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Stability of functional equation obtained through a fixed-point alternative in intuitionistic fuzzy normed spaces

SA Mohiuddine* and Mohammed A Alghamdi

*Correspondence: mohiuddine@gmail.com
Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia

Abstract

In this paper, we determine the stability of a generalized Hyers-Ulam-Rassias-type theorem concerning the additive functional equation $2f(\frac{x+y+z}{2}) = f(x) + f(y) + f(z)$ in the framework of intuitionistic fuzzy normed spaces through the fixed-point alternative. Further, we prove some stability results of an additive functional equation in this setup through the direct method.

MSC: 39B52; 39B82; 46S40

Keywords: t-norm; t-conorm; additive functional equation; intuitionistic fuzzy normed space; fixed point

1 Introduction

The study of the stability problem of functional equations originated from a question of S.M. Ulam [30] concerning the stability of group homomorphisms.

Let $(G,*)$ be a group and $(G',\circ,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G \to G'$ satisfies the inequality $d(h(x*y), h(x)\circ h(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $H : G \to G'$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G$?

If the answer is affirmative, we would say the equation of homomorphism $H(x*y) = H(x)\circ H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers theorem was generalized by Aoki [3] for additive mappings and by Rassias [26] for linear mappings by considering an unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$ for all $\epsilon > 0$ and $p \in [0,1)$. Following the same approach as Rassias, Gajda [7] gave an affirmative solution of this problem for $p > 1$ and also proved that it is possible to solve the Rassias-type theorem for $p = 1$. A further generalization was obtained by Găvruta [8], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x,y)$. The paper of Rassias has significantly influenced the development of what we now call the Hyers-Ulam-Rassias stability of functional equations. Since then, several stability problems for various functional equations have been investigated in [1, 2, 6, 9–12, 27, 32]. Quite recently, the stability problem for the Pexiderized quadratic functional equation, Jensen functional equation, cubic functional equation, functional equations associated with inner product spaces, and a mixed type additive-cubic functional equation were considered in [15, 17, 21, 29], and

[31], respectively, in the intuitionistic fuzzy normed spaces; while the idea of intuitionistic fuzzy normed space was introduced in [28], and further studied in [18–20, 22–24, 34] to deal with some summability problems.

In 2003, Radu [25] proposed that the fixed-point alternative method is very useful for obtaining the solution of the Ulam problem and obtained the stability of the Cauchy functional equation in Banach spaces through the fixed-point method. Since then, several stability problems of this concept have been established by various authors, *e.g.*, [13, 14, 16, 33] and references therein.

The aim of this paper is to present a relationship between three various disciplines: the theory of fuzzy spaces, the theory of functional equations, and fixed-point theory. We determine the stability of the additive functional equation

$$2f\left(\frac{x+y+z}{2}\right)=f(x)+f(y)+f(z) \tag{1.1}$$

in the setting of intuitionistic fuzzy normed spaces by using the fixed-point alternative theorem. Also, we investigate the stability of this functional equation through the direct method.

2 Definitions, notations and preliminary results

In this section, we recall some notations, basic definitions, and preliminary results used in this paper.

A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:

(a) $*$ is associative and commutative,
(b) $*$ is continuous,
(c) $a*1=a$ for all $a\in[0,1]$,
(d) $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$ for each $a,b,c,d\in[0,1]$.

A binary operation $\diamondsuit:[0,1]\times[0,1]\to[0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:

(a′) $\diamondsuit$ is associative and commutative,
(b′) $\diamondsuit$ is continuous,
(c′) $a\diamondsuit 0=a$ for all $a\in[0,1]$,
(d′) $a\diamondsuit b\leq c\diamondsuit d$ whenever $a\leq c$ and $b\leq d$ for each $a,b,c,d\in[0,1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [28] have recently introduced the concepts of intuitionistic fuzzy normed space and defined convergence and Cauchy sequences in this setting as follows.

Definition 2.1 The five-tuple $(X,\mu,\nu,*,\diamondsuit)$ is said to be an *intuitionistic fuzzy normed spaces* (for short, IFN-Spaces) if X is a vector space, $*$ is a continuous t-norm, $\diamondsuit$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions. For every $x,y\in X$ and $s,t>0$

(i) $\mu(x,t)+\nu(x,t)\leq 1$,
(ii) $\mu(x,t)>0$,
(iii) $\mu(x,t)=1$ if and only if $x=0$,

(iv) $\mu(\alpha x,t) = \mu(x,\frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y,t+s)$,
(vi) $\mu(x,\cdot):(0,\infty) \to [0,1]$ is continuous,
(vii) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
(viii) $\nu(x,t) < 1$,
(ix) $\nu(x,t) = 0$ if and only if $x = 0$,
(x) $\nu(\alpha x,t) = \nu(x,\frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(xi) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y,t+s)$,
(xii) $\nu(x,\cdot):(0,\infty) \to [0,1]$ is continuous,
(xiii) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case, (μ,ν) is called an *intuitionistic fuzzy norm*. For simplicity in notation, we denote the intuitionistic fuzzy normed spaces by (X,μ,ν) instead of $(X,\mu,\nu,*,\diamond)$. For example, let $(X,\|\cdot\|)$ be a normed space, and let $a * b = ab$ and $a \diamond b = \min\{a+b,1\}$ for all $a,b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) := \frac{t}{t+\|x\|} \quad \text{and} \quad \nu(x,t) := \frac{\|x\|}{t+\|x\|}.$$

Then (X,μ,ν) is an intuitionistic fuzzy normed space.

