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A fractional order nonlinear dynamical model of interpersonal relationships

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Abstract

In this paper, a fractional order nonlinear dynamical model of interpersonal relationships has been introduced. The stability of equilibrium points is studied. Numerical simulations are also presented to verify the obtained results.

Keywords: fractional model; fractional differential equations; stability; numerical solution

1 Introduction

In recent decades the study of interpersonal relationships has begun to be popular. Interpersonal relationships appear in many contexts such as family, kinship, acquaintance, work, and clubs [1]. Mathematical modeling in interpersonal relationships is very important for capturing the dynamics of people. But there are few models in this area, and models have been restricted to integer order differential equations. Since experiments in this area are difficult to design and may be constrained by ethical considerations, mathematical models can play a vital role in studying the dynamics of relationships and their behavioral features [2]. In this paper, we consider a system of nonlinear fractional differential equations. This fractional system of equations is obtained by replacing a derivative term by a fractional derivative of order $\alpha > 0$. The integer order model reported in [2] is given as

$$\frac{dX_1}{dt} = -\alpha_1 X_1 + \beta_1 X_2\left(1 - \varepsilon X_2^2\right) + A_1,$$
$$\frac{dX_2}{dt} = -\alpha_2 X_2 + \beta_2 X_1\left(1 - \varepsilon X_1^2\right) + A_2.$$

A fractional order system instead of its integer order counterpart has been considered because fractional order differential equations are generalizations of integer order differential equations and fractional order models possess memory. Also, the fact that interpersonal relationships are influenced by memory makes fractional modeling appropriate for this kind of dynamical systems [3].

In this paper, firstly a fractional order nonlinear dynamical model of interpersonal relationships has been introduced. A detailed analysis for the asymptotic stability of equilibrium points has been given. Finally, numerical simulations are presented to verify the obtained results.

2 Model

First of all, we recall the definitions of fractional order integrals and derivatives [4].

Definition 1 The Riemann-Liouville type fractional integral of order $\alpha > 0$ for a function $f:(0,\infty) \to R$ is defined by

$$\mathbf{I}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau.$$

Here and elsewhere $\Gamma(\cdot)$ denotes the gamma function.

Definition 2 The Riemann-Liouville type fractional derivative of order $\alpha > 0$ for a function $f:(0,\infty) \to R$ is defined by

$$\mathbf{D}^{\alpha} f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau,$$

where $n = \lceil \alpha \rceil + 1$ and $\lceil \alpha \rceil$ is the integer part of α.

Definition 3 The Caputo-type fractional derivative of order $\alpha > 0$ for a function $f:(0,\infty) \to R$ is defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{n}(\tau)\, d\tau,$$

where $n = \lceil \alpha \rceil + 1$ and $\lceil \alpha \rceil$ is the integer part of α.

Generally, in mathematical modeling we use Caputo's definition. The main advantage of Caputo's definition is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations [4].

The model that we study in this paper is a fractional order nonlinear dynamical model of interpersonal relationships. The model is a nonlinear dynamical system with two state variables. The variables X_1 and X_2 are the measures of love of individuals 1 and 2 for their respective partners, where positive and negative measures represent feelings. The following model is considered:

$$\begin{aligned} D^{\alpha} X_1(t) &= -\alpha_1 X_1 + \beta_1 X_2 \left(1 - \varepsilon X_2^2\right) + A_1, \\ D^{\alpha} X_2(t) &= -\alpha_2 X_2 + \beta_2 X_1 \left(1 - \varepsilon X_1^2\right) + A_2 \end{aligned} \tag{1}$$

with initial conditions

$$X_1(0) = X_{01}, \qquad X_2(0) = X_{02}, \tag{2}$$

where $0 < \alpha \leq 1$, $\alpha_i > 0$, α_i, β_i, and A_i $(i = 1, 2)$ are real constants. These parameters are oblivion, reaction, and attraction constants. In the equations above, we assume that feelings decay exponentially fast in the absence of partners. The parameters specify the romantic style of individuals 1 and 2. For instance, α_i describes the extent to which individual i is encouraged by his/her own feeling. In other words, α_i indicates the degree to which an individual has internalized a sense of his/her self-worth. In addition, it can be used as the level of anxiety and dependency on other person's approval in romantic relationships. The parameter β_i represents the extent to which individual i is encouraged by his/her partner, and/or expects his/her partner to be supportive. It measures the tendency to seek or avoid closeness in a romantic relationship. Therefore, the term $-\alpha_i X_i$ says that the love measure of i, in the absence of the partner, decays exponentially and $\frac{1}{\alpha_i}$ is the time required for love to decay.

