B, if $ab \neq 0$ and $a = cb$ ($0 < c < 1$), it is easy to see that $\sigma(A_1 e^{az}) = \sigma(A_2 e^{bz}) = 1$ and $\tau(A_1 e^{az}) = |a| < \tau(A_2 e^{bz}) = |b|$. By Theorem 1.1, for every solution $f \not\equiv 0$ of (3) and for any entire function $\varphi(z) \not\equiv 0$ with $\sigma_2(\varphi) < 1$, we have $\overline{\lambda}_2(f - \varphi) = \overline{\lambda}_2(f' - \varphi) = \overline{\lambda}_2(f'' - \varphi) = 1$. Therefore, Theorem 1.1 is also a partial extension of Theorem B. Theorem 1.2 is the improvement of Theorem C. Theorem 1.3 and Corollary 1.3 are the improvements of Theorem D.

2 Some lemmas
To prove our theorems, we require the following lemmas.

Lemma 2.1 *Assume $f \not\equiv 0$ is a solution of Equation (3), set $g = f - \varphi$, then g satisfies the equation*

$$g^{(k)} + A_{k-1}g^{(k-1)} + \cdots + A_0 g = -\left[\varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_0\varphi\right]. \tag{4}$$

Proof Since $g = f - \varphi$, we have $g' = f' - \varphi', \ldots, g^{(k)} = f^{(k)} - \varphi^{(k)}$. Substituting them into Equation (3), we have (4). $\square$

Lemma 2.2 *Assume $f \not\equiv 0$ is a solution of Equation (3), set $g_1 = f' - \varphi$, then g_1 satisfies the equation*

$$g_1^{(k)} + U_{k-1}^1 g_1^{(k-1)} + \cdots + U_0^1 g_1 = -[\varphi^{(k)} + U_{k-1}^1 \varphi^{(k-1)} + \cdots + U_0^1 \varphi], \tag{5}$$

where $U_j^1 = A_{j+1}' + A_j - \frac{A_0'}{A_0} A_{j+1}, j = 0, 1, 2, \ldots, k-1$ and $A_k \equiv 1$.

Proof Since $g_1 = f' - \varphi$, we have

$$f' = g_1 + \varphi, \qquad f'' = g_1' + \varphi', \ldots, f^{(k+1)} = g_1^{(k)} + \varphi^{(k)}. \tag{6}$$

And the derivation of (3) is

$$f^{(k+1)} + A_{k-1}f^{(k)} + (A_{k-1}' + A_{k-2})f^{(k-1)} + \cdots + (A_1' + A_0)f' + A_0'f = 0. \tag{7}$$

We can rewrite (3) as

$$f = -\frac{1}{A_0}(f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f'). \tag{8}$$

Substituting (8) into (7), we have

$$\begin{aligned}
&f^{(k+1)} + \left(A_{k-1} - \frac{A_0'}{A_0}\right)f^{(k)} + \left(A_{k-1}' + A_{k-2} - \frac{A_0'}{A_0}A_{k-1}\right)f^{(k-1)} \\
&\quad + \cdots + \left(A_1' + A_0 - \frac{A_0'}{A_0}A_1\right)f' = 0.
\end{aligned} \tag{9}$$

Substituting (6) into (9), we have

$$\begin{aligned}
&g_1^{(k)} + \left(A_{k-1} - \frac{A_0'}{A_0}\right)g_1^{(k-1)} + \left(A_{k-1}' + A_{k-2} - \frac{A_0'}{A_0}A_{k-1}\right)g_1^{(k-2)} \\
&\quad + \cdots + \left(A_1' + A_0 - \frac{A_0'}{A_0}A_1\right)g_1 \\
&= -\left[\varphi^{(k)} + \left(A_{k-1} - \frac{A_0'}{A_0}\right)\varphi^{(k-1)} + \left(A_{k-1}' + A_{k-2} - \frac{A_0'}{A_0}A_{k-1}\right)\varphi^{(k-2)} \right.\\
&\quad \left. + \cdots + \left(A_1' + A_0 - \frac{A_0'}{A_0}A_1\right)\varphi\right].
\end{aligned} \tag{10}$$

Set $U_j^1 = A_{j+1}' + A_j - \frac{A_0'}{A_0}A_{j+1}, j = 0, 1, 2, \ldots, k-1$ and $A_k \equiv 1$. Then from (9) and (10), we can get

$$f^{(k+1)} + U_{k-1}^1 f^{(k)} + U_{k-2}^1 f^{(k-1)} + \cdots + U_0^1 f' = 0 \tag{11}$$

and (5).

Thus, we complete the proof of Lemma 2.2. $\square$

Lemma 2.3 *Assume $f \not\equiv 0$ is a solution of Equation (3), set $g_2 = f'' - \varphi$, then g_2 satisfies the equation*

$$g_2^{(k)} + U_{k-1}^2 g_2^{(k-1)} + \cdots + U_0^2 g_2 = -\big[\varphi^{(k)} + U_{k-1}^2 \varphi^{(k-1)} + \cdots + U_0^2 \varphi\big], \tag{12}$$

where $U_j^2 = U_{j+1}^{1\,\prime} + U_j^1 - \dfrac{U_0^{1\,\prime}}{U_0^1} U_{j+1}^1,\ j = 0, 1, 2, \ldots, k-1$ and $U_k^1 \equiv 1$.

