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Chaos control and Hopf bifurcation analysis of the Genesio system with distributed delays feedback

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Abstract
In this paper, the Genesio system with distributed time delay feedback is investigated. Firstly, the stability of the equilibria of the system is investigated by analyzing the characteristic equation, and then the existence of Hopf bifurcations is verified by choosing the mean time delay as a bifurcation parameter. Subsequent to that, the direction and stability of the bifurcating periodic solutions are determined by using the normal form theory and the center manifold theorem. Finally, some numerical simulations are presented to verify the effectiveness of the theoretical results.
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Keywords: chaos control; Hopf bifurcation; distributed time delay; feedback

1 Introduction

Chaos control has attracted considerable attention since the pioneering work of Ott and Grebogi [1]. It is well known that in many practical applications, chaos is undesirable and needs to be controlled. Therefore, the investigation of controlling chaos is of great significance. Many schemes have been presented to implement chaos control, among which using time-delayed controlling forces proves to be a simple and viable method for a continuous dynamical system [2]. It is noteworthy that time-delayed feedback controller can also be used to realize the control of a bifurcation, see [3–6] and references therein. It is known that if the steady state is stable or the bifurcating periodic solutions are orbitally asymptotically stable, then the chaotic system will not exhibit chaotic dynamical behaviors. As a consequence, bifurcation control in this sense may also help to control chaos.

In order to better model some complicated practical phenomena, recently, distributed time delay has been introduced into many modeling systems. There are extensive literature works dealing with such systems [7–11]. As the distributed time delay is incorporated in a system, some interesting dynamical behaviors occur near the equilibrium point. Inspired by these previous works, in this paper, we intend to introduce the distributed time delay as a feedback controller into the chaotic Genesio system with the aim to realize the control of chaos. The rest of this paper is organized as follows. In the next section, we present the mathematical models of the Genesio system with distributed time delay feedback and consider its local stability and Hopf bifurcation. In Section 3, the stability of the bifurcating periodic solutions and the direction of the Hopf bifurcation at the critical values of mean time delay are determined by using the normal form method and the center manifold

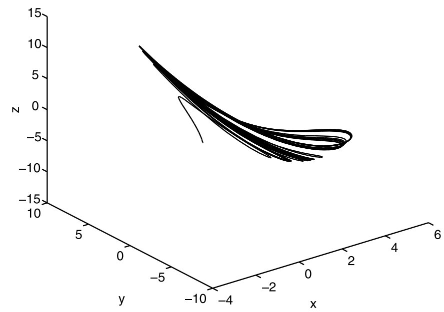

Figure 1 The attractor of Genesio system with $a = -6$, $b = -2.92$, $c = -1.2$.

reduction due to Hassard *et al.* [12]. In Section 4, a numerical example is provided to verify the theoretical results. Finally, some concluding remarks are given in Section 5.

2 Stability analysis and Hopf bifurcation of the Genesio system with distributed delay feedback

The Genesio system, proposed by Genesio and Tesi [13] and studied extensively in recent years [8, 14–19], is described by the following three-dimensional autonomous system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = ax + by + cz + x^2, \end{cases} \tag{2.1}$$

where $a, b, c < 0$ are parameters. System (2.1) exhibits chaotic dynamical behaviors when $a = -6$, $b = -2.92$, $c = -1.2$, as illustrated in Figure 1.

In order to apply feedback control, we consider system (2.1) with continuous distributed delay feedback described by

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = z(t) + M\int_{-\infty}^{0}(y(t) - y(t+s))k(-s)\,\mathrm{d}s, \\ \dot{z}(t) = ax(t) + by(t) + cz(t) + x^2(t), \end{cases} \tag{2.2}$$

where $M \in R$, $a, b, c < 0$, $\int_0^{+\infty} k(s)\,\mathrm{d}s = 1$, $\int_0^{+\infty} sk(s)\,\mathrm{d}s < +\infty$.

It is easy to see that systems (2.2) and (2.1) have the same equilibrium points $E_0(0, 0, 0)$, $E_1(-a, 0, 0)$. Without loss of generality, let (x^*, y^*, z^*) be the equilibrium point of system (2.2), and let $y_1(t) = x(t) - x^*$, $y_2(t) = y(t) - y^*$, $y_3(t) = z(t) - z^*$. Substituting them into system (2.2) yields

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = y_3(t) + My_2(t) - M\int_{-\infty}^{0} y_2(t+s)k(-s)\,\mathrm{d}s, \\ \dot{y}_3(t) = ay_1(t) + by_2(t) + cy_3(t) + y_1^2(t) + 2x^*y_1(t). \end{cases} \tag{2.3}$$

Rewrite system (2.3) as follows:

$$\dot{y}(t) = Ly(t) + \int_{-\infty}^{0} F(s)y(t+s)\,\mathrm{d}s + H(y), \tag{2.4}$$

where

$$y(t)=\begin{pmatrix} y_1(t)\\ y_2(t)\\ y_3(t)\end{pmatrix},\qquad L=\begin{pmatrix} 0 & 1 & 0\\ 0 & M & 1\\ a+2x^* & b & c\end{pmatrix},$$
$$F(s)=\begin{pmatrix} 0 & 0 & 0\\ 0 & -Mk(-s) & 0\\ 0 & 0 & 0\end{pmatrix},\qquad H(y)=\begin{pmatrix} 0\\ 0\\ y_1^2(t)\end{pmatrix}.$$

The corresponding characteristic equation appears as

$$\lambda(\lambda-c)\left(\lambda-M+\int_{-\infty}^{0} Mk(-s)e^{\lambda s}\,\mathrm{d}s\right)-b\lambda-\left(a+2x^*\right)=0. \tag{2.5}$$

In this paper, we consider the weak kernel case , *i.e.*, $k(s)=\alpha e^{-\alpha s}$, where $\alpha>0$. The analysis for the general gamma kernel case is similar. We define the initial condition of system (2.3) as follows:

$$\begin{pmatrix} y_1(s)\\ y_2(s)\\ y_3(s)\end{pmatrix}=\begin{pmatrix} \phi_1(s)\\ \phi_2(s)\\ \phi_3(s)\end{pmatrix},\quad -\infty<s\le 0.$$