3 Equilibrium points and their asymptotic stability

Let $\alpha \in (0,1]$ and consider the system

$$D^\alpha X_1(t) = f_1(X_1, X_2),$$
$$D^\alpha X_2(t) = f_2(X_1, X_2),$$

with the initial values $X_1(0) = X_{01}$ and $X_2(0) = X_{02}$. Here $f_1(X_1, X_2) = -\alpha_1 X_1 + \beta_1 X_2(1-\varepsilon X_2^2) + A_1$ and $f_2(X_1, X_2) = -\alpha_2 X_2 + \beta_2 X_1(1-\varepsilon X_1^2) + A_2$.

To evaluate the equilibrium points, let

$$D^\alpha X_i(t) = 0 \quad \Longrightarrow \quad f_i\bigl(X_1^*, X_2^*\bigr) = 0, \quad i = 1,2,$$

from which we can get the equilibrium points $K_0 = (0,0)$ for $A_1 = A_2 = 0$ and $K_1 = (X_1^*, X_2^*)$.

The Jacobian matrix $J(X_1^*, X_2^*)$ for the system given in (1) is

$$J\bigl(X_1^*, X_2^*\bigr) = \frac{\partial(f_1, f_2)}{\partial(X_1^*, X_2^*)} = \begin{bmatrix} \frac{\partial f_1}{\partial X_1^*} & \frac{\partial f_1}{\partial X_2^*} \\ \frac{\partial f_2}{\partial X_1^*} & \frac{\partial f_2}{\partial X_2^*} \end{bmatrix} = \begin{bmatrix} -\alpha_1 & \beta_1(1-3\varepsilon X_2^{*2}) \\ \beta_2(1-3\varepsilon X_1^{*2}) & -\alpha_2 \end{bmatrix}.$$

Mutual apathy equilibrium for the system given in (1) is $K_0 = (X_1^*, X_2^*) = (0,0)$ and the Jacobian matrix at this point is as follows:

$$J(0,0) = \begin{bmatrix} -\alpha_1 & \beta_1 \\ \beta_2 & -\alpha_2 \end{bmatrix}.$$

Theorem 1 *If one of the conditions below holds, the mutual apathy equilibrium* $K_0 = (0,0)$ *of the system given in* (1) *and* (2) *is asymptotically stable.*

(i) $1 < \frac{-4\beta_1\beta_2}{(\alpha_1-\alpha_2)^2}$,

(ii) $\frac{\beta_1\beta_2}{\alpha_1\alpha_2} < 1$.

The equilibrium point $K_0 = (0,0)$ *is otherwise unstable.*

Proof The mutual apathy equilibrium is asymptotically stable if all of the eigenvalues, λ_i, $i = 1,2$ of $J(K_0)$, satisfy the condition [5, 6]

$$|\arg \lambda_i| > \frac{\alpha\pi}{2}. \tag{3}$$

These eigenvalues can be determined by solving the characteristic equation

$$\det\bigl(J(K_0) - \lambda I\bigr) = 0,$$

which leads to the equation

$$\lambda^2 + B\lambda + C = 0,$$

where

$$B = (\alpha_1 + \alpha_2),$$
$$C = \alpha_1\alpha_2 - \beta_1\beta_2.$$

The roots of the characteristic equation are

$$\lambda_{1,2} = \frac{-B}{2} \pm \frac{\sqrt{B^2 - 4C}}{2}.$$

The equilibrium point $K_0 = (X_1^*, X_2^*) = (0, 0)$ is asymptotically stable if both the eigenvalues of the Jacobian matrix for the system given in (1) are negative ($|\arg(\lambda_1)| > \frac{\alpha\pi}{2}$, $|\arg(\lambda_2)| > \frac{\alpha\pi}{2}$). It is clear that $B = (\alpha_1 + \alpha_2) > 0$. If $B^2 - 4C < 0$, then all of the eigenvalues, λ_i, $i = 1, 2$, are negative and satisfy the condition given by (i). If $B^2 > B^2 - 4C$, then both of the eigenvalues, λ_i, $i = 1, 2$, are negative and satisfy the conditions given by (ii). If one of the conditions above does not hold, model gives rise to unbounded feeling, which is obviously unrealistic. □