Definition 2.2 Let (X,μ,ν) be an intuitionistic fuzzy normed space. Then a sequence $x = (x_k)$ is said to be

(i) *convergent* to $L \in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L,t) > 1-\epsilon$ and $\nu(x_k - L,t) < \epsilon$ for all $k \geq k_0$. In this case, we write (μ,ν)-$\lim x_k = L$ or $x_k \stackrel{(\mu,\nu)}{\longrightarrow} L$ as $k \to \infty$.
(ii) *Cauchy sequence* with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_\ell,t) > 1-\epsilon$ and $\nu(x_k - x_\ell,t) < \epsilon$ for all $k,\ell \geq k_0$. IFN-space (X,μ,ν) is said to be *complete* if every Cauchy sequence in (X,μ,ν) is convergent in IFN-space. In this case, (X,μ,ν) is called *intuitionistic fuzzy Banach space.*

Remark 2.3 Let $(X,\|\cdot\|)$ be a real normed linear space,

$$\mu := \frac{t}{t+\|x\|} \quad \text{and} \quad \nu := \frac{\|x\|}{t+\|x\|}$$

for all $x \in X$ and $t > 0$. Then $x_n \stackrel{\|\cdot\|}{\longrightarrow} x$ if and only if $x_n \stackrel{(\mu,\nu)}{\longrightarrow} x$.

Recall the following results related to the concept of fixed point.

Theorem 2.4 (Banach's contraction principle) *Let (X,d) be a complete generalized metric space and consider a mapping $J: X \to X$ be a strictly contractive mapping, that is,*

$$d(Jx,Jy) \leq Ld(x,y), \quad \forall x,y \in X$$

for some (Lipschitz constant) $L < 1$. Then

(i) The mapping J has one and only one fixed point $x^* = J(x^*)$;

(ii) *The fixed-point x^* is globally attractive, that is,*

$$\lim_{n\to\infty} J^n x = x^*,$$

for any starting point $x \in X$;

(iii) *One has the following estimation inequalities for all $x \in X$ and $n \geq 0$:*

$$d(J^n x, x^*) \leq L^n d(x, x^*), \tag{2.1}$$

$$d(J^n x, x^*) \leq \frac{1}{1-L} d(J^n x, J^{n+1} x), \tag{2.2}$$

$$d(x, x^*) \leq \frac{1}{1-L} d(x, Jx). \tag{2.3}$$

Theorem 2.5 (The alternative of fixed point [4]) *Suppose we are given a complete generalized metric space (X, d) and a strictly contractive mapping $J : X \to X$, with Lipschitz constant L. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \quad \forall n \geq 0 \tag{2.4}$$

or

$$d(J^n x, J^{n+1} x) < +\infty, \quad \forall n \geq n_\circ \tag{2.5}$$

for some natural number $n_\circ$. Moreover, if the second alternative holds then

(i) *The sequence $(J^n x)$ is convergent to a fixed point y^* of J;*

(ii) *y^* is the unique fixed point of J in the set $Y = \{y \in X, d(J^{n_\circ} x, y) < +\infty\}$;*

(iii) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$, $y \in Y$.*

3 Stability of the additive functional equation through the fixed-point alternative

Using the fixed point alternative, here we can prove the stability of the Hyers-Ulam-Rassias-type theorem in IFN-spaces. First, we prove the following lemma which will be used in our main result.

Lemma 3.1 *Let X be a linear space, (Y, μ, ν) be an IFN-space and $\varphi : X \times X \times X \to [0, \infty)$ be a function. Consider a set $G = \{g : X \to Y\}$ and define*

$$d_s(g, h) = \inf\left\{\gamma \in \mathbb{R}^+ : \mu(g(x) - h(x), \gamma t) \geq \frac{t}{t + \varphi(x, 2x, x)} \text{ and } \nu(g(x) - h(x), \gamma t) \leq \frac{\varphi(x, 2x, x)}{t + \varphi(x, 2x, x)}\right\}, \tag{3.1}$$

for all $g, h \in G$, $x \in X$ and $t > 0$. Then d_s is a complete generalized metric on G.

Proof Let $g, h, k \in G$, $d_s(g, h) < \gamma_1$ and $d_s(h, k) < \gamma_2$. Then, for all $x \in X$ and $t > 0$, we have

$$\mu(g(x) - h(x), \gamma_1 t) \geq \frac{t}{t + \varphi(x, 2x, x)}, \qquad \mu(h(x) - k(x), \gamma_2 t) \geq \frac{t}{t + \varphi(x, 2x, x)};$$

and

$$\nu\big(g(x)-h(x),\gamma_1 t\big) \le \frac{\varphi(x,2x,x)}{t+\varphi(x,2x,x)}, \qquad \nu\big(h(x)-k(x),\gamma_2 t\big) \le \frac{\varphi(x,2x,x)}{t+\varphi(x,2x,x)}.$$

Therefore,

$$\mu\big(g(x)-k(x),(\gamma_1+\gamma_2)t\big) \ge \mu\big(g(x)-h(x),\gamma_1 t\big) * \mu\big(h(x)-k(x),\gamma_2 t\big) \ge \frac{t}{t+\varphi(x,2x,x)}$$

and

$$\nu\big(g(x)-k(x),(\gamma_1+\gamma_2)t\big) \le \nu\big(g(x)-h(x),\gamma_1 t\big) \diamond \nu\big(h(x)-k(x),\gamma_2 t\big) \le \frac{\varphi(x,2x,x)}{t+\varphi(x,2x,x)},$$

for each $x \in X$ and $t > 0$. Thus, $d_s(g,k) \le \gamma_1 + \gamma_2$, which is a triangle inequality for d_s. The rest of the conditions follow directly from the definition. □