Proof Since $g_2 = f'' - \varphi$, we can get

$$f'' = g_2 + \varphi, \qquad f''' = g_2' + \varphi', \qquad \ldots, \qquad f^{(k+2)} = g_2^{(k)} + \varphi^{(k)}. \tag{13}$$

The derivation of (11) is

$$f^{(k+2)} + U_{k-1}^1 f^{(k+1)} + \big(U_{k-1}^{1\,\prime} + U_{k-2}^1\big)f^{(k)} + \cdots + \big(U_1^{1\,\prime} + U_0^1\big)f'' + U_0^{1\,\prime}f' = 0. \tag{14}$$

Substituting (11) into (14), we have

$$f^{(k+2)} + \left(U_{k-1}^1 - \frac{U_0^{1\,\prime}}{U_0^1}\right)f^{(k+1)} + \left(U_{k-1}^{1\,\prime} + U_{k-2}^1 - \frac{U_0^{1\,\prime}}{U_0^1}U_{k-1}^1\right)f^{(k)}$$
$$+ \cdots + \left(U_1^{1\,\prime} + U_0^1 - \frac{U_0^{1\,\prime}}{U_0^1}U_1^1\right)f'' = 0. \tag{15}$$

Set $U_j^2 = U_{j+1}^{1\,\prime} + U_j^1 - \dfrac{U_0^{1\,\prime}}{U_0^1}U_{j+1}^1,\ j = 0, 1, 2, \ldots, k-1$ and $U_k^1 \equiv 1$, then from (14) we have

$$f^{(k+2)} + U_{k-1}^2 f^{(k+1)} + U_{k-2}^2 f^{(k)} + \cdots + U_0^2 f'' = 0. \tag{16}$$

Substituting (13) into (16), we can get (12).

Thus, this completes the proof of Lemma 2.3. □

Lemma 2.4 *Assume $f \not\equiv 0$ is a solution of Equation (3), set $g_3 = f''' - \varphi$, then g_3 satisfies the equation*

$$g_3^{(k)} + U_{k-1}^3 g_3^{(k-1)} + \cdots + U_0^3 g_3 = -\big[\varphi^{(k)} + U_{k-1}^3 \varphi^{(k-1)} + \cdots + U_0^3 \varphi\big], \tag{17}$$

where $U_j^3 = U_{j+1}^{2\,\prime} + U_j^2 - \dfrac{U_0^{2\,\prime}}{U_0^2} U_{j+1}^2,\ j = 0, 1, 2, \ldots, k-1$ and $U_k^2 \equiv 1$.

Proof Since $g_3 = f''' - \varphi$, we have

$$f''' = g_3 + \varphi, \qquad f^{(4)} = g_3' + \varphi', \qquad \ldots, \qquad f^{(k+3)} = g_3^{(k)} + \varphi^{(4)}. \tag{18}$$

The derivation of (16) is

$$f^{(k+3)} + U_{k-1}^2 f^{(k+2)} + \big(U_{k-1}^{2\,\prime} + U_{k-2}^2\big)f^{(k+1)} + \cdots + \big(U_1^{2\,\prime} + U_0^2\big)f''' + U_0^{2\,\prime}f'' = 0. \tag{19}$$

Substituting (16) into (18), we have

$$
f^{(k+3)} + \left(U_{k-1}^2 - \frac{U_0^{2'}}{U_0^2}\right)f^{(k+2)} + \left(U_{k-1}^2{}' + U_{k-2}^2 - \frac{U_0^{2'}}{U_0^2}U_{k-1}^2\right)f^{(k+1)}
$$
$$
+ \cdots + \left(U_1^{2'} + U_0^2 - \frac{U_0^{2'}}{U_0^2}U_1^2\right)f''' = 0. \tag{20}
$$

Set $U_j^3 = U_{j+1}^2{}' + U_j^2 - \frac{U_0^{2'}}{U_0^2}U_{j+1}^2$, $j = 0, 1, 2, \ldots, k-1$ and $U_k^2 \equiv 1$, then from (20) we have

$$
f^{(k+3)} + U_{k-1}^3 f^{(k+2)} + U_{k-2}^3 f^{(k+1)} + \cdots + U_0^3 f''' = 0. \tag{21}
$$

Substituting (18) into (21), we can get (17).

Thus, we complete the proof of Lemma 2.4. □

Lemma 2.5 *Assume $f \not\equiv 0$ is a solution of Equation (3), set $g_i = f^{(i)} - \varphi$, then g_i satisfies the equation*

$$
g_i^{(k)} + U_{k-1}^i g_i^{(k-1)} + \cdots + U_0^i g_i = -\left[\varphi^{(k)} + U_{k-1}^i \varphi^{(k-1)} + \cdots + U_0^i \varphi\right], \tag{22}
$$

where $U_j^i = U_{j+1}^{i-1}{}' + U_j^{i-1} - \frac{U_0^{i-1'}}{U_0^{i-1}}U_{j+1}^{i-1}$, $j = 0, 1, 2, \ldots, k-1$, $U_k^{i-1} \equiv 1$ and $i \in \mathbb{N}$.

Proof The inductive method will be used to prove it.

At first, from Lemmas 2.2-2.4, we get that (22) holds for $i = 1, 2, 3$.

Next, suppose that $g_i = f^{(i)} - \varphi$, $i \leq n$, $n \in \mathbb{N}$ satisfy (22). Thus, $g_n = f^{(n)} - \varphi$ satisfies the equation

$$
g_n^{(k)} + U_{k-1}^n g_n^{(k-1)} + \cdots + U_0^n g_n = -\left[\varphi^{(k)} + U_{k-1}^n \varphi^{(k-1)} + \cdots + U_0^n \varphi\right], \tag{23}
$$

where $U_j^n = U_{j+1}^{n-1}{}' + U_j^{n-1} - \frac{U_0^{n-1'}}{U_0^{n-1}}U_{j+1}^{n-1}$, $j = 0, 1, 2, \ldots, k-1$ and $U_k^{n-1} \equiv 1$.