The characteristic equation (2.5) under the weak kernel case then takes the form

$$\lambda^4+n_1(\alpha)\lambda^3+n_2(\alpha)\lambda^2+n_3(\alpha)\lambda+n_4(\alpha)=0, \tag{2.6}$$

where

$$\begin{cases} n_1(\alpha)=\alpha-M-c,\\ n_2(\alpha)=Mc-c\alpha-b,\\ n_3(\alpha)=-b\alpha-a-2x^*,\\ n_4(\alpha)=-(a+2x^*)\alpha.\end{cases}$$

It follows from the well-known Routh-Hurwitz criterion that all the roots of Eq. (2.6) have negative real parts if the following conditions are satisfied:

$$\begin{cases} D_1(\alpha)\equiv n_1(\alpha)=\alpha-M-c>0,\\ D_2(\alpha)\equiv n_1(\alpha)n_2(\alpha)-n_3(\alpha)=(\alpha-M-c)(Mc-c\alpha-b)+b\alpha+a+2x^*>0,\\ D_3(\alpha)\equiv n_3(\alpha)D_2(\alpha)-n_1^2(\alpha)n_4(\alpha)=(-b\alpha-a-2x^*)[(\alpha-M-c)(Mc-c\alpha-b)\\ \qquad +b\alpha+a+2x^*]+\alpha(a+2x^*)(\alpha-M-c)^2>0,\\ D_4(\alpha)\equiv n_4(\alpha)D_3(\alpha)>0.\end{cases}$$

It is easy to see that at $E_1(-a,0,0)$, we always have $n_4(a)=a\alpha<0$, thus the equilibrium point $E_1(-a,0,0)$ is unstable. In what follows, we only analyze the equilibrium point $E_0(0,0,0)$. Straightforwardly, we have the following result.

Theorem 1 *The equilibrium point $E_0(0,0,0)$ of system (2.2), where $k(s)$ represents the weak kernel, is locally asymptotically stable if the following conditions hold:*

$$\begin{cases} \alpha - M - c > 0, \\ (\alpha - M - c)(Mc - c\alpha - b) + b\alpha + a > 0, \\ (-b\alpha - a)[(\alpha - M - c)(Mc - c\alpha - b) + b\alpha + a] + a\alpha(\alpha - M - c)^2 > 0. \end{cases} \tag{2.7}$$

Let λ_i $(i = 1,2,3,4)$ be the roots of Eq. (2.6), then we have

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -n_1(\alpha),$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = n_2(\alpha),$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 = -n_3(\alpha),$$

$$\lambda_1\lambda_2\lambda_3\lambda_4 = n_4(\alpha).$$

If there exists an $\alpha_0 \in R^+$ such that $D_3(\alpha_0) = 0$ and $\mathrm{d}D_3(\alpha)/\mathrm{d}\alpha|_{\alpha=\alpha_0} \neq 0$, then by the Routh-Hurwitz criterion, there exists a pair of purely imaginary roots, say $\lambda_1 = \overline{\lambda_2} = \mathrm{i}\omega_0$ $(\omega_0 \neq 0)$, and the other two roots λ_3, λ_4 satisfy: if λ_3, λ_4 are real, then $\lambda_3 < 0$, $\lambda_4 < 0$; if λ_3, λ_4 are complex conjugate, then $\operatorname{Re}\lambda_3 = \operatorname{Re}\lambda_4 = -n_1(\alpha)/2$. It is easy to calculate that

$$\left.\frac{\mathrm{d}(\operatorname{Re}\lambda_1)}{\mathrm{d}\alpha}\right|_{\alpha_0} = -\frac{n_1(\alpha)}{2[n_1^3(\alpha)n_3(\alpha) + (n_1(\alpha)n_2(\alpha) - 2n_3(\alpha))^2]} \cdot \left.\frac{\mathrm{d}D_3(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha_0},$$

thus the Hopf bifurcation occurs at E_0 as α passes through α_0.

3 Direction and stability of bifurcating periodic solutions

In this section, we investigate the direction, stability and period of bifurcating periodic solutions from the steady state by applying the normal form theory and the center manifold theorem developed by Hassard *et al.* in [12]. Let $\mu = \alpha - \alpha_0$, then system (2.3) undergoes the Hopf bifurcation at $E_0(0,0,0)$ near $\mu = 0$. Assume that $\pm\mathrm{i}\omega_0$ is the corresponding purely imaginary roots of Eq. (2.6) at steady state $E_0(0,0,0)$ for $\mu = 0$. We transform system (2.4) into an FDE in $C((-\infty,0],\mathbf{R}^3)$ as

$$\dot{y}_t = A(\mu)y_t + R(\mu)y_t, \tag{3.1}$$

where $y_t(\theta) = y(t+\theta)$, $\theta \in (-\infty,0)$, $y = (y_1,y_2,y_3)^T$, and operators A and R are defined as

$$A(\mu)\phi(\theta) = \begin{cases} \frac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & -\infty < \theta < 0, \\ L\phi(\theta) + \int_{-\infty}^{0} F(s)\phi(s)\,\mathrm{d}s, & \theta = 0, \end{cases}$$

$$R(\mu)\phi(\theta) = \begin{cases} (0,0,0)^T, & -\infty < \theta < 0, \\ (0,0,f_3)^T, & \theta = 0, \end{cases}$$

where

$$f_3 = \phi_1^2(0).$$

For $\psi \in C([0,+\infty),(\mathbf{R}^3)^*)$, $\psi(s) = (\psi_1(s),\psi_2(s),\psi_3(s))^T \in C[0,+\infty)$, the adjoint operator of A denoted by A^* is defined as

$$A^*(\mu)\psi(s) = \begin{cases} -\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & 0 < s < +\infty, \\ L^T\psi(0) + \int_{-\infty}^0 F^T(t)\psi(-t)\,\mathrm{d}t, & s = 0. \end{cases} \tag{3.2}$$