Theorem 3.2 *Let X be a linear space and f be a mapping from X to an intuitionistic fuzzy Banach space (Y,μ,ν). Suppose that $\varphi : X \times X \times X \to [0,\infty)$ is a function such that*

$$\left.\begin{aligned} &\mu\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z),t\right) \ge \frac{t}{t+\varphi(x,y,z)} \quad \text{and} \\ &\nu\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z),t\right) \le \frac{\varphi(x,y,z)}{t+\varphi(x,y,z)}, \end{aligned}\right\} \tag{3.2}$$

for all $x,y,z \in X$ and $t > 0$. If $\varphi(x,y,z) \le \frac{\alpha}{2}\varphi(2x,2y,2z)$ holds for some real number α with $\alpha < 1$ then there exists a unique additive mapping $T : X \to Y$ such that $T(x) = (\mu,\nu)\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$,

$$\begin{aligned} &\mu\big(f(x)-T(x),t\big) \ge \frac{(2-2\alpha)t}{(2-2\alpha)t+\alpha\varphi(x,2x,x)} \quad \text{and} \\ &\nu\big(f(x)-T(x),t\big) \le \frac{\alpha\varphi(x,2x,x)}{(2-2\alpha)t+\alpha\varphi(x,2x,x)}, \end{aligned} \tag{3.3}$$

for all $x \in X$ and $t > 0$.

Proof Putting $y = 2x$ and $z = x$ in (3.2). Then for $x \in X$ and $t > 0$

$$\begin{aligned} &\mu\big(f(2x)-2f(x),t\big) \ge \frac{t}{t+\varphi(x,2x,x)} \quad \text{and} \\ &\nu\big(f(2x)-2f(x),t\big) \le \frac{\varphi(x,2x,x)}{t+\varphi(x,2x,x)}. \end{aligned}$$

Replacing x by $x/2$, we get

$$\begin{aligned} &\mu\left(2f\left(\frac{x}{2}\right)-f(x),t\right) \ge \frac{t}{t+\varphi(\frac{x}{2},x,\frac{x}{2})} \quad \text{and} \\ &\mu\left(2f\left(\frac{x}{2}\right)-f(x),t\right) \ge \frac{\varphi(\frac{x}{2},x,\frac{x}{2})}{t+\varphi(\frac{x}{2},x,\frac{x}{2})}. \end{aligned} \tag{3.4}$$

Consider the set $G = \{g : X \to Y\}$ and the mapping d defined on $G \times G$ by

$$d_s(g,h) = \inf\left\{\gamma \in \mathbb{R}^+ : \mu\big(g(x) - h(x), \gamma t\big) \geq \frac{t}{t + \varphi(x,2x,x)} \text{ and } \nu\big(g(x) - h(x), \gamma t\big) \leq \frac{\varphi(x,2x,x)}{t + \varphi(x,2x,x)}\right\}$$

for all $x \in X$ and $t > 0$. It is known that $d_s(g,h)$ is a complete generalized metric on G by Lemma 3.1. Now we consider the linear mapping $J : G \to G$ such that $Jg(x) = 2g(\frac{x}{2})$ for all $x \in X$. Let $g, h \in G$ be such that $d_s(g,h) = \xi$. Then, for all $x \in X$ and $t > 0$, we have

$$\mu\big(g(x) - h(x), \xi t\big) \geq \frac{t}{t + \varphi(x,2x,x)} \quad \text{and} \quad \nu\big(g(x) - h(x), \xi t\big) \leq \frac{\varphi(x,2x,x)}{t + \varphi(x,2x,x)}.$$

Using the hypothesis of the function φ and a mapping J, we obtain

$$\mu\big(Jg(x) - Jh(x), \alpha\xi t\big) = \mu\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{\alpha\xi t}{2}\right) \geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \varphi(\frac{x}{2}, x, \frac{x}{2})} \geq \frac{\frac{\alpha t}{2}}{\frac{\alpha t}{2} + \frac{\alpha}{2}\varphi(x,2x,x)},$$

this implies

$$\mu\big(Jg(x) - Jh(x), \alpha\xi t\big) \geq \frac{t}{t + \varphi(x,2x,x)}$$

and similarly

$$\nu\big(Jg(x) - Jh(x), \alpha\xi t\big) = \nu\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{\alpha\xi t}{2}\right) \geq \frac{\varphi(\frac{x}{2}, x, \frac{x}{2})}{\frac{\alpha t}{2} + \varphi(\frac{x}{2}, x, \frac{x}{2})} \geq \frac{\varphi(x,2x,x)}{t + \varphi(x,2x,x)},$$

for all $x \in X$ and $t > 0$. From above, we conclude that $d_s(g,h) = \xi$ implies $d_s(Jg, Jh) \leq \alpha\xi$. Hence,

$$d_s(Jg, Jh) \leq \alpha d_s(g,h),$$

for all $g, h \in G$. Using the hypothesis of the function φ and from (3.4), we have

$$\begin{aligned} &\mu\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{\frac{2t}{\alpha}}{\frac{2t}{\alpha} + \varphi(x,2x,x)} \quad \text{and} \\ &\nu\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{\varphi(x,2x,x)}{\frac{2t}{\alpha} + \varphi(x,2x,x)}, \end{aligned} \tag{3.5}$$

for all $x \in X$, $t > 0$ and $\alpha < 1$. Replacing t by $\frac{\alpha t}{2}$ in (3.5), we get

$$\begin{aligned} &\mu\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{\alpha t}{2}\right) \geq \frac{t}{t + \varphi(x,2x,x)} \quad \text{and} \\ &\nu\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{\alpha t}{2}\right) \geq \frac{\varphi(x,2x,x)}{t + \varphi(x,2x,x)}, \end{aligned}$$

for all $x \in X$, $t > 0$ and $\alpha < 1$. It follows that

$$d_s(f, Jf) \leq \frac{\alpha}{2}.$$

Using the fixed-point alternative we deduce the existence of a fixed point of J, that is, the existence of a mapping $T : X \to Y$ such that

$$T\left(\frac{x}{2}\right) = \frac{T(x)}{2}$$

for all $x \in X$. The mapping T is a unique fixed point of J in the set $E = \{h \in G : d_s(g, h) < \infty\}$. It follows that T is the unique fixed point of J with the property that there exists $c \in (0, \infty)$ such that