Since $g_n = f^{(n)} - \varphi$, we have

$$
f^{(n+1)} = g_n' + \varphi', \qquad f^{(n+2)} = g_n'' + \varphi'', \qquad \ldots, \qquad f^{(k+n)} = g_n^{(k)} + \varphi^{(k)}. \tag{24}
$$

From (23) and (24), we have

$$
f^{(k+n)} + U_{k-1}^n f^{(k+n-1)} + U_{k-2}^n f^{(k+n-2)} + \cdots + U_0^n f^{(n)} = 0. \tag{25}
$$

Now we will prove that $g_{n+1} = f^{(n+1)} - \varphi$ satisfies (22). Since $g_{n+1} = f^{(n+1)} - \varphi$, we have

$$
f^{(n+2)} = g_{n+1}' + \varphi', \qquad f^{(n+3)} = g_{n+1}'' + \varphi'', \qquad \ldots, \qquad f^{(n+k+1)} = g_{n+1}^{(k)} + \varphi^{(k)}. \tag{26}
$$

The derivation of (25) is

$$
f^{(k+n+1)} + U_{k-1}^n f^{(k+n)} + \left(U_{k-1}^n{}' + U_{k-2}^n\right)f^{(k+n-1)} + \cdots + U_0^{n'} f^{(n)} = 0. \tag{27}
$$

Substituting (25) into (27), we have

$$f^{(k+n+1)} + \left(U_{k-1}^n - \frac{U_0^{n'}}{U_0^n} \right) f^{(k+n)} + \left(U_{k-1}^n{}' + U_{k-2}^n - \frac{U_0^{n'}}{U_0^n} U_{k-1}^n \right) f^{(k+n-1)}$$
$$+ \cdots + \left(U_1^n{}' + U_0^n - \frac{U_0^{n'}}{U_0^n} U_1^n \right) f^{(n)} = 0. \tag{28}$$

From the definition of U_j^i and (28), we have

$$f^{(k+n+1)} + U_{k-1}^{n+1} f^{(k+n)} + \cdots + U_1^{n+1} f^{(n+1)} + U_0^{n+1} f^{(n)} = 0, \tag{29}$$

where $U_j^{n+1} = U_{j+1}^n{}' + U_j^n - \frac{U_0^{n'}}{U_0^n} U_{j+1}^n$, $j = 0, 1, 2, \ldots, k-1$ and $U_k^n \equiv 1$. Substituting (26) into (29), we have

$$g_{n+1}^{(k)} + U_{k-1}^{n+1} g_{n+1}^{(k-1)} + \cdots + U_0^{n+1} g_{n+1} = -\left[\varphi^{(k)} + U_{k-1}^{n+1} \varphi^{(k-1)} + \cdots + U_0^{n+1} \varphi \right]. \tag{30}$$

This completes the proof of Lemma 2.5. □

Similar to the proof of [19, Lemma 3.8], we can get the following lemma.

Lemma 2.6 *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma \geq 0$, then there exists a set $E \subset [1, +\infty)$ with infinite logarithmic measure such that for all $r \in E$, we have*

$$\lim_{r \to \infty} \frac{\log T(r, f)}{\log r} = \sigma, \quad r \in E.$$

Lemma 2.7 *Let $A_0(z)$, $A_1(z)$, $\ldots$, $A_{k-1}(z)$ be entire functions with finite order and satisfy $\max\{\sigma(A_j) : j = 1, 2, \ldots, k-1\} = \sigma_1 < \sigma(A_0) < \infty$, and set*

$$U_j^1 = A_{j+1}' + A_j - \frac{A_0'}{A_0} A_{j+1}$$

and

$$U_j^i = U_{j+1}^{i-1}{}' + U_j^{i-1} - \frac{U_0^{i-1'}}{U_0^{i-1}} U_{j+1}^{i-1},$$

where $j = 0, 1, 2, \ldots, k-1$, $A_k \equiv 1$, $U_k^{i-1} \equiv 1$ and $i \in \mathbb{N}$. Then there exists a set E with infinite logarithmic measure such that

$$\lim_{r \to \infty} \frac{\log m(r, U_0^i)}{\log r} = \sigma(A_0) > \limsup_{r \to \infty} \frac{\max_{1 \leq j \leq k-1} \{\log m(r, U_j^i)\}}{\log r} = \sigma_1, \quad r \in E. \tag{31}$$

Proof The inductive method will be used to prove it.

First, when $i = 1$, *i.e.*, $U_j^1 = A_{j+1}' + A_j - \frac{A_0'}{A_0} A_{j+1}$, $j = 0, 1, 2, \ldots, k-1$ and $A_k \equiv 1$.

When $j = 0$, that is $U_0^1 = A_1' + A_0 - \frac{A_0'}{A_0} A_1$. Then, we have

$$m(r, U_0^1) \leq m(r, A_1) + m(r, A_0) + m\left(r, \frac{A_1'}{A_1} \right) + m\left(r, \frac{A_0'}{A_0} \right) + O(1). \tag{32}$$

From $A_0 = -A_1' + U_0^1 + \frac{A_0'}{A_0} A_1$, we have

$$m(r, A_0) \leq m(r, A_1) + m(r, U_0^1) + m\left(r, \frac{A_1'}{A_1}\right) + m\left(r, \frac{A_0'}{A_0}\right) + O(1). \tag{33}$$

When $j \neq 0$, $j = 1, 2, \ldots, k-1$, from the definitions of U_j^1, we have

$$m(r, U_j^1) \leq m(r, A_{j+1}) + m(r, A_j) + m\left(r, \frac{A_0'}{A_0}\right) + m\left(r, \frac{A_j'}{A_j}\right) + O(1). \tag{34}$$