For $\phi \in C(-\infty,0]$ and $\psi \in C[0,+\infty)$, a bilinear inner product is defined as

$$\langle\psi,\phi\rangle = \bar{\psi}^T(0)\phi(0) - \int_{\theta=-\infty}^0 \int_{\xi=0}^{\theta} \bar{\psi}^T(\xi-\theta)F(\theta)\phi(\xi)\,\mathrm{d}\xi\,\mathrm{d}\theta.$$

In what follows, we need to calculate the eigenvector q of A associated with the eigenvalue $\mathrm{i}\omega_0$ and the eigenvector q^* of A^* associated with the eigenvalue $-\mathrm{i}\omega_0$. Assume that $q(\theta) = (1,\beta,\gamma)^T e^{\mathrm{i}\omega_0\theta}$ is the eigenvector of $A(0)$ corresponding to $\mathrm{i}\omega_0$, then $A(0)q(0) = \mathrm{i}\omega_0 q(0)$, namely

$$\begin{aligned} &Lq(0) + \int_{-\infty}^0 F(s)q(s)\,\mathrm{d}s \\ &\quad= \begin{pmatrix} 0 & 1 & 0 \\ 0 & M & 1 \\ a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ \beta \\ \gamma \end{pmatrix} + \int_{-\infty}^0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -Mk(-s) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\qquad\cdot \begin{pmatrix} 1 \\ \beta \\ \gamma \end{pmatrix} e^{\mathrm{i}\omega_0 s}\,\mathrm{d}s = \begin{pmatrix} \beta \\ M\beta + \gamma - M\beta + \gamma - M\beta J^{(1)} \\ a + b\beta + c\gamma \end{pmatrix} = \begin{pmatrix} \mathrm{i}\omega_0 \\ \mathrm{i}\omega_0\beta \\ \mathrm{i}\omega_0\gamma \end{pmatrix}, \end{aligned}$$

where

$$J^{(1)} = \int_{-\infty}^0 k(-s)e^{\mathrm{i}\omega_0 s}\,\mathrm{d}s = \frac{\alpha}{\alpha + \mathrm{i}\omega_0}.$$

It is easy to calculate from the above equality that

$$\begin{cases} \beta = \mathrm{i}\omega_0, \\ \gamma = \frac{\alpha + \mathrm{i}\omega_0 b}{\mathrm{i}\omega_0 - c} = -\omega_0^2 - \mathrm{i}\omega_0 M + \mathrm{i}\omega_0 M J^{(1)}. \end{cases}$$

Assume that $q^*(\zeta) = N(1,\beta^*,\gamma^*)^T e^{\mathrm{i}\omega_0\zeta}$, $0 \le \zeta < +\infty$, then $A^*(0)q^*(0) = -\mathrm{i}\omega_0 q^*(0)$, that is,

$$\begin{aligned} &L^T q^*(0) + \int_{-\infty}^0 F^*(s)q^*(-s)\,\mathrm{d}s \\ &\quad= \begin{pmatrix} 0 & 0 & a \\ 1 & M & b \\ 0 & 1 & c \end{pmatrix} \begin{pmatrix} N \\ N\beta^* \\ N\gamma^* \end{pmatrix} + \int_{-\infty}^0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -Mk(-s) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\qquad\cdot \begin{pmatrix} N \\ N\beta^* \\ N\gamma^* \end{pmatrix} e^{-\mathrm{i}\omega_0 s}\,\mathrm{d}s = \begin{pmatrix} aN\gamma^* \\ N + MN\beta^* + bN\gamma^* - MN\beta^* J^{(-1)} \\ \beta^* N + cN\gamma^* \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\omega_0 N \\ \mathrm{i}\omega_0\beta^* N \\ \mathrm{i}\omega_0\gamma^* N \end{pmatrix}, \end{aligned}$$

where

$$J^{(-1)} = \int_{-\infty}^{0} k(-s)e^{-\mathrm{i}\omega_0 s}\,\mathrm{d}s = \frac{\alpha}{\alpha - \mathrm{i}\omega_0}.$$

Hence we have

$$\begin{cases} \beta^* = \frac{\omega_0^2+\mathrm{i}\omega_0 c}{a} = \frac{a-\mathrm{i}\omega_0}{a(\mathrm{i}\omega_0 - M + MJ^{(-1)})}, \\ \gamma^* = \frac{-\mathrm{i}\omega_0}{a}. \end{cases}$$

We choose $N = \frac{1}{1+\beta^*\bar{\beta}+\gamma^*\bar{\gamma}+M\beta^*\bar{\beta}\int_{\theta=-\infty}^{0}\theta e^{-\mathrm{i}\omega_0\theta}k(-\theta)\,\mathrm{d}\theta}$, then $\langle q^*, q\rangle = 1$, $\langle q^*, \bar{q}\rangle = 0$ hold. In what follows, we follow the same notations as in [12]. We first construct the coordinates of the center manifold Ω_0 at $\mu = 0$. Let

$$z(t) = \langle q^*, u_t\rangle, \qquad w(t,\theta) = u_t - 2\,\mathrm{Re}\{z(t)q(\theta)\}.$$

On the center manifold Ω_0, we have

$$w(t,\theta) = w(z,\bar{z},\theta),$$

where

$$w(z,\bar{z},\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{3.3}$$

and z, $\bar{z}$ are the local coordinates of the center manifold Ω_0 in the directions of q^* and $\bar{q}^*$ respectively.

Note that w is real if u_t is real. We only consider real solutions. For the solution $u_t \in \Omega_0$ of (3.1), since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= \mathrm{i}\omega_0 z(t) + \langle q^*, R(w + 2\,\mathrm{Re}\{z(t)q(\theta)\})\rangle \\ &= \mathrm{i}\omega_0 z(t) + \overline{q^*}^T(0)R(w(z,\bar{z},0) + 2\,\mathrm{Re}\{z(t)q(0)\}). \end{aligned}$$

Rewrite the above equation as

$$\dot{z}(t) = \mathrm{i}\omega_0 z(t) + g(z,\bar{z}),$$

where

$$g(z,\bar{z}) = \overline{q^*}^T(0)R(w(z,\bar{z},0) + 2\,\mathrm{Re}\{z(t)q(0)\}). \tag{3.4}$$