Theorem 2 *We now discuss the asymptotic stability of the* $K_1 = (X_1^*, X_2^*)$ *equilibrium of the system given by* (1). *The Jacobian matrix* $J(K_1)$ *evaluated at the* (X_1^*, X_2^*) *equilibrium is given as*

$$J(X_1^*, X_2^*) = \begin{bmatrix} -\alpha_1 & \beta_1(1 - 3\varepsilon X_2^{*2}) \\ \beta_2(1 - 3\varepsilon X_1^{*2}) & -\alpha_2 \end{bmatrix}.$$

The characteristic equation of the linearized system is as follows:

$$P(\lambda) = \lambda^2 + D\lambda + K = 0, \tag{4}$$

where

$$\begin{aligned} K &= \alpha_1\alpha_2 - \beta_1\beta_2\left(1 - 3\varepsilon X_2^{*2}\right)\left(1 - 3\varepsilon X_1^{*2}\right), \\ D &= (\alpha_1 + \alpha_2). \end{aligned} \tag{4a}$$

The roots of the characteristic equation are

$$\lambda_{1,2} = \frac{-D}{2} \pm \frac{\sqrt{D^2 - 4K}}{2}. \tag{5}$$

The equilibrium point $K_1 = (X_1^*, X_2^*)$ *of the system given in* (1) *and* (2) *is asymptotically stable if one of the following conditions holds for eigenvalues which are given as* (5).

(i) $1 < \frac{-4\beta_1\beta_2}{(\alpha_1-\alpha_2)^2}(1 - 3\varepsilon X_2^{*2})(1 - 3\varepsilon X_1^{*2})$,

(ii) $\frac{\beta_1\beta_2}{\alpha_1\alpha_2}(1 - 3\varepsilon X_2^{*2})(1 - 3\varepsilon X_1^{*2}) < 1$.

The proof of Theorem 2 is similar to that of Theorem 1.

Theorem 3 *Let* $K = \alpha_1\alpha_2 - \beta_1\beta_2(1 - 3\varepsilon X_2^{*2})(1 - 3\varepsilon X_1^{*2})$ *be as given in* (4a). *If* $K < 0$, *then the positive equilibrium point* $K_1 = (X_1^*, X_2^*)$ *of the system given in* (1) *and* (2) *is unstable.*

Proof If $K < 0$, from Descartes' rule of signs, it is clear that the characteristic equation $P(\lambda)$ has at least one positive real root. So, the equilibrium point $K_1 = (X_1^*, X_2^*)$ of the system given in (1) and (2) is unstable. □

4 Numerical method

Consider the initial value problem (IVP) with Caputo-type FDE given by

$$D^\alpha x(t) = f\bigl(t, x(t)\bigr), \quad x(0) = x_0, \tag{6}$$

where $f \in C([0,T] \times R, R)$, $0 < \alpha < 1$. Since f is assumed to be a continuous function, every solution of IVP given by (6) is also a solution of the following Volterra fractional integral equation:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\bigl(\tau, x(\tau)\bigr)\,d\tau, \quad t \in [0,T]. \tag{7}$$

Moreover, every solution of (7) is a solution of IVP (6) [8] (Lakshmikantham and Vatsala 2007).

The following theorems are given for solving differential equations of fractional order in [7].

Theorem 4 *Let* $\|\cdot\|$ *denote any convenient norm on* R^n. *Assume that* $f \in C[R_1, R^n]$, *where* $R_1 = [(t,X) : 0 \le t \le a$ *and* $\|X - X_0\| \le b]$, $f = (f_1, f_2, \ldots, f_n)^T$ *and* $X = (x_1, x_2, \ldots, x_n)^T$, *and let* $\|f(t,X)\| \le M$ *on* R_1. *Then there exists at least one solution for the system of FDE given by*

$$D^\alpha X(t) = f\bigl(t, X(t)\bigr) \tag{8}$$

with the initial condition

$$X(0) = X_0 \tag{9}$$

on $0 \le t \le \beta$, *where* $\beta = \min(a, [\frac{b}{M}\Gamma(\alpha+1)]^{\frac{1}{\alpha}})$, $0 < \alpha < 1$.