Since $A_j(z)$ are entire functions with $\max\{\sigma(A_j) : j = 1, 2, \ldots, k-1\} < \sigma(A_0) < \infty$ and (34), we have

$$\max_{1 \leq j \leq k-1} \{m(r, U_j^1)\} \leq \max_{1 \leq j \leq k-1} \{m(r, A_j) + o(m(r, A_0)) + O(\log r)\}. \tag{35}$$

From (32), (33), (35) and Lemma 2.6, there exists a set $E \subset [1, +\infty)$ with infinite logarithmic measure such that

$$\lim_{r \to \infty} \frac{\log m(r, U_0^1)}{\log r} = \sigma(A_0) > \sigma_1 = \limsup_{r \to \infty} \frac{\max_{1 \leq j \leq k-1}\{\log m(r, A_j)\}}{\log r}$$

$$\geq \limsup_{r \to \infty} \frac{\max_{1 \leq j \leq k-1}\{\log m(r, U_j^1)\}}{\log r}, \quad r \in E. \tag{36}$$

Now, suppose that (31) holds for $i \leq n$, $n \in \mathbb{N}$, that is, there exists a set E with infinite logarithmic measure such that

$$\lim_{r \to \infty} \frac{\log m(r, U_0^n)}{\log r} = \sigma(A_0) > \limsup_{r \to \infty} \frac{\max_{1 \leq j \leq k-1}\{\log m(r, U_j^n)\}}{\log r} = \sigma_1. \tag{37}$$

Next, we prove that (31) holds for $i = n+1$. Since $i = n+1$, we have $U_j^{n+1} = U_{j+1}^{n}{}' + U_j^n - \frac{U_0^{n}{}'}{U_0^n} U_{j+1}^n$ where $j = 0, 1, 2, \ldots, k-1$, and $U_k^n \equiv 1$. When $j = 0$, When $j = 0$, that is $U_0^{n+1} = U_1^{n}{}' + U_0^n - \frac{U_0^{n}{}'}{U_0^n} U_1^n$. Then, we have

$$m(r, U_0^{n+1}) \leq m(r, U_0^n) + m(r, U_1^n) + m\left(r, \frac{U_0^{n}{}'}{U_0^n}\right) + m\left(r, \frac{U_1^{n}{}'}{U_1^n}\right) + O(1). \tag{38}$$

And since $U_0^n = U_1^{n}{}' + U_0^{n+1} - \frac{U_0^{n}{}'}{U_0^n} U_1^n$, we have

$$m(r, U_0^n) \leq m(r, U_0^{n+1}) + m(r, U_1^n) + m\left(r, \frac{U_0^{n}{}'}{U_0^n}\right) + m\left(r, \frac{U_1^{n}{}'}{U_1^n}\right) + O(1). \tag{39}$$

When $j \neq 0$, from the definitions of U_j^{n+1}, $j = 1, 2, \ldots, k-1$ and $U_k^n \equiv 1$, we have

$$m(r, U_j^{n+1}) \leq m(r, U_{j+1}^n) + m(r, U_j^n) + m\left(r, \frac{U_{j+1}^{n}{}'}{U_{j+1}^n}\right) + m\left(r, \frac{U_0^{n}{}'}{U_0^n}\right) + O(1). \tag{40}$$

From (37)-(40), there exists a set E with infinite logarithmic measure such that

$$
\begin{aligned}
\lim_{r \to \infty} \frac{\log m(r, U_0^{n+1})}{\log r} = \lim_{r \to \infty} \frac{\log m(r, U_0^n)}{\log r} &= \sigma(A_0) > \sigma_1 \\
&= \limsup_{r \to \infty} \frac{\max_{1 \le j \le k-1}\{\log m(r, U_j^n)\}}{\log r} \\
&= \limsup_{r \to \infty} \frac{\max_{1 \le j \le k-1}\{\log m(r, U_j^{n+1})\}}{\log r}.
\end{aligned}
\tag{41}
$$

Thus, the proof of Lemma 2.7 is completed. □

Lemma 2.8 *Let $H_j(z)$ ($j = 0, 1, \ldots, k-1$) be meromorphic functions of finite order. If*

$$
\limsup_{r \to \infty} \frac{\max_{1 \le j \le k-1}\{\log m(r, H_j)\}}{\log r} = \beta_1
$$

and there exists a set E_1 with infinite logarithmic measure such that

$$
\lim_{r \to \infty} \frac{\log m(r, H_0)}{\log r} = \beta_2 > \beta_1
$$

holds for all $r \in E_1$, then every meromorphic solution of

$$
f^{(k)} + H_{k-1} f^{(k-1)} + \cdots + H_1 f' + H_0 f = 0
\tag{42}
$$

satisfies $\sigma_2(f) \ge \beta_2$.

Proof Assume that $f(z)$ is a meromorphic solution of (42). From (42), we have

$$
m(r, H_0) \le m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{f^{(k-1)}}{f}\right) + \cdots + m\left(r, \frac{f'}{f}\right) + \sum_{j=1}^{k-1} m(r, H_j) + O(1).
\tag{43}
$$

By the lemma on logarithmic derivative and (43), we have

$$
m(r, H_0) \le O\{\log r T(r, f)\} + \sum_{j=1}^{k-1} m(r, H_j), \quad r \notin E_2,
\tag{44}
$$

where $E_2 \subset [1, +\infty)$ is a set with finite linear measure. From the assumptions of Lemma 2.6, there exists a set E_1 with infinite logarithmic measure such that for all $|z| = r \in E_1 - E_2$ we have

$$
r^{\beta_2 - \varepsilon} \le O\{\log r T(r, f)\} + (k-1) r^{\beta_1 + \varepsilon},
\tag{45}
$$

where $0 < 2\varepsilon < \beta_2 - \beta_1$. From (45), we have $\sigma_2(f) \ge \beta_2$. □