Expand the function $g(z,\bar{z})$ on the center manifold Ω_0 as

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.5}$$

By (3.1) and (3.3), we have

$$\dot{w} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = Aw - 2\,\mathrm{Re}\{g(z,\bar{z})q(\theta)\} + R(w + 2\,\mathrm{Re}\{z(t)q(\theta)\}).$$

Rewrite this as

$$\dot{w} = Aw + H(z,\bar{z},\theta),$$

where

$$H(z,\bar{z},\theta) = -2\,\mathrm{Re}\{g(z,\bar{z})q(\theta)\} + R\bigl(w + 2\,\mathrm{Re}\{z(t)q(\theta)\}\bigr). \tag{3.6}$$

Expand the function $H(z,\bar{z},\theta)$ on the center manifold Ω_0 as

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

While

$$w + zq(\theta) + \bar{z}\bar{q}(\theta) = \begin{pmatrix} w^{(1)}(\theta) + ze^{\mathrm{i}\omega_0\theta} + \bar{z}e^{-\mathrm{i}\omega_0\theta} \\ w^{(2)}(\theta) + z\beta e^{\mathrm{i}\omega_0\theta} + \bar{z}\bar{\beta}e^{-\mathrm{i}\omega_0\theta} \\ w^{(3)}(\theta) + z\gamma e^{\mathrm{i}\omega_0\theta} + \bar{z}\bar{\gamma}e^{-\mathrm{i}\omega_0\theta} \end{pmatrix}.$$

Thus

$$R\bigl(w + 2\,\mathrm{Re}\{z(t)q(\theta)\}\bigr) = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, & -\infty < \theta < 0, \\ \begin{pmatrix} 0 \\ 0 \\ f_0^{(3)} \end{pmatrix}, & \theta = 0, \end{cases}$$

where

$$f_0^{(3)} = \bigl[w^{(1)}(0) + z + \bar{z}\bigr]^2.$$

It follows from (3.4) that

$$g(z,\bar{z}) = \overline{N}\bigl(1, \overline{\beta^*}, \overline{\gamma^*}\bigr)\begin{pmatrix} 0 \\ 0 \\ f_0^{(3)} \end{pmatrix} = \overline{N}\overline{\gamma^*}f_0^{(3)}(z,\bar{z}).$$

Hence

$$H(z,\bar{z},\theta) = -2\,\mathrm{Re}\{\overline{N}\overline{\gamma^*}f_0^{(3)}(z,\bar{z})q(\theta)\} + \begin{cases} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, & -\infty < \theta < 0, \\ \begin{pmatrix} 0 \\ 0 \\ f_0^{(3)} \end{pmatrix}, & \theta = 0. \end{cases}$$

From (3.4), we obtain

$$H_{20}(\theta) = -2\,\mathrm{Re}\big\{\overline{N}\overline{\gamma^*} f^{(3)}_{0,z^2} q(\theta)\big\} + \begin{cases} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, & -\infty < \theta < 0, \\ \begin{pmatrix} 0 \\ 0 \\ f^{(3)}_{0,z^2} \end{pmatrix}, & \theta = 0. \end{cases}$$

Notice that $f^{(3)}_{0,z^2} = 2$, we have

$$H_{20}(\theta) = -4\,\mathrm{Re}\big\{\overline{N}\overline{\gamma^*} q(\theta)\big\} + \begin{cases} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, & -\infty < \theta < 0, \\ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, & \theta = 0. \end{cases} \tag{3.7}$$

Similarly, we have

$$H_{11}(\theta) = 4\,\mathrm{Re}\big\{\overline{N}\overline{\gamma^*} q(\theta)\big\} + \begin{cases} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, & -\infty < \theta < 0, \\ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, & \theta = 0. \end{cases}$$

On the other hand, on the center manifold Ω_0 near the origin, we have

$$\dot{w}(z,\overline{z}) = w_z \dot{z} + w_{\overline{z}} \dot{\overline{z}}.$$

Expanding the above equation and comparing the corresponding coefficients, we get

$$\begin{aligned} &(A - 2\mathrm{i}\omega_0 I) w_{20}(\theta) = -H_{20}(\theta), \\ &A w_{11}(\theta) = -H_{11}(\theta). \end{aligned} \tag{3.8}$$

Define

$$w_{20}(\theta) = \begin{pmatrix} w^{(1)}_{20}(\theta) \\ w^{(2)}_{20}(\theta) \\ w^{(3)}_{20}(\theta) \end{pmatrix}, \quad -\infty < \theta < 0.$$

Substituting (3.2) and (3.7) into (3.8), when $-\infty < \theta < 0$, we have

$$\begin{pmatrix} 2\mathrm{i}\omega_0 - \frac{\mathrm{d}}{\mathrm{d}\theta} & 0 & 0 \\ 0 & 2\mathrm{i}\omega_0 - \frac{\mathrm{d}}{\mathrm{d}\theta} & 0 \\ 0 & 0 & 2\mathrm{i}\omega_0 - \frac{\mathrm{d}}{\mathrm{d}\theta} \end{pmatrix} \begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta) \\ w_{20}^{(3)}(\theta) \end{pmatrix} = \begin{pmatrix} -4\,\mathrm{Re}\{\overline{N\gamma^*}e^{\mathrm{i}\omega_0\theta}\} \\ -4\,\mathrm{Re}\{\overline{N\gamma^*}\beta e^{\mathrm{i}\omega_0\theta}\} \\ -4\,\mathrm{Re}\{\overline{N\gamma^*}\gamma e^{\mathrm{i}\omega_0\theta}\} \end{pmatrix}. \tag{3.9}$$

When $\theta = 0$, we obtain

$$\begin{aligned} &\begin{pmatrix} 2\mathrm{i}\omega_0 & -1 & 0 \\ 0 & 2\mathrm{i}\omega_0 - M & -1 \\ -a & -b & 2\mathrm{i}\omega_0 - c \end{pmatrix} \begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0) \\ w_{20}^{(3)}(0) \end{pmatrix} - \int_{-\infty}^{0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -Mk(-s) & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} w_{20}^{(1)}(s) \\ w_{20}^{(2)}(s) \\ w_{20}^{(3)}(s) \end{pmatrix} \mathrm{d}s = \begin{pmatrix} H_{20}^{(1)}(0) \\ H_{20}^{(2)}(0) \\ H_{20}^{(3)}(0) \end{pmatrix} = \begin{pmatrix} -4\,\mathrm{Re}\{\overline{N\gamma^*}\} \\ -4\,\mathrm{Re}\{\overline{N\gamma^*}\beta\} \\ -4\,\mathrm{Re}\{\overline{N\gamma^*}\gamma\} + 2 \end{pmatrix}. \end{aligned} \tag{3.10}$$

In order to guarantee the continuity of solutions, we further assume that $\begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta) \\ w_{20}^{(3)}(\theta) \end{pmatrix}$ is continuous at $\theta = 0$.