$$\mu\big(g(x) - h(x), ct\big) \geq \frac{t}{t + \varphi(x, 2x, x)} \quad \text{and} \quad \nu\big(g(x) - h(x), ct\big) \leq \frac{\varphi(x, 2x, x)}{t + \varphi(x, 2x, x)},$$

for all $x \in X$ and $t > 0$. Moreover, we have $d(J^n f, T) \to 0$ as $n \to \infty$ which implies

$$(\mu, \nu)\text{-}\lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) = T(x),$$

for all $x \in X$. Also $d_s(f, T) \leq \frac{1}{1-\alpha} d_s(f, Jf)$ implies $d_s(f, T) \leq \frac{\alpha}{2-2\alpha}$. This means that (3.3) holds. For all $x, y, z \in X$ and $t > 0$, write

$$\begin{aligned}
&\mu\left(2T\left(\frac{x+y+z}{2}\right) - T(x) - T(y) - T(z), t\right) \\
&\quad \geq \mu\left(2T\left(\frac{x+y+z}{2}\right) - 2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right), \frac{t}{5}\right) * \mu\left(2^n f\left(\frac{x}{2^n}\right) - T(x), \frac{t}{5}\right) \\
&\qquad * \mu\left(2^n f\left(\frac{y}{2^n}\right) - T(y), \frac{t}{5}\right) * \mu\left(2^n f\left(\frac{z}{2^n}\right) - T(z), \frac{t}{5}\right) \\
&\qquad * \mu\left(2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{z}{2^n}\right), \frac{t}{5}\right).
\end{aligned} \tag{3.6}$$

Letting $n \to \infty$ in (3.6) and using (3.2), we get

$$\mu\left(2T\left(\frac{x+y+z}{2}\right) - T(x) - T(y) - T(z), t\right) = 1.$$

Similarly, we obtain

$$\nu\left(2T\left(\frac{x+y+z}{2}\right) - T(x) - T(y) - T(z), t\right) = 0,$$

for all $x, y, z \in X$ and $t > 0$. Thus, the mapping T satisfies (1.1) and so it is additive. □

Corollary 3.3 *Let X be a normed linear space and (Y, μ, ν) be an intuitionistic fuzzy Banach space. Let θ be a positive real number and r is a real number with $r > 1$. If a mapping*

$f: X \to Y$ satisfies the conditions

$$\mu\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \geq \frac{t}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)} \quad \text{and}$$
$$\nu\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \leq \frac{\theta(\|x\|^r + \|y\|^r + \|z\|^r)}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)},$$

for all $x, y, z \in X$ and $t > 0$, then there exists a unique additive mapping $T: X \to Y$ such that $T(x) = (\mu, \nu)\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$,

$$\mu\big(f(x) - T(x), t\big) \geq \frac{(2^r - 1)t}{(2^r - 1)t + (2^{r-1} + 1)\theta\|x\|^r} \quad \text{and}$$
$$\nu\big(f(x) - T(x), t\big) \leq \frac{(2^{r-1} + 1)\theta\|x\|^r}{(2^r - 1)t + (2^{r-1} + 1)\theta\|x\|^r},$$

for all $x \in X$ and $t > 0$.

Proof Taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 3.2, for all $x, y, z \in X$, and choosing $\alpha = 2^{-r}$, we get the desired result. □

Theorem 3.4 *Let X be a linear space and $\varphi: X \times X \times X \to [0, \infty)$ be a function such that there exists $\alpha < 1$ with $\varphi(2x, 2y, 2z) \leq 2\alpha\varphi(x, y, z)$ for all $x, y, z \in X$ and $t > 0$. Suppose f is a mapping from X to an intuitionistic fuzzy Banach space (Y, μ, ν) satisfying (3.2). Then there exists a unique additive mapping $T: X \to Y$ such that $T(x) = (\mu, \nu)\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$,*

$$\mu\big(f(x) - T(x), t\big) \geq \frac{(2 - 2\alpha)t}{(2 - 2\alpha)t + \varphi(x, 2x, x)} \quad \text{and}$$
$$\nu\big(f(x) - T(x), t\big) \leq \frac{\varphi(x, 2x, x)}{(2 - 2\alpha)t + \varphi(x, 2x, x)}, \tag{3.7}$$

for all $x \in X$ and $t > 0$.

Proof Consider a complete generalized metric space (G, d_s) same as in the proof of Theorem 3.2. We define a linear mapping $J: G \to G$ such that

$$Jg(x) = \frac{g(2x)}{2}, \tag{3.8}$$

for all $x \in X$. Indeed, for given g and h in G, $d_s(g, h) = \xi$. Then

$$\mu\big(g(x) - h(x), \xi t\big) \geq \frac{t}{t + \varphi(x, 2x, x)} \quad \text{and} \quad \nu\big(g(x) - h(x), \xi t\big) \leq \frac{\varphi(x, 2x, x)}{t + \varphi(x, 2x, x)},$$

for all $x \in X$ and $t > 0$. By the given hypothesis and using (3.8), we have

$$\mu\big(Jg(x) - Jh(x), \alpha\xi t\big) = \mu\big(g(2x) - h(2x), 2\alpha\xi t\big) \geq \frac{2\alpha t}{2\alpha t + \varphi(2x, 4x, 2x)} \geq \frac{t}{t + \varphi(x, 2x, x)}$$

and

$$\nu\big(Jg(x) - Jh(x), \alpha\xi t\big) = \nu\big(g(2x) - h(2x), 2\alpha\xi t\big) \leq \frac{\varphi(2x, 4x, 2x)}{2\alpha t + \varphi(2x, 4x, 2x)} \geq \frac{\varphi(x, 2x, x)}{t + \varphi(x, 2x, x)},$$