Lemma 2.9 (see [12]) *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, $\Gamma = \{(k_1, j_1), \ldots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers which satisfy $k_i > j_i \ge 0$ for $i = 1, \ldots, m$. And let $\varepsilon > 0$ be a given constant, then there exists a set $E_3 \subset (1, \infty)$ that has*

finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_3$ and $(k, j) \in \Gamma$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 2.10 (see [20]) *Let $f(z)$ be an entire function with $\sigma(f) = \sigma$, $\tau(f) = \tau$, $0 < \sigma < \infty$, $0 < \tau < \infty$, then for any given $\beta < \tau$, there exists a set $E_4 \subset [1, +\infty)$ that has infinite logarithmic measure such that for all $r \in E_4$, we have*

$$\log M(r, f) > \beta r^{\sigma}.$$

Lemma 2.11 *Let $A_0(z)$, $A_1(z)$, ..., $A_{k-1}(z)$ be entire functions with finite order and satisfy $0 < \sigma(A_0) = \sigma(A_1) = \cdots = \sigma(A_{k-1}) = \sigma_2 < \infty$ and $\max\{\tau(A_j) : j = 1, 2, \ldots, k-1\} = \tau_1 < \tau(A_0) = \tau$, and let U_j^1, U_j^i be stated as in Lemma 2.7, then for any given ε $(0 < 2\varepsilon < \tau - \tau_1)$, there exists a set E_5 with infinite logarithmic measure such that*

$$\left| U_j^i \right| \leq \exp\{(\tau_1 + \varepsilon)r^{\sigma_2}\}, \qquad \left| U_0^i \right| \geq \exp\{(\tau - \varepsilon)r^{\sigma_2}\}, \tag{46}$$

where $i \in \mathbb{N}$ and $j = 1, 2, \ldots, k-1$.

Proof The inductive method will be used to prove it.

(i) We first prove that U_j^i $(j = 0, 1, \ldots, k-1)$ satisfy (46) when $i = 1$. From the definition of $U_j^1 = A_{j+1}' + A_j - \frac{A_0'}{A_0}A_{j+1}$ $(j \neq 0)$ and $U_0^1 = A_1' + A_0 - \frac{A_0'}{A_0}A_1$, we have

$$\left| U_0^1 \right| \geq |A_0| - |A_1|\left(\left| \frac{A_1'}{A_1} \right| + \left| \frac{A_0'}{A_0} \right| \right) \tag{47}$$

and

$$\left| U_j^1 \right| \leq |A_j| + |A_{j+1}|\left(\left| \frac{A_{j+1}'}{A_{j+1}} \right| + \left| \frac{A_0'}{A_0} \right| \right), \quad j = 1, 2, \ldots, k-1, \qquad A_k \equiv 1. \tag{48}$$

From Lemma 2.9, Lemma 2.10 and (47)-(48), for any ε $(0 < 4\varepsilon < \tau - \tau_1)$, there exists a set E_5 with infinite logarithmic measure such that

$$\left| U_0^1 \right| \geq \exp\left\{ \left(\tau - \frac{\varepsilon}{4} \right)r^{\sigma_2} \right\} - 2\exp\left\{ \left(\tau_1 + \frac{\varepsilon}{4} \right)r^{\sigma_2} \right\}r^M \geq \exp\left\{ \left(\tau - \frac{\varepsilon}{2} \right)r^{\sigma_2} \right\} \tag{49}$$

and

$$\begin{aligned}
\left| U_j^1 \right| &\leq \exp\left\{ \left(\tau_1 + \frac{\varepsilon}{4} \right)r^{\sigma_2} \right\} + 2\exp\left\{ \left(\tau_1 + \frac{\varepsilon}{4} \right)r^{\sigma_2} \right\}r^M \\
&\leq \exp\left\{ \left(\tau_1 + \frac{\varepsilon}{2} \right)r^{\sigma_2} \right\}, \quad j \neq 0,
\end{aligned} \tag{50}$$

where $M > 0$ is a constant, not necessarily the same at each occurrence.

(ii) Next, we show that U_j^i ($j = 0, 1, 2, \ldots, k-1$) satisfy (46) when $i = 2$. From $U_0^2 = U_1^{1'} + U_0^1 - \frac{U_0^{1'}}{U_0^1} U_1^1$ and $U_j^2 = U_{j+1}^{1}{}' + U_j^1 - \frac{U_0^{1'}}{U_0^1} U_{j+1}^1$ ($j = 0, 1, \ldots, k-1$) and $U_k^1 \equiv 1$, we have

$$\left| U_0^2 \right| \geq \left| U_0^1 \right| - \left| U_1^1 \right| \left(\left| \frac{U_1^{1'}}{U_1^1} \right| + \left| \frac{U_0^{1'}}{U_0^1} \right| \right) \tag{51}$$

and

$$\left| U_j^2 \right| \leq \left| U_j^1 \right| + \left| U_{j+1}^1 \right| \left(\left| \frac{U_{j+1}^{1}{}'}{U_{j+1}^1} \right| + \left| \frac{U_0^{1'}}{U_0^1} \right| \right), \quad j = 1, 2, \ldots, k-1. \tag{52}$$