It follows from (3.9) that

$$\begin{cases} \frac{\mathrm{d}w_{20}^{(1)}(\theta)}{\mathrm{d}\theta} = 2\mathrm{i}\omega_0 w_{20}^{(1)}(\theta) + 4\,\mathrm{Re}\{\overline{N\gamma^*}e^{\mathrm{i}\omega_0\theta}\}, \\ \frac{\mathrm{d}w_{20}^{(2)}(\theta)}{\mathrm{d}\theta} = 2\mathrm{i}\omega_0 w_{20}^{(2)}(\theta) + 4\,\mathrm{Re}\{\overline{N\gamma^*}\beta e^{\mathrm{i}\omega_0\theta}\}, \\ \frac{\mathrm{d}w_{20}^{(3)}(\theta)}{\mathrm{d}\theta} = 2\mathrm{i}\omega_0 w_{20}^{(3)}(\theta) + 4\,\mathrm{Re}\{\overline{N\gamma^*}\gamma\} - 2. \end{cases}$$

The solutions of the above equations take the form

$$\begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta) \\ w_{20}^{(3)}(\theta) \end{pmatrix} = \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} e^{2\mathrm{i}\omega_0\theta} + \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} e^{\mathrm{i}\omega_0\theta} + \begin{pmatrix} l_0 \\ m_0 \\ n_0 \end{pmatrix} e^{-\mathrm{i}\omega_0\theta}, \tag{3.11}$$

where

$$\begin{cases} l_0 = \frac{8N\gamma^* e^{-\mathrm{i}\omega_0\theta}}{3\mathrm{i}\omega_0}, \\ m_0 = \beta l_0, \\ n_0 = \gamma l_0, \end{cases} \qquad \begin{cases} l_1 = \frac{8\overline{N\gamma^*}e^{\mathrm{i}\omega_0\theta}}{\mathrm{i}\omega_0}, \\ m_1 = \beta l_1, \\ n_1 = \gamma l_1, \end{cases} \qquad \begin{cases} l_2 = w_{20}^{(1)}(0) - l_0 - l_1, \\ m_2 = w_{20}^{(2)}(0) - m_0 - m_1, \\ n_2 = w_{20}^{(3)}(0) - n_0 - n_1. \end{cases}$$

Substituting (3.11) into (3.10) yields

$$\begin{pmatrix} 2\mathrm{i}\omega_0 & -1 & 0 \\ 0 & 2\mathrm{i}\omega_0 - M + MJ^{(2)} & -1 \\ -a & -b & 2\mathrm{i}\omega_0 - c \end{pmatrix} \begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0) \\ w_{20}^{(3)}(0) \end{pmatrix} = \begin{pmatrix} G_{20}^{(1)} \\ G_{20}^{(2)} \\ G_{20}^{(13)} \end{pmatrix},$$

where

$$\begin{cases} G_{20}^{(1)} = H_{20}^{(1)}(0), \\ G_{20}^{(2)} = H_{20}^{(2)}(0) + M[(m_1 + m_0)J^{(2)} - m_1 J^{(1)} - m_0 J^{(-1)}], \\ G_{20}^{(3)} = H_{20}^{(3)}(0), \end{cases}$$

and

$$J^{(2)} = \int_{-\infty}^{0} k(-s)e^{2\mathrm{i}\omega_0 s}\,\mathrm{d}s = \frac{\alpha}{\alpha + 2\mathrm{i}\omega_0}.$$

Let

$$B^{-1} = \begin{pmatrix} 2\mathrm{i}\omega_0 & -1 & 0 \\ 0 & 2\mathrm{i}\omega_0 - M + MJ^{(2)} & -1 \\ -a & -b & 2\mathrm{i}\omega_0 - c \end{pmatrix}.$$

Then

$$B = \frac{1}{\Lambda}\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix},$$

where $\Lambda = \det(B^{-1}) = -8\mathrm{i}\omega_0^3 + 4(c - M + MJ^{(2)})\omega_0^2 + 2\mathrm{i}(Mc - MJ^{(2)} - b)\omega_0 - a$, and

$$\begin{cases} B_{11} = (2\mathrm{i}\omega_0 - M + MJ^{(2)})(2\mathrm{i}\omega_0 - c) - b, \\ B_{12} = 2\mathrm{i}\omega_0 - c, \\ B_{13} = 1, \end{cases} \qquad \begin{cases} B_{21} = a, \\ B_{22} = -4\omega_0^2 - 2\mathrm{i}\omega_0 c, \\ B_{23} = 2\mathrm{i}\omega_0, \end{cases}$$

$$\begin{cases} B_{31} = -a(2\mathrm{i}\omega_0 - M + MJ^{(2)}), \\ B_{32} = 2\mathrm{i}\omega_0 b + a, \\ B_{33} = 2\mathrm{i}\omega_0(2\mathrm{i}\omega_0 - M + MJ^{(2)}). \end{cases}$$

Therefore, the following can be determined:

$$\begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0) \\ w_{20}^{(3)}(0) \end{pmatrix} = \frac{1}{\Lambda}\begin{pmatrix} B_{11}G_{20}^{(1)} + B_{12}G_{20}^{(2)} + B_{13}G_{20}^{(3)} \\ B_{21}G_{20}^{(1)} + B_{22}G_{20}^{(2)} + B_{23}G_{20}^{(3)} \\ B_{31}G_{20}^{(1)} + B_{32}G_{20}^{(2)} + B_{33}G_{20}^{(3)} \end{pmatrix}.$$