for all $x \in X$ and $t > 0$. This means that $d_s(Jg, Jh) \le \alpha\xi$. Thus, $d_s(Jg, Jh) \le \alpha d_s(g, h)$ for all g and h in G. It follows from (3.2) that

$$\begin{aligned} &\mu\left(\frac{f(2x)}{2} - f(x), \frac{t}{2}\right) \ge \frac{t}{t + \varphi(x, 2x, x)} \quad \text{and} \\ &\nu\left(\frac{f(2x)}{2} - f(x), \frac{t}{2}\right) \ge \frac{\varphi(x, 2x, x)}{t + \varphi(x, 2x, x)}, \end{aligned} \tag{3.9}$$

for all $x \in X$ and $t > 0$. From the definition of complete generalized metric space, we have $d_s(f, Jf) \le \frac{1}{2}$. Using the fixed-point alternative, we deduce the existence of a fixed point of J, that is, the existence of a mapping $T : X \to Y$ such that $2T(x) = T(2x)$ for all $x \in X$. Moreover, we have $d(J^n f, T) \to 0$ which implies

$$(\mu, \nu)\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = T(x),$$

for all $x \in X$. Also $d_s(f, T) \le \frac{1}{1-\alpha} d_s(f, Jf)$ implies $d_s(f, T) \le \frac{1}{2-2\alpha}$. The rest of the proof can be done by the same way as in Theorem 3.2. □

Corollary 3.5 *Let X be a normed linear space and (Y, μ, ν) be an intuitionistic fuzzy Banach space. Let θ be a positive real number and r is a real number with $0 < r < \frac{1}{3}$. If a mapping $f : X \to Y$ satisfies the conditions,*

$$\begin{aligned} &\mu\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \ge \frac{t}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)} \quad \text{and} \\ &\nu\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \le \frac{\theta(\|x\|^r + \|y\|^r + \|z\|^r)}{t + \theta(\|x\|^r + \|y\|^r + \|z\|^r)}, \end{aligned}$$

for all $x, y, z \in X$ and $t > 0$, then there exists a unique additive mapping $T : X \to Y$ such that $T(x) = (\mu, \nu)\text{-}\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$,

$$\begin{aligned} &\mu\big(f(x) - T(x), t\big) \ge \frac{(2^{3r} - 1)t}{(2^{3r} - 1)t + (2^{3r} + 2^{4r-1})\theta\|x\|^r} \quad \text{and} \\ &\nu\big(f(x) - T(x), t\big) \le \frac{(2^{3r} + 2^{4r-1})\theta\|x\|^r}{(2^{3r} - 1)t + (2^{3r} + 2^{4r-1})\theta\|x\|^r}, \end{aligned}$$

for all $x \in X$ and $t > 0$.

Proof Taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 3.4, for all $x, y, z \in X$, and choosing $\alpha = 2^{-3r}$, we get the desired result. □

4 Stability of the additive functional equation through the direct method

In this section, we deal with the stability results concerning the additive functional equation *via* direct method in intuitionistic fuzzy normed spaces.

Theorem 4.1 *Let X be a linear space and (Z,μ',ν') be an IFN-space. Suppose that $\varphi: X\times X\times X\to Z$ is a function such that for some real number α with $0<|\alpha|<1/2$*

$$\begin{aligned}&\mu'\left(\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right),t\right)\geq\mu'\left(\varphi(x,y,z),\frac{t}{|\alpha|}\right)\quad\text{and}\\&\nu'\left(\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right),t\right)\leq\nu'\left(\varphi(x,y,z),\frac{t}{|\alpha|}\right),\end{aligned}\tag{4.1}$$

for all $x,y,z\in X$ and $t>0$. Let f be a mapping from X to an intuitionistic fuzzy Banach space (Y,μ,ν) such that

$$\left.\begin{aligned}&\mu\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z),t\right)\geq\mu'\big(\varphi(x,y,z),t\big)\quad\text{and}\\&\nu\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z),t\right)\leq\nu'\big(\varphi(x,y,z),t\big),\end{aligned}\right\}\tag{4.2}$$

for all $x,y,z\in X$ and $t>0$. Then there exists a unique additive mapping $T:X\to Y$ such that

$$\left.\begin{aligned}&\mu\big(f(x)-T(x),t\big)\geq\mu'\left(\varphi(x,2x,x),\frac{(1-2|\alpha|)t}{|\alpha|}\right)\quad\text{and}\\&\nu\big(f(x)-T(x),t\big)\leq\nu'\left(\varphi(x,2x,x),\frac{(1-2|\alpha|)t}{|\alpha|}\right),\end{aligned}\right\}\tag{4.3}$$

for all $x\in X$ and $t>0$.

Proof Put $y=2x$ and $z=x$ in (4.2). Then for all $x\in X$ and $t>0$

$$\begin{aligned}&\mu\big(f(2x)-2f(x),t\big)\geq\mu'\big(\varphi(x,2x,x),t\big)\quad\text{and}\\&\nu\big(f(2x)-2f(x),t\big)\leq\nu'\big(\varphi(x,2x,x),t\big).\end{aligned}\tag{4.4}$$