By the conclusions of (i) and Lemma 2.9, (49)-(52), for all $|z| = r \in E_5$,

$$\left| U_0^2 \right| \geq \exp\left\{ \left(\tau - \frac{\varepsilon}{2} \right) r^{\sigma_2} \right\} - 2 \exp\left\{ \left(\tau_1 + \frac{\varepsilon}{2} \right) r^{\sigma_2} \right\} r^M \geq \exp\left\{ (\tau - \varepsilon) r^{\sigma_2} \right\} \tag{53}$$

and

$$\left| U_j^2 \right| \leq \exp\left\{ \left(\tau_1 + \frac{\varepsilon}{2} \right) r^{\sigma_2} \right\} + 2 \exp\left\{ \left(\tau_1 + \frac{\varepsilon}{2} \right) r^{\sigma_2} \right\} r^M \leq \exp\left\{ (\tau_1 + \varepsilon) r^{\sigma_2} \right\}, \quad j \neq 0. \tag{54}$$

(iii) Now, suppose that (46) holds for $i \leq n$, $n \in \mathbb{N}$, that is, for any given ε ($0 < 4\varepsilon < \tau - \tau_1$), there exists a set E_5 with infinite logarithmic measure such that

$$\left| U_j^i \right| \leq \exp\left\{ (\tau_1 + \varepsilon) r^{\sigma_2} \right\}, \qquad \left| U_0^i \right| \geq \exp\left\{ (\tau - \varepsilon) r^{\sigma_2} \right\} \quad (i \leq n, j = 1, 2, \ldots, k-1). \tag{55}$$

From $U_0^{n+1} = U_1^{n'} + U_0^n - \frac{U_0^{n'}}{U_0^n} U_1^n$ and $U_j^{n+1} = U_{j+1}^{n}{}' + U_j^n - \frac{U_0^{n'}}{U_0^n} U_{j+1}^n$ ($j = 0, 1, \ldots, k-1$) and $U_k^n \equiv 1$, we have

$$\left| U_0^{n+1} \right| \geq \left| U_0^n \right| - \left| U_1^n \right| \left(\left| \frac{U_1^{n'}}{U_1^n} \right| + \left| \frac{U_0^{n'}}{U_0^n} \right| \right) \tag{56}$$

and

$$\left| U_j^{n+1} \right| \leq \left| U_j^n \right| + \left| U_{j+1}^n \right| \left(\left| \frac{U_{j+1}^{n}{}'}{U_{j+1}^n} \right| + \left| \frac{U_0^{n'}}{U_0^n} \right| \right), \quad j = 1, 2, \ldots, k-1. \tag{57}$$

Then, from Lemma 2.9 and (55)-(57), for all $|z| = r \in E_5$,

$$\left| U_j^{n+1} \right| \leq \exp\left\{ (\tau_1 + \varepsilon) r^{\sigma_2} \right\} + 2 \exp\left\{ (\tau_1 + \varepsilon) r^{\sigma_2} \right\} r^M \leq \exp\left\{ (\tau_1 + 2\varepsilon) r^{\sigma_2} \right\}, \quad j \neq 0 \tag{58}$$

and

$$\left| U_0^{n+1} \right| \geq \exp\left\{ (\tau - \varepsilon) r^{\sigma_2} \right\} - 2 \exp\left\{ (\tau_1 + \varepsilon) r^{\sigma_2} \right\} r^M \geq \exp\left\{ (\tau - 2\varepsilon) r^{\sigma_2} \right\}. \tag{59}$$

Thus, we complete the proof of Lemma 2.11. □

Lemma 2.12 *Let* $B_j(z)$, $j = 0, 1, \ldots, k - 1$ *be meromorphic functions with* $\max\{\sigma(B_j) : j = 1, 2, \ldots, k-1\} = \sigma_4 < \sigma(B_0) = \sigma_3$ *and* $\delta(\infty, B_0) = \underline{\lim}_{r \to \infty} \frac{m(r, B_0)}{T(r, B_0)} > 0$. *Then every meromorphic solution f of the equation*

$$f^{(k)} + B_{k-1}f^{(k-1)} + \cdots + B_1 f' + B_0 f = 0 \tag{60}$$

satisfies $\sigma_2(f) \geq \sigma_3$.

Proof Let f be a meromorphic solution of Equation (60), by (60), we have

$$m(r, B_0) \leq m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{f^{(k-1)}}{f}\right) + \cdots + m\left(r, \frac{f'}{f}\right) + \sum_{j=1}^{k-1} m(r, B_j)$$

$$\leq O\{\log r T(r, f)\} + \sum_{j=1}^{k-1} T(r, B_j), \quad r \notin E_6, \tag{61}$$

where $E_6 \subset [1, +\infty)$ is a set with finite linear measure. By Lemma 2.6, there exists a set E with infinite logarithmic measure such that for all $|z| = r \in E$, we have

$$\lim_{r \to \infty} \frac{\log T(r, B_0)}{\log r} = \sigma_3, \quad r \notin E. \tag{62}$$

Since $\delta(\infty, B_0) > 0$, then for any given ε ($0 < 2\varepsilon < \sigma_3 - \sigma_4$) and for all $r \in E$, by (62), we have

$$m(r, B_0) \geq r^{\sigma_3 - \varepsilon}. \tag{63}$$

From (61) and (63), we have

$$r^{\sigma_3 - \varepsilon} \leq O\{\log r T(r, f)\} + (k - 1)r^{\sigma_4 + \varepsilon}, \quad r \in E - E_6. \tag{64}$$

From (64), we can get $\sigma_2(f) \geq \sigma_3 = \sigma(B_0)$. □

Lemma 2.13 (see [12]) *Let $f(z)$ be a transcendental meromorphic function and $\alpha > 1$ be a given constant, for any given $\varepsilon > 0$, there exists a set $E_7 \subset [1, \infty)$ that has finite logarithmic measure and a constant $M > 0$ that depends only on α and (m, n) ($m, n \in \{0, \ldots, k\}$ with $m < n$) such that for all z satisfying $|z| = r \notin [0, 1] \cup E_7$, we have*

$$\left| \frac{f^{(n)}(z)}{f^{(m)}(z)} \right| \leq M \left(\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f) \right)^{n - m}.$$