Following the similar analysis presented above, we have

$$\begin{pmatrix} w_{11}^{(1)}(\theta) \\ w_{11}^{(2)}(\theta) \\ w_{11}^{(3)}(\theta) \end{pmatrix} = \begin{pmatrix} p_2 \\ q_2 \\ r_2 \end{pmatrix} + \begin{pmatrix} p_1 \\ q_1 \\ r_1 \end{pmatrix} e^{\mathrm{i}\omega_0\theta} + \begin{pmatrix} p_0 \\ q_0 \\ r_0 \end{pmatrix} e^{-\mathrm{i}\omega_0\theta},$$

where

$$\begin{cases} p_0 = \frac{2N\gamma^*}{\mathrm{i}\omega_0}, \\ q_0 = \beta p_0, \\ r_0 = \gamma p_0, \end{cases} \qquad \begin{cases} p_1 = -\frac{2\overline{N}\overline{\gamma^*}}{\mathrm{i}\omega_0}, \\ q_1 = \beta p_1, \\ r_1 = \gamma p_1, \end{cases} \qquad \begin{cases} p_2 = w_{11}^{(1)}(0) - p_0 - p_1, \\ q_2 = w_{11}^{(2)}(0) - q_0 - q_1, \\ r_2 = w_{11}^{(3)}(0) - r_0 - r_1. \end{cases}$$

The following can be calculated:

$$\begin{pmatrix} w_{11}^{(1)}(0) \\ w_{11}^{(2)}(0) \\ w_{11}^{(3)}(0) \end{pmatrix} = \frac{1}{-a} \begin{pmatrix} C_{11}G_{11}^{(1)} + C_{12}G_{11}^{(2)} + C_{13}G_{11}^{(3)} \\ C_{21}G_{11}^{(1)} + C_{22}G_{11}^{(2)} + C_{23}G_{11}^{(3)} \\ C_{31}G_{11}^{(1)} + C_{32}G_{11}^{(2)} + C_{33}G_{11}^{(3)} \end{pmatrix},$$

where

$$\begin{cases} G_{11}^{(1)} = H_{11}^{(1)}(0), \\ G_{11}^{(2)} = H_{11}^{(2)}(0) + M[q_0 + q_1 - q_0 J(-1) - q_1 J(1)], \\ G_{11}^{(3)} = H_{11}^{(3)}(0), \end{cases}$$

and

$$\begin{cases} C_{11} = Mc - MJ^{(2)}c, \\ C_{12} = -c, \\ C_{13} = 1, \end{cases} \qquad \begin{cases} C_{21} = a, \\ C_{22} = 0, \\ C_{23} = 0, \end{cases} \qquad \begin{cases} C_{31} = -Ma + MJ^{(2)}a, \\ C_{32} = a, \\ C_{33} = 0. \end{cases}$$

Next, we consider $R(w(z,\overline{z},0) + 2\operatorname{Re}\{z(t)q(0)\})$, noticing

$$\begin{aligned} & w(z,\overline{z},0) + 2\operatorname{Re}\{z(t)q(0)\} \\ &= w_{20}(0)\frac{z^2}{2} + w_{11}(0)z\overline{z} + w_{02}(0)\frac{\overline{z}^2}{2} + \cdots + 2\operatorname{Re}\{z(t)q(0)\} \\ &= \begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0) \\ w_{20}^{(3)}(0) \end{pmatrix} \frac{z^2}{2} + \begin{pmatrix} w_{11}^{(1)}(0) \\ w_{11}^{(2)}(0) \\ w_{11}^{(3)}(0) \end{pmatrix} z\overline{z} + \begin{pmatrix} w_{02}^{(1)}(0) \\ w_{02}^{(2)}(0) \\ w_{02}^{(3)}(0) \end{pmatrix} \frac{\overline{z}^2}{2} + \cdots + 2\operatorname{Re}\{z(t)q(0)\}. \end{aligned}$$

While

$$f_0^{(3)} = \left[w_{20}^{(1)}(0)\frac{z^2}{2} + w_{11}^{(1)}(0)z\overline{z} + w_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots + z + \overline{z} \right]^2.$$

Hence

$$\begin{aligned} g(z,\overline{z}) &= \overline{q^*}^T(0)R\big(w(z,\overline{z},0) + 2\operatorname{Re}\{z(t)q(0)\}\big) \\ &= \overline{N}\big(1, \overline{\beta^*}, \overline{\gamma^*}\big) \begin{pmatrix} 0 \\ 0 \\ f_0^{(3)} \end{pmatrix} \\ &= \overline{N}\overline{\gamma^*} f_0^{(3)}. \end{aligned}$$

Comparing the coefficients with (3.5), we obtain

$$\begin{aligned}
g_{20} &= 2\overline{N}\overline{\gamma^*},\\
g_{11} &= 2\overline{N}\overline{\gamma^*},\\
g_{02} &= 2\overline{N}\overline{\gamma^*},\\
g_{21} &= 2\overline{N}\overline{\gamma^*}\big(w_{20}^{(1)}(0) + 2w_{11}^{(1)}(0)\big).
\end{aligned}$$

Therefore, the following values can be calculated

$$\begin{cases}
c_1(0) = \frac{\mathrm{i}}{2\omega_0}[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2] + \frac{g_{21}}{2},\\
\mu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda_1'(\alpha_0)\}},\\
\tau_2 = -\frac{\mathrm{Im}\{c_1(0)\}+\mu_2\,\mathrm{Im}\{\lambda_1'(\alpha_0)\}}{\omega_0},\\
\beta_2 = 2\,\mathrm{Re}\{c_1(0)\},
\end{cases} \tag{3.12}$$

which determine the quantities of bifurcating periodic solutions on the center manifold Ω_0 at the critical value α_0, *i.e.*, μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\alpha > \alpha_0$; τ_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $\tau_2 > 0$ ($\tau_2 < 0$); β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$).