Replacing x by $\frac{x}{2^{j+1}}$ in (4.4) and using (4.1), we obtain

$$\mu\left(2^{j+1}f\left(\frac{x}{2^{j+1}}\right)-2^jf\left(\frac{x}{2^j}\right),2^jt\right)\geq\mu'\left(\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^j},\frac{x}{2^{j+1}}\right),t\right)\geq\mu'\left(\varphi(x,y,z),\frac{t}{|\alpha|^{j+1}}\right)$$

and

$$\nu\left(2^{j+1}f\left(\frac{x}{2^{j+1}}\right)-2^jf\left(\frac{x}{2^j}\right),2^jt\right)\leq\nu'\left(\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^j},\frac{x}{2^{j+1}}\right),t\right)\leq\nu'\left(\varphi(x,y,z),\frac{t}{|\alpha|^{j+1}}\right),$$

for all $x\in X$, $t>0$ and an integer $j\geq0$. By replacing $t=|\alpha|^{j+1}t$, we get

$$\left.\begin{aligned}&\mu\left(2^{j+1}f\left(\frac{x}{2^{j+1}}\right)-2^jf\left(\frac{x}{2^j}\right),2^j|\alpha|^{j+1}t\right)\geq\mu'\big(\varphi(x,y,z),t\big)\quad\text{and}\\&\nu\left(2^{j+1}f\left(\frac{x}{2^{j+1}}\right)-2^jf\left(\frac{x}{2^j}\right),2^j|\alpha|^{j+1}t\right)\leq\nu'\big(\varphi(x,y,z),t\big).\end{aligned}\right\}\tag{4.5}$$

It follows from

$$2^n f\left(\frac{x}{2^n}\right) - f(x) = \sum_{j=0}^{n-1}\left(2^{j+1} f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right)\right)$$

and (4.5) that

$$\begin{aligned}
\mu\left(2^n f\left(\frac{x}{2^n}\right) - f(x), \sum_{j=0}^{n-1} 2^j |\alpha|^{j+1} t\right) &\geq \prod_{j=0}^{n-1} \mu\left(2^{j+1} f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right), 2^j |\alpha|^{j+1} t\right) \\
&\geq \mu'\big(\varphi(x, 2x, x), t\big) \quad \text{and} \\
\nu\left(2^n f\left(\frac{x}{2^n}\right) - f(x), \sum_{j=0}^{n-1} 2^j |\alpha|^{j+1} t\right) &\leq \coprod_{j=0}^{n-1} \nu\left(2^{j+1} f\left(\frac{x}{2^{j+1}}\right) - 2^j f\left(\frac{x}{2^j}\right), 2^j |\alpha|^{j+1} t\right) \\
&\leq \nu'\big(\varphi(x, 2x, x), t\big),
\end{aligned}$$

for all $x \in X$, $t > 0$ and $n > 0$, where $\prod_{i=1}^n a_i = a_1 * a_2 * \cdots * a_n$ and $\coprod_{i=1}^n a_i = a_1 \diamond a_2 \diamond \cdots \diamond a_n$. Replacing x by $\frac{x}{2^p}$ in the last inequalities, we have

$$\begin{aligned}
&\mu\left(2^{n+p} f\left(\frac{x}{2^{n+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), \sum_{j=0}^{n-1} 2^{j+p} |\alpha|^{j+1} t\right) \\
&\quad \geq \mu'\left(\varphi\left(\frac{x}{2^p}, \frac{2x}{2^p}, \frac{x}{2^p}\right), t\right) \geq \mu'\left(\varphi(x, 2x, x), \frac{t}{|\alpha|^p}\right) \quad \text{and} \\
&\nu\left(2^{n+p} f\left(\frac{x}{2^{n+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), \sum_{j=0}^{n-1} 2^{j+p} |\alpha|^{j+1} t\right) \\
&\quad \leq \nu'\left(\varphi\left(\frac{x}{2^p}, \frac{2x}{2^p}, \frac{x}{2^p}\right), t\right) \leq \nu'\left(\varphi(x, 2x, x), \frac{t}{|\alpha|^p}\right),
\end{aligned}$$

whence

$$\begin{aligned}
&\mu\left(2^{n+p} f\left(\frac{x}{2^{n+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), \sum_{j=0}^{n-1} 2^{j+p} |\alpha|^{j+p+1} t\right) \geq \mu'\big(\varphi(x, 2x, x), t\big) \quad \text{and} \\
&\nu\left(2^{n+p} f\left(\frac{x}{2^{n+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), \sum_{j=0}^{n-1} 2^{j+p} |\alpha|^{j+p+1} t\right) \leq \nu'\big(\varphi(x, 2x, x), t\big),
\end{aligned}$$

for all $x \in X$, $t > 0$, $n > 0$ and $p \geq 0$. Hence,

$$\left.\begin{aligned}
&\mu\left(2^{n+p} f\left(\frac{x}{2^{n+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), t\right) \geq \mu'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=p}^{n+p-1} 2^j |\alpha|^{j+1}}\right) \quad \text{and} \\
&\nu\left(2^{n+p} f\left(\frac{x}{2^{n+p}}\right) - 2^p f\left(\frac{x}{2^p}\right), t\right) \leq \nu'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=p}^{n+p-1} 2^j |\alpha|^{j+1}}\right).
\end{aligned}\right\} \tag{4.6}$$

Since $0 < |\alpha| < \frac{1}{2}$, we have $\sum_{j=0}^{\infty} (2|\alpha|)^{j+1} < \infty$. This shows that $(2^n f(\frac{x}{2^n}))$ is a Cauchy sequence in an intuitionistic fuzzy Banach space (Y, μ, ν) and so it converges to some point

$T(x) \in Y$. Thus, we define a mapping $T : X \to Y$ such that

$$T(x) = (\mu, \nu)\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right).$$

Hence, for all $x \in X$ and $t > 0$, we have

$$\mu\left(2^n f\left(\frac{x}{2^n}\right) - T(x), t\right) = 1 \quad \text{and} \quad \nu\left(2^n f\left(\frac{x}{2^n}\right) - T(x), t\right) = 0.$$