Lemma 2.14 *Let $B_j(z)$, $j = 0, 1, \ldots, k - 1$ be meromorphic functions of finite order. If there exist positive constants σ_5, β_3, β_4 ($0 < \beta_3 < \beta_4$) and a set E_8 with infinite logarithmic measure such that*

$$\max\{|B_j(z)| : j = 1, 2, \ldots, k - 1\} \leq \exp\{\beta_3 r^{\sigma_5}\}, \quad |B_0(z)| \geq \exp\{\beta_4 r^{\sigma_5}\}$$

hold for all $|z| = r \in E_8$, then every meromorphic solution of (60) satisfies $\sigma_2(f) \geq \sigma_5$.

Proof Suppose that f is a meromorphic solution of (60), by (60), we have

$$\left|B_0(z)\right| \leq \left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1}\left|B_j(z)\right|\left|\frac{f^{(j)}}{f}\right|. \tag{65}$$

By Lemma 2.13, there exists a set E_7 with finite logarithmic measure such that for all $|z| = r \notin E_7$, we have

$$\left|\frac{f^{(j)}}{f}\right| \leq M\left[T(2r,f)\right]^j, \quad j = 1, 2, \ldots, k, \tag{66}$$

where $M > 0$ is a constant. By (65), (66) and the assumptions of Lemma 2.14, for all $|z| = r \in E_8 - E_7$, we have

$$\exp\left\{\beta_4 r^{\sigma_5}\right\} \leq M\left[T(2r,f)\right]^k \exp\left\{\beta_3 r^{\sigma_5}\right\}. \tag{67}$$

Since $0 < \beta_3 < \beta_4$, by (67), we have $\sigma_2(f) \geq \sigma_5$. □

Lemma 2.15 (see [19]) *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be meromorphic functions, if f is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = F,$$

then we have the following statements:

  (i) if $\max\{\sigma(F), \sigma(A_j); j = 0, 1, \ldots, k-1\} < \sigma(f) = \sigma \leq \infty$, then $\sigma(f) = \lambda(f) = \overline{\lambda}(f)$;
  (ii) if $\max\{\sigma_2(F), \sigma_2(A_j); j = 0, 1, \ldots, k-1\} < \sigma_2(f) = \sigma$, then $\sigma_2(f) = \lambda_2(f) = \overline{\lambda}_2(f)$.

Lemma 2.16 (see [20, Theorem 1]) *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying $0 < \sigma(A_{k-1}) = \cdots = \sigma(A_1) = \sigma(A_0) < \infty$, $\max\{\tau(A_j) : j = 1, 2, \ldots, k-1\} < \tau(A_0) < \infty$, then every solution $f \not\equiv 0$ of (3) satisfies $\sigma_2(f) = \sigma(A_0)$.*

3 Proofs of theorems

3.1 The proof of Theorem 1.1

To prove the conclusions of Theorem 1.1, we will consider two cases as follows.

   Case 1. Suppose that $\max\{\sigma(A_j) : j = 1, 2, \ldots, k-1\} < \sigma(A_0) < \infty$.

   (i) First, we prove that $\overline{\lambda}_2(f - \varphi) = \sigma_2(f)$. Assume that $f \not\equiv 0$ is a solution of (3), from ref. [9], we have $\sigma_2(f) = \sigma(A_0)$. Set $g = f - \varphi$. Since $\sigma_2(\varphi) < \sigma(A_0)$, then $\sigma_2(g) = \sigma_2(f) = \sigma(A_0)$ and $\overline{\lambda}_2(g) = \overline{\lambda}_2(f - \varphi)$. By Lemma 2.1, we get that g satisfies Equation (4). Set $F = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \cdots + A_0\varphi$. If $F \equiv 0$, from ref. [9], we have $\sigma_2(\varphi) = \sigma(A_0)$, we can get a contradiction. Then $F \not\equiv 0$, by Lemma 2.15, we have $\overline{\lambda}_2(g) = \lambda_2(g) = \sigma_2(g) = \sigma(A_0)$. Thus, we have $\overline{\lambda}_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = \sigma(A_0)$.

   (ii) Second, we prove that $\overline{\lambda}_2(f' - \varphi) = \sigma_2(f)$. Set $g_1 = f' - \varphi$, then $\sigma_2(g_1) = \sigma_2(f) = \sigma(A_0)$. From Lemma 2.2, we get that g_1 satisfies Equation (5). Set $F_1 = \varphi^{(k)} + U_{k-1}^1\varphi^{(k-1)} + \cdots + U_0^1\varphi$, where U_j^1 $(j = 0, 1, \ldots, k-1)$ are stated as in Lemma 2.2. If $F_1 \equiv 0$, from Lemma 2.7 and Lemma 2.8, we have $\sigma_2(\varphi) \geq \sigma(A_0)$, a contradiction with $\sigma_2(\varphi) < \sigma(A_0)$. Hence $F_1 \not\equiv 0$. By Lemma 2.15, we get $\overline{\lambda}_2(f' - \varphi) = \lambda_2(f' - \varphi) = \sigma_2(f)$.