4 Numerical simulations

In this section, we shall perform some numerical simulations to verify the analytical results presented in the previous sections. Let us take $a = -6$, $b = -2.92$, $c = -1.2$, $M = -1$ in system (2.2) and consider the weak kernel case, *i.e.*,

$$\begin{cases}
\dot{x}(t) = y(t),\\
\dot{y}(t) = z(t) - \int_{-\infty}^{0}(y(t) - y(t+s))k(-s)\,\mathrm{d}s,\\
\dot{z}(t) = -6x(t) - 2.92y(t) - 1.2z(t) + x^2(t),
\end{cases} \tag{4.1}$$

with $k(s) = \alpha e^{-\alpha s}$, $\alpha > 0$. The initial conditions are given as $x(0) = 0.3$, $y(0) = 0.1$, $z(0) = 0.3$; $y(t \le 0) = 0.1$. We can easily determine by these parameters that $\alpha_0 \doteq 1.40017$. For a very small α, when $\alpha = 0.00001$, we see in Figure 2 that the attractor of system (4.1) still exists. But when $\alpha = 1.3 < 1.40017$, as all the conditions in (2.7) are satisfied, the equilibrium point $E_0(0,0,0)$ in system (4.1) is asymptotically stable, which is illustrated in Figure 3. When $\alpha = \alpha_0 \doteq 1.40017$, system (2.2) undergoes the Hopf bifurcation at $E_0(0,0,0)$, as illustrated in Figure 4. Moreover, from the formulae (3.12) presented in Section 3, it follows that $\mu_2 > 0$, $\beta_2 < 0$, $\tau_2 > 0$, the Hopf bifurcation is supercritical and the direction of the bifurcation is $\alpha > \alpha_0$. However, as α increases, when $\alpha = 1.5 > 1.40017$, the third condition in (2.7) does not hold, hence the equilibrium point $E_0(0,0,0)$ in system (4.1) is unstable, as illustrated in Figure 5.

As compared with the former method, a chaotic model with distributed delay feedback is more general than that with discrete delay feedback [3–5], because the distributed delay

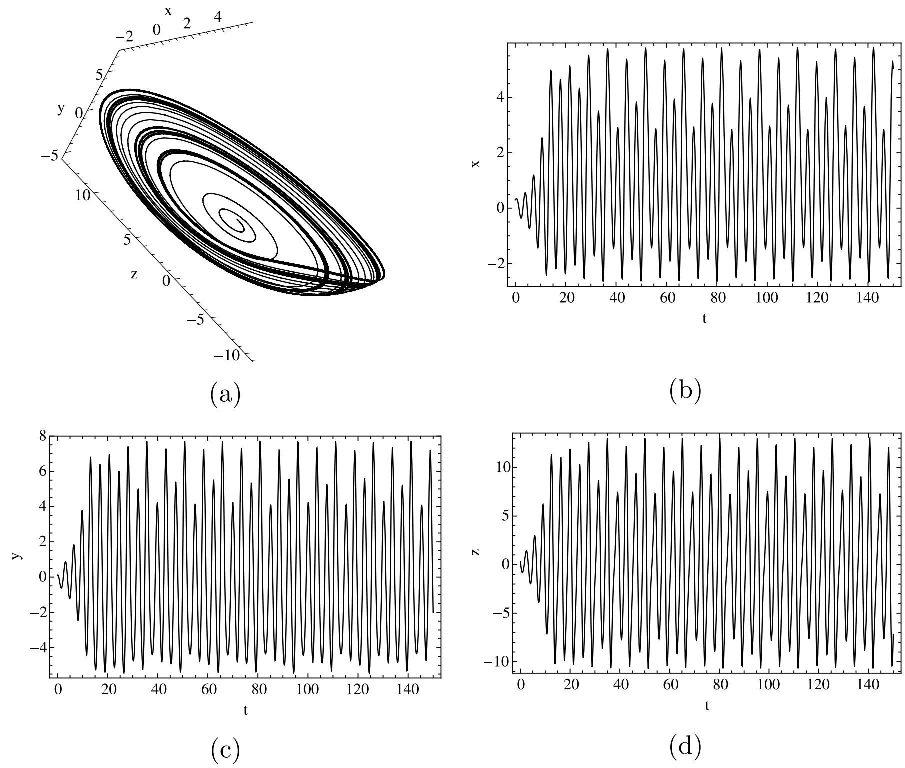

Figure 2 The attractor of system (4.1) still exists when $\alpha = 0.00001$.

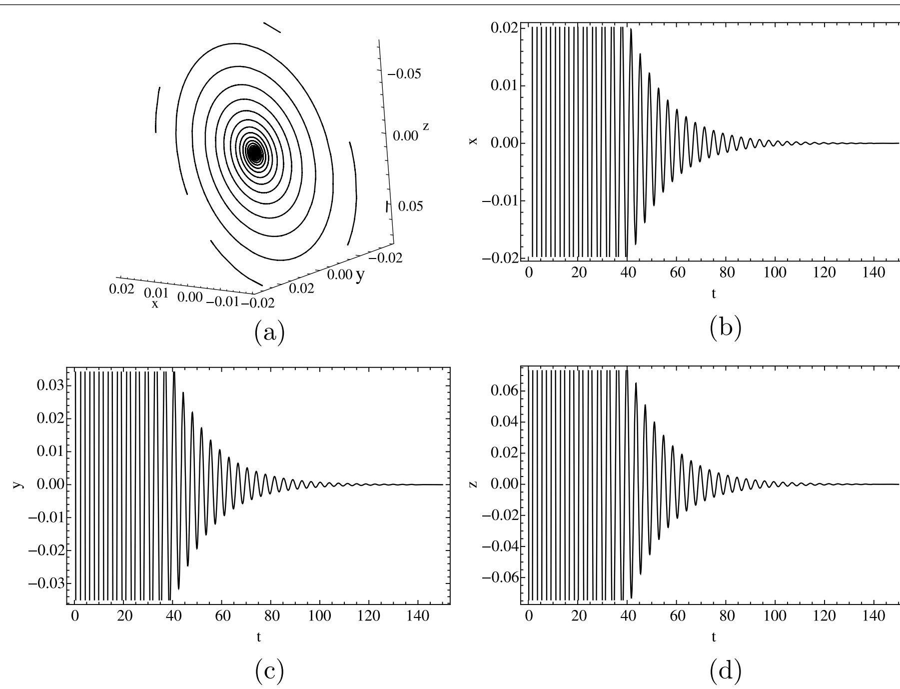

Figure 3 The equilibrium point $E_0(0,0,0)$ of system (4.1) is asymptotically stable when $\alpha = 1.3 < \alpha_0 = 1.40017$.