Moreover, if we put $p = 0$ in (4.6), we get

$$\mu\left(2^n f\left(\frac{x}{2^n}\right) - f(x), t\right) \geq \mu'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=0}^{n-1} 2^j |\alpha|^{j+1}}\right) \quad \text{and}$$
$$\nu\left(2^n f\left(\frac{x}{2^n}\right) - f(x), t\right) \leq \nu'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=0}^{n-1} 2^j |\alpha|^{j+1}}\right),$$

for all $x \in X$, $t > 0$ and $n > 0$. Therefore,

$$\begin{aligned}\mu\big(T(x) - f(x), t\big) &\geq \mu\left(T(x) - 2^n f\left(\frac{x}{2^n}\right), t/2\right) * \mu\left(2^n f\left(\frac{x}{2^n}\right) - f(x), t/2\right)\\ &\geq \mu'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=0}^{n-1} (2|\alpha|)^{j+1}}\right)\end{aligned}$$

and

$$\begin{aligned}\nu\big(T(x) - f(x), t\big) &\leq \nu\left(T(x) - 2^n f\left(\frac{x}{2^n}\right), t/2\right) \diamond \nu\left(2^n f\left(\frac{x}{2^n}\right) - f(x), t/2\right)\\ &\leq \nu'\left(\varphi(x, 2x, x), \frac{t}{\sum_{j=0}^{n-1} (2|\alpha|)^{j+1}}\right),\end{aligned}$$

for all $x \in X$, $t > 0$ and $n > 0$. Letting $n \to \infty$ in the above inequalities, we obtain

$$\mu\big(T(x) - f(x), t\big) \geq \mu'\left(\varphi(x, 2x, x), \frac{(1 - 2|\alpha|)t}{|\alpha|}\right) \quad \text{and}$$
$$\nu\big(T(x) - f(x), t\big) \leq \nu'\left(\varphi(x, 2x, x), \frac{(1 - 2|\alpha|)t}{|\alpha|}\right).$$

Hence, T satisfies (4.3). Let $x, y, z \in X$. Then

$$\begin{aligned}&\mu\left(2T\left(\frac{x+y+z}{2}\right) - T(x) - T(y) - T(z), t\right)\\ &\quad\geq \mu\left(2T\left(\frac{x+y+z}{2}\right) - 2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right), \frac{t}{5}\right) * \mu\left(2^n f\left(\frac{x}{2^n}\right) - T(x), \frac{t}{5}\right)\\ &\qquad * \mu\left(2^n f\left(\frac{y}{2^n}\right) - T(y), \frac{t}{5}\right) * \mu\left(2^n f\left(\frac{z}{2^n}\right) - T(z), \frac{t}{5}\right)\\ &\qquad * \mu\left(2^{n+1} f\left(\frac{x+y+z}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - 2^n f\left(\frac{z}{2^n}\right), \frac{t}{5}\right) \qquad (4.7)\end{aligned}$$

and by using (4.2)

$$\begin{aligned} &\mu\left(2^{n+1}f\left(\frac{x+y+z}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right)-2^nf\left(\frac{y}{2^n}\right)-2^nf\left(\frac{z}{2^n}\right),t\right) \\ &\quad\geq \mu'\left(\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right),\frac{t}{2^n}\right)\geq \mu'\left(\varphi(x,y,z),\frac{t}{2^n|\alpha|^n}\right). \end{aligned} \tag{4.8}$$

Letting $n\to\infty$ in (4.7) and (4.8), we get

$$\mu\left(2T\left(\frac{x+y+z}{2}\right)-T(x)-T(y)-T(z),t\right)=1.$$

Similarly, we obtain

$$\nu\left(2T\left(\frac{x+y+z}{2}\right)-T(x)-T(y)-T(z),t\right)=0,$$

for all $x,y,z\in X$ and $t>0$. This means that T satisfies (1.1) and so it is additive. To prove the uniqueness of T, assume that S be another additive mapping from X into Y, which satisfies (4.3). For $x\in X$, clearly $T(x)=2^nT(\frac{x}{2^n})$ and $S(x)=2^nS(\frac{x}{2^n})$ for all n. It follows from (4.3) that

$$\begin{aligned} \mu\bigl(T(x)-S(x),t\bigr) &= \mu\left(2^nT\left(\frac{x}{2^n}\right)-2^nS\left(\frac{x}{2^n}\right),t\right) \\ &\geq \mu\left(2^nT\left(\frac{x}{2^n}\right)-2^nf\left(\frac{x}{2^n}\right),\frac{t}{2}\right)*\mu\left(2^nf\left(\frac{x}{2^n}\right)-2^nS\left(\frac{x}{2^n}\right),\frac{t}{2}\right) \\ &\geq \mu'\left(\varphi\left(\frac{x}{2^n},\frac{2x}{2^n},\frac{x}{2^n}\right),\frac{(1-2|\alpha|)t}{2^{n+1}|\alpha|}\right) \\ &\geq \mu'\left(\varphi(x,2x,x),\frac{(1-2|\alpha|)t}{2^{n+1}|\alpha|^{n+1}}\right) \end{aligned} \tag{4.9}$$

and similarly

$$\nu\bigl(T(x)-S(x),t\bigr)\leq \nu'\left(\varphi(x,2x,x),\frac{(1-2|\alpha|)t}{2^{n+1}|\alpha|^{n+1}}\right). \tag{4.10}$$

We see that the right-hand side of (4.9) and (4.10) tending to 1 and 0, respectively, as $n\to\infty$. Therefore, $\mu(T(x)-S(x),t)=1$ and $\nu(T(x)-S(x),t)=0$ for all $x\in X$ and $t>0$. Hence, $T(x)=S(x)$. □

Corollary 4.2 *Let X be a normed linear space and $(\mathbb{R},\mu',\nu')$ be an intuitionistic fuzzy Banach space. Let θ be a positive real number and r is a real number with $0<r<1$. If a mapping $f:X\to Y$ satisfies the conditions,*

$$\begin{aligned} &\mu\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z),t\right)\geq \mu'\bigl(\theta\bigl(\|x\|^r+\|y\|^r+\|z\|^r\bigr),t\bigr) \quad \textit{and} \\ &\nu\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z),t\right)\leq \nu'\bigl(\theta\bigl(\|x\|^r+\|y\|^r+\|z\|^r\bigr),t\bigr), \end{aligned}$$

for all $x, y, z \in X$ and $t > 0$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\mu\big(f(x) - T(x), t\big) \geq \mu'\left(\theta\|x\|^r, \frac{2t}{2^\alpha + 2}\right) \quad \text{and} \quad \nu\big(f(x) - T(x), t\big) \leq \nu'\left(\theta\|x\|^r, \frac{2t}{2^\alpha + 2}\right),$$

for all $x \in X$ and $t > 0$.