(iii) Now, we prove that $\overline{\lambda}_2(f'' - \varphi) = \sigma_2(f)$. Set $g_2 = f'' - \varphi$, then $\sigma_2(g_2) = \sigma_2(f) = \sigma(A_0)$. From Lemma 2.3, we get that g_2 satisfies Equation (12). Set $F_2 = \varphi^{(k)} + U_{k-1}^2 \varphi^{(k-1)} + \cdots + U_0^2 \varphi$, where U_j^2 $(j = 0, 1, \ldots, k-1)$ are stated as in Lemma 2.3. If $F_2 \equiv 0$, from Lemma 2.7 and Lemma 2.8, we have $\sigma_2(\varphi) \geq \sigma(A_0)$, a contradiction with $\sigma_2(\varphi) < \sigma(A_0)$. Hence $F_2 \not\equiv 0$. By Lemma 2.15, we get $\overline{\lambda}_2(f'' - \varphi) = \lambda_2(f'' - \varphi) = \sigma_2(f)$.

(iv) Set $g_3 = f''' - \varphi$. From Lemmas 2.4, 2.7, 2.8 and Lemma 2.15, using the same argument as in Case 1(iii), we can get $\overline{\lambda}_2(f''' - \varphi) = \lambda_2(f''' - \varphi) = \sigma_2(f)$ easily.

(v) We will prove that $\overline{\lambda}_2(f^{(i)} - \varphi) = \sigma_2(f)$ $(i > 3, i \in \mathbb{N})$. Set $g_i = f^{(i)} - \varphi$, then $\sigma_2(g_i) = \sigma_2(f) = \sigma(A_0)$. From Lemma 2.5, we have g_i satisfies Equation (22). Set $F_i = \varphi^{(k)} + U_{k-1}^i \varphi^{(k-1)} + \cdots + U_0^i \varphi$, where U_j^i $(j = 0, 1, \ldots, k-1; i \in \mathbb{N})$ are stated as in Lemma 2.5. If $F_i \equiv 0$, from Lemma 2.7 and Lemma 2.8, we have $\sigma_2(\varphi) \geq \sigma(A_0)$, a contradiction with $\sigma_2(\varphi) < \sigma(A_0)$. Hence $F_i \not\equiv 0$. By Lemma 2.15, we get $\overline{\lambda}_2(f^{(i)} - \varphi) = \lambda_2(f^{(i)} - \varphi) = \sigma_2(f)$.

Case 2. Suppose that $0 < \sigma(A_{k-1}) = \cdots = \sigma(A_1) = \sigma(A_0) < \infty$ and $\max\{\tau(A_j) : j = 1, 2, \ldots, k-1\} = \tau_1 < \tau(A_0) = \tau$.

(i) We first prove that $\overline{\lambda}_2(f - \varphi) = \sigma_2(f)$. Assume that $f \not\equiv 0$ is a solution of (3), by Lemma 2.16, we have $\sigma_2(f) = \sigma(A_0) > 0$. Set $g = f - \varphi$. Since $\varphi \not\equiv 0$ is an entire function satisfying $\sigma_2(\varphi) < \sigma(A_0)$, then we have $\sigma_2(g) = \sigma_2(f) = \sigma(A_0)$ and $\overline{\lambda}_2(g) = \overline{\lambda}_2(f - \varphi)$. From Lemma 2.1, we get that g satisfies Equation (4). We will affirm $F \not\equiv 0$. If $F \equiv 0$, by Lemma 2.16, we get $\sigma_2(\varphi) = \sigma_2(A_0)$, a contradiction. Hence $F \not\equiv 0$. From the assumptions of Theorem 1.1, we get

$$\max\{\sigma_2(F), \sigma_2(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_2(g) = \sigma(A_0).$$

From Lemma 2.15(ii), we have $\overline{\lambda}_2(f - \varphi) = \lambda_2(f - \varphi) = \sigma_2(f) = \sigma(A_0)$.

(ii) Now we prove that $\overline{\lambda}_2(f' - \varphi) = \sigma_2(f)$. Let $g_1 = f' - \varphi$. Since $\sigma_2(\varphi) < \sigma(A_0)$, we have $\sigma_2(g_1) = \sigma_2(f) = \sigma(A_0)$. By Lemma 2.2, we have g_1 satisfies Equation (5). If $F_1 \equiv 0$, from Lemma 2.11 and Lemma 2.14, we have $\sigma_2(\varphi) \geq \sigma(A_0)$. Then we can get a contradiction with $\sigma_2(\varphi) < \sigma(A_0)$. Therefore, we have $F_1 \not\equiv 0$. By (5) and Lemma 2.15, we have $\overline{\lambda}_2(f' - \varphi) = \lambda_2(f' - \varphi) = \sigma_2(f) = \sigma(A_0)$.

Similar to the arguments as in Case 1(iii)-(v) and by using Lemmas 2.3-2.5, 2.11 and Lemma 2.14, we can get

$$\overline{\lambda}_2(f^{(i)} - \varphi) = \lambda_2(f^{(i)} - \varphi) = \sigma_2(f) = \sigma(A_0) \quad (i \in \mathbb{N}).$$

Thus, the proof of Theorem 1.1 is completed.

3.2 The proof of Theorem 1.2

Since $A_j(z)$ $(j = 1, 2, \ldots, k-1)$ are polynomials and $A_0(z)$ is a transcendental entire function, then we have that $A_j(z)$ $(j = 0, 1, 2, \ldots, k-1)$ satisfy the condition of Theorem 1.1. By using the same argument as in Theorem 1.1 and Lemma 2.15(i), we can get the conclusions of Theorem 1.2 easily.

Thus, we omit the process of proving Theorem 1.2.

3.3 The proof of Theorem 1.3

According to the conditions of Theorem 1.3, we can easily get the conclusions of Theorem 1.3 by using the similar argument as in Theorem 1.1 and Lemma 2.15.

Authors' contributions

HYX completed the main part of this article, HYX, JT and XMZ corrected the main theorems. All authors read and approved the final manuscript.

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