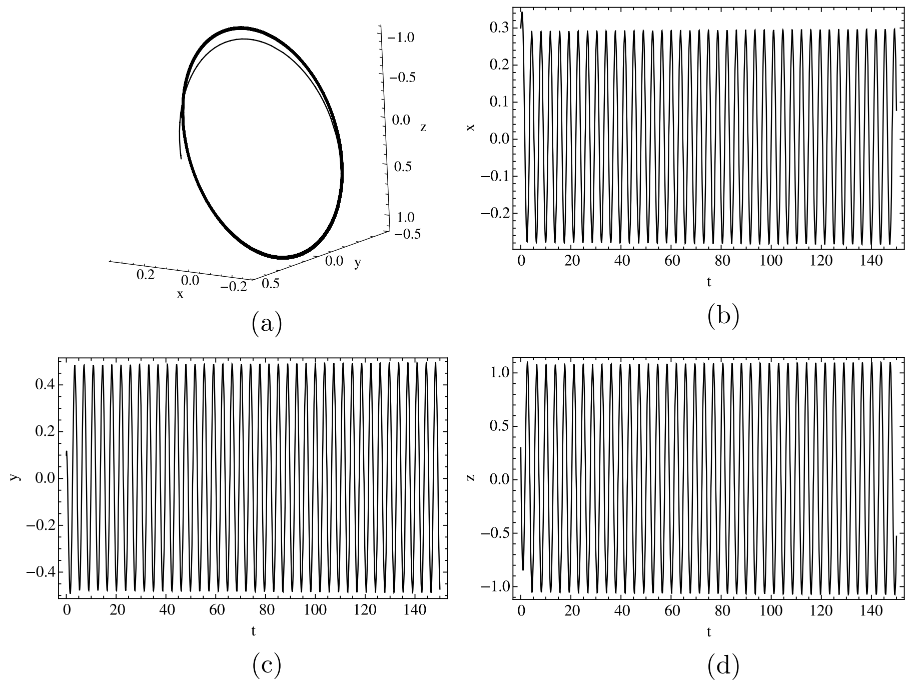

Figure 4 When $\alpha = \alpha_0 \doteq 1.40017$, a Hopf bifurcation occurs near the equilibrium point $E_0(0,0,0)$ of system (4.1).

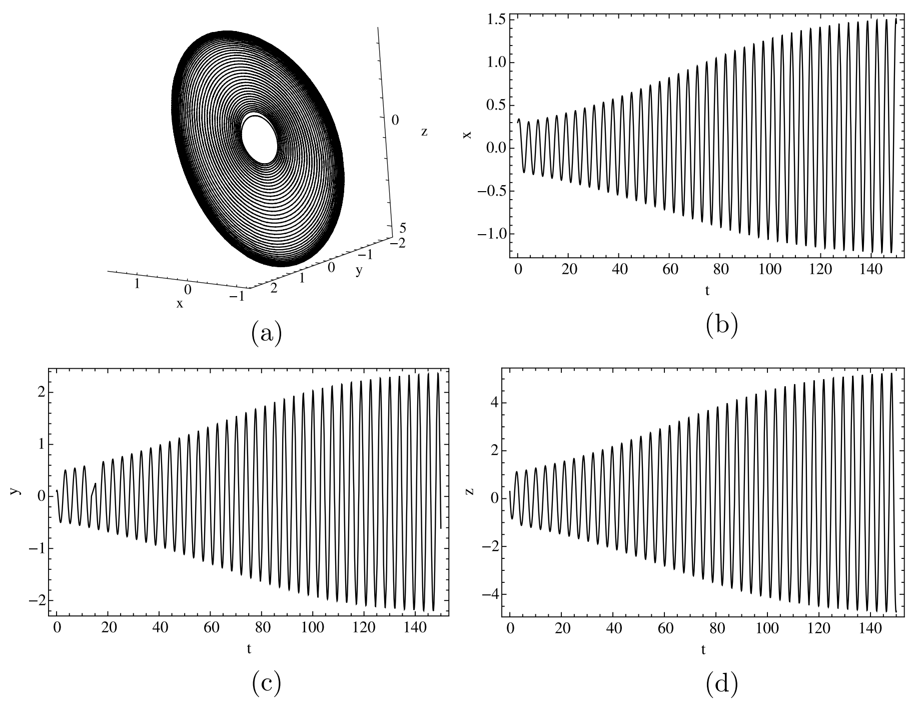

Figure 5 The equilibrium point $E_0(0,0,0)$ of system (4.1) is unstable when $\alpha = 1.5 > \alpha_0 = 1.40017$.

becomes a discrete delay when the delay kernel is a delta function at a certain time. The distributed delay has found widespread applications in many fields such as neural network [8, 10], complicated real models [9], the modeling of aggregative processes involving the flow of entities with random transit times through a given process [20], and so on. Therefore, it is of considerable significance to propose distributed delays as control input to control the chaotic system.

From the numerical simulations, we see, as the distributed delay feedback is incorporated in the chaotic Genesio system, a rich spectrum of dynamical behaviors can occur by adjusting the mean time delay values. Chaotic behaviors vanish and the orbitally asymptotically stable Hopf bifurcation occurs as the mean time delay reaches a certain value. Also, we can determine the critical mean time delay value that the Hopf bifurcation occurs at, which is of great help when choosing appropriate parameter values to realize Hopf bifurcation control.

5 Concluding remarks

In this paper, the Genesio system with distributed time delay feedback has been studied. It has been demonstrated that the Hopf bifurcation occurs near the steady state as the average time delay crosses the critical value. The explicit formulae for determining the direction, stability and period of bifurcating periodic solutions have been presented by using the normal form theory and the center manifold theorem. A numerical example is provided to verify the theoretical results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JG drafted the manuscript. FC and GW checked the writing and revised the manuscript. All authors read and approved the final manuscript.

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