Proof Taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 4.1, for all $x, y, z \in X$, and choosing $|\alpha| = 1/4$, we get the desired result. □

Theorem 4.3 *Let X be a linear space and (Z, μ', ν') be an IFN-space. Suppose that $\varphi : X \times X \times X \to Z$ is a function such that for some real number α with $0 < |\alpha| < 2$*

$$\begin{aligned} &\mu'\big(\varphi(2x, 2y, 2z), t\big) \geq \mu'\big(|\alpha|\varphi(x, y, z), t\big) \quad \text{and} \\ &\nu'\big(\varphi(2x, 2y, 2z), t\big) \leq \nu'\big(|\alpha|\varphi(x, y, z), t\big), \end{aligned} \tag{4.11}$$

for all $x, y, z \in X$ and $t > 0$. Let (Y, μ, ν) an intuitionistic fuzzy Banach space and a map $f : X \to Y$ satisfies (4.2). Then there exists a unique additive mapping $T : X \to Y$ such that

$$\begin{aligned} &\mu\big(f(x) - T(x), t\big) \geq \mu'\big(\varphi(x, 2x, x), (2 - |\alpha|)t\big) \quad \text{and} \\ &\nu\big(f(x) - T(x), t\big) \leq \nu'\big(\varphi(x, 2x, x), (2 - |\alpha|)t\big), \end{aligned} \tag{4.12}$$

for all $x \in X$ and $t > 0$.

Proof From (4.4), it is easy to see that

$$\begin{aligned} &\mu\left(\frac{f(2x)}{2} - f(x), \frac{t}{2}\right) \geq \mu'\big(\varphi(x, 2x, x), t\big) \quad \text{and} \\ &\nu\left(\frac{f(2x)}{2} - f(x), \frac{t}{2}\right) \leq \nu'\big(\varphi(x, 2x, x), t\big), \end{aligned} \tag{4.13}$$

for all $x \in X$ and $t > 0$. Replacing x by $2^n x$, we get

$$\begin{aligned} &\mu\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \frac{t}{2^{n+1}}\right) \geq \mu'\big(\varphi(2^n x, 2^{n+1}x, 2^n x), t\big) \geq \mu'\left(\varphi(x, 2x, x), \frac{t}{|\alpha|^n}\right) \quad \text{and} \\ &\nu\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \frac{t}{2^{n+1}}\right) \leq \nu'\big(\varphi(2^n x, 2^{n+1}x, 2^n x), t\big) \leq \nu'\left(\varphi(x, 2x, x), \frac{t}{|\alpha|^n}\right). \end{aligned}$$

It follows that, for all $x \in X$ and $t > 0$, we have

$$\begin{aligned} &\mu\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \frac{|\alpha|^n t}{2^{n+1}}\right) \geq \mu'\big(\varphi(x, 2x, x), t\big) \quad \text{and} \\ &\nu\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, \frac{|\alpha|^n t}{2^{n+1}}\right) \leq \nu'\big(\varphi(x, 2x, x), t\big). \end{aligned}$$

Proceeding the same lines as in the proof of Theorem 4.1, we get

$$\mu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{j=0}^{n-1} \frac{|\alpha|^j t}{2^{j+1}}\right) \geq \mu'\big(\varphi(x,2x,x),t\big) \quad \text{and}$$

$$\nu\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{j=0}^{n-1} \frac{|\alpha|^j t}{2^{j+1}}\right) \leq \nu'\big(\varphi(x,2x,x),t\big),$$

for all $x \in X$, $t > 0$ and $n > 0$. Thus,

$$\mu\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \geq \mu'\left(\varphi(x,2x,x), \frac{t}{\sum_{j=0}^{n-1} \frac{|\alpha|^j}{2^{j+1}}}\right) \geq \mu'\big(\varphi(x,2x,x), \big(2-|\alpha|\big)t\big) \quad \text{and}$$

$$\nu\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \leq \nu'\left(\varphi(x,2x,x), \frac{t}{\sum_{j=0}^{n-1} \frac{|\alpha|^j}{2^{j+1}}}\right) \leq \nu'\big(\varphi(x,2x,x), \big(2-|\alpha|\big)t\big).$$

Rest of the proof can be done by the same way as in Theorem 4.1. □

Corollary 4.4 *Let X be a normed linear space and $(\mathbb{R}, \mu', \nu')$ be an intuitionistic fuzzy Banach space. Let θ be a positive real number and r is a real number with $0 < r < \frac{1}{3}$. If a mapping $f : X \to Y$ satisfies the conditions*

$$\mu\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \geq \mu'\big(\theta\big(\|x\|^r + \|y\|^r + \|z\|^r\big), t\big) \quad \textit{and}$$

$$\nu\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z), t\right) \leq \nu'\big(\theta\big(\|x\|^r + \|y\|^r + \|z\|^r\big), t\big),$$

for all $x, y, z \in X$ and $t > 0$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\mu\big(f(x) - T(x), t\big) \geq \mu'\left(\theta\|x\|^r, \frac{t}{2^r + 2}\right) \quad \textit{and} \quad \nu\big(f(x) - T(x), t\big) \leq \nu'\left(\theta\|x\|^r, \frac{t}{2^r + 2}\right),$$

for all $x \in X$ and $t > 0$.

Proof Taking $\varphi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 4.3, for all $x, y, z \in X$, and choosing $|\alpha| = 1$, we get the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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