Theorem 5 *Consider IVP given by* (8)-(9) *of order* α $(0 < \alpha < 1)$. *Let*

$$g\bigl(\nu, X_*(\nu)\bigr) = f\bigl(t - \bigl(t^\alpha - \nu\Gamma(\alpha+1)\bigr)^{1/\alpha}, X\bigl(t - \bigl(t^\alpha - \nu\Gamma(\alpha+1)\bigr)^{1/\alpha}\bigr)\bigr)$$

and assume that the conditions of Theorem 4 *hold. Then a solution* $X(t)$ *of* (6) *can be given by*

$$X(t) = X_*\bigl(t^\alpha/\Gamma(\alpha+1)\bigr),$$

where $X_*(\nu)$ is a solution of the system of integer order differential equations

$$\frac{d(X_*(\nu))}{d\nu} = g(\nu, X_*(\nu))$$

with the initial condition

$$X_*(0) = X_0.$$

5 Numerical solutions and simulations

Letting

$$\alpha_1 = 0.001, \quad \alpha_2 = 0.004, \quad \beta_1 = 0.005, \quad \beta_2 = -0.001,$$
$$\varepsilon = 0.01, \quad A_1 = 0.02, \quad A_2 = 0.03, \quad \alpha = 0.95$$

we consider the system (1)-(2)

$$\begin{aligned} D^\alpha X_1 &= -0.001X_1 + 0.005X_2(1 - 0.01X_2^2) + 0.02, \\ D^\alpha X_2 &= -0.001X_1(1 - 0.01X_1^2) - 0.004X_2 + 0.03. \end{aligned} \tag{10}$$

Let the initial conditions be

$$X_1(0) = 10, \quad X_2(0) = 8. \tag{11}$$

A positive equilibrium point for the problem (10)-(11) is calculated as

$$X_1^* = 13.7604, \quad X_2^* = 10.5737.$$

For the numerical solution of (10)-(11), Theorem 5 has been used. The corresponding integer order system given in Theorem 5 is

$$\begin{aligned} \frac{dX_1^*}{d\nu} &= -0.001X_1^* + 0.005X_2^*(1 - 0.01X_2^{*2}) + 0.02, \\ \frac{dX_2^*}{d\nu} &= -0.001X_1^*(1 - 0.01X_1^{*2}) - 0.004X_2^* + 0.03. \end{aligned}$$

If the solution of this integer order system is in $(X_1^*(\nu), X_2^*(\nu))$, then the solution of IVP (10)-(11) is $(X_1^*(t^{0.95}/\Gamma(1.95)), X_2^*(t^{0.95}/\Gamma(1.95)))$. We get the numerical solution of the integer order system by applying the fourth-order Runge-Kutta method. The approximate solutions $X_1(t)$ and $X_2(t)$ are displayed in Figure 1 for $\alpha = 0.95$. Figure 2 shows the asymptotic approximation of $(X_1(t), X_2(t))$ to the equilibrium point $(13.7604, 10.5737)$.

6 Concluding remarks

We have formulated and analyzed a fractional order nonlinear dynamical model of interpersonal relationships. We have obtained a stability condition for equilibrium points. We have also given a numerical example and verified our results.

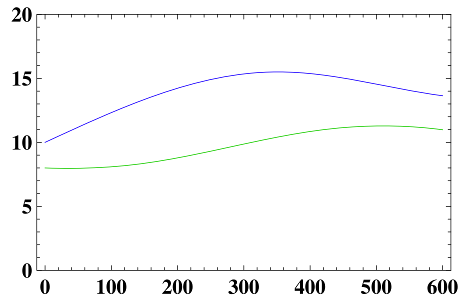

Figure 1 The graphs of $X_1(t)$ (above) and $X_2(t)$ (below).

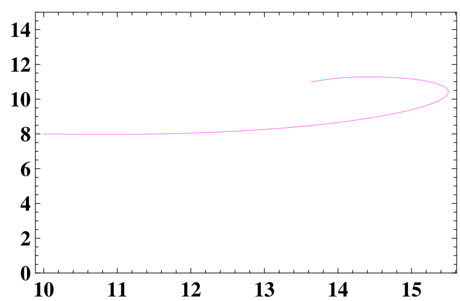

Figure 2 Approximation to the equilibrium point.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have achieved equal contributions. All authors read and approved the final version of the manuscript.

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