Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Fibonacci sequences in groupoids

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Abstract

In this article, we consider several properties of Fibonacci sequences in arbitrary groupoids (i.e., binary systems). Such sequences can be defined in a left-hand way and a right-hand way. Thus, it becomes a question of interest to decide when these two ways are equivalent, i.e., when they produce the same sequence for the same inputs. The problem has a simple solution when the groupoid is flexible. The Fibonacci sequences for several groupoids and for the class of groups as special cases are also discussed.

2000 Mathematics Subject Classification: 20N02; 11B39.

Keywords: fibonacci sequences, groupoid, flexible, wrapped around

1 Introduction

In this article, we consider several properties of Fibonacci sequences in arbitrary groupoids (i.e., binary systems). Such sequences can be defined in a left-hand way and a right-hand way. Thus, it becomes a question of interest to decide when these two ways are equivalent, i.e., when they produce the same sequence for the same inputs. The problem has a simple solution when the groupoid is flexible. In order to construct sufficiently large classes of flexible groupoids to make the results interesting, the notion of a groupoid $(X, *)$ wrapping around a groupoid $(X, \bullet)$ is employed to construct flexible groupoids $(X, \square)$. Among other examples this leads to solving the problem indicated for the selective groupoids associated with certain digraphs, providing many flexible groupoids and indicating that the general groupoid problem of deciding when a groupoid $(X, *)$ is wrapped around a groupoid $(X, \bullet)$ is of independent interest as well.

Given the usual Fibonacci-sequences [1,2] and other sequences of this type, one is naturally interested in considering what may happen in more general circumstances. Thus, one may consider what happens if one replaces the (positive) integers by the modulo an integer n or what happens in even more general circumstances. The most general circumstance we shall deal with is the situation where $(X, *)$ is actually a groupoid, i.e., the product operation $*$ is a binary operation, where we assume no restrictions a priori.

2 Fibonacci sequences in groupoids

Given a sequence $<\phi_0, \phi_1, ..., \phi_n, ... >$ of elements of X, it is a *left-*-Fibonacci sequence* if $\phi_{n+2} = \phi_{n+1} * \phi_n$ for $n \geq 0$, and a *right-*-Fibonacci sequence* if $\phi_{n+2} = \phi_n * \phi_{n+1}$ for $n \geq 0$. Unless $(X, *)$ is commutative, i.e., $x * y = y * x$ for all $x, y \in X$, there is no reason to assume that left-*-Fibonacci sequences are right-*-Fibonacci sequences and conversely. We shall begin with a collection of examples to note what if anything can be concluded about such sequences.

Example 2.1. Let $(X, *)$ be a left-zero-semigroup, i.e., $x * y := x$ for any $x, y \in X$. Then $\phi_2 = \phi_1 * \phi_0 = \phi_1$, $\phi_3 = \phi_2 * \phi_1 = \phi_2 = \phi_1$, $\phi_4 = \phi_3 * \phi_2 = \phi_3 = \phi_1$, ... for any ϕ_0, $\phi_1 \in X$. It follows that $<\phi_n>_L = <\phi_0, \phi_1, \phi_1, ...>$. Similarly, $\phi_2 = \phi_0 * \phi_1 = \phi_0$, $\phi_3 = \phi_1 * \phi_2 = \phi_1$, $\phi_4 = \phi_2 * \phi_3 = \phi_2 = \phi_0$, ... for any $\phi_0, \phi_1 \in X$. It follows that $<\phi_n>_R = <\phi_0, \phi_1, \phi_0, \phi_1, \phi_0, \phi_1, ...>$. In particular, if we let $\phi_0 := 0$, $\phi_1 := 1$, then $<\phi_n>_L = <0, 1, 1, 1, 1, ...>$ and $<\phi_n>_R = <0, 1, 0, 1, 0, 1, ...>$.

A groupoid $(X, *)$ is said to be a *leftoid* if $x * y = l(x)$, a function of x in X, for all $x, y \in X$. We denote it by $(X, *, l)$.

Proposition 2.2. *Let* $(X, *, l)$ *be a leftoid and let* $<\phi_n>$ *be a left-*-Fibonacci sequence on X. Then* $<\phi_n> = <\phi_0, \phi_1, l(\phi_1), l^{(2)}(\phi_1), ..., l^{(n)}(\phi_1), ...>$, *where* $l^{(n+1)}(x) = l(l^n(x))$.

Proof. If $<\phi_n>$ is a left-*-Fibonacci sequence on X, then $\phi_2 = \phi_1 * \phi_0 = l(\phi_1)$ and $\phi_3 = \phi_2 * \phi_1 = l(\phi_2) = l^{(2)}(\phi_1)$. It follows that $\phi_{k+1} = \phi_k * \phi_{k-1} = 1(\phi_k) = l^{(k)}(\phi_1)$. □

A groupoid $(X, *)$ is said to be a *rightoid* if $x * y = r(y)$, a function of y in X, for all $x, y \in X$. We denote it by $(X, *, r)$.

Proposition 2.2'. *Let* $(X, *, r)$ *be a rightoid and let* $<\phi_n>$ *be a right-*-Fibonacci sequence on X. Then* $<\phi_n> = <\phi_0, \phi_1, r(\phi_0), \mathrm{r}(\phi_0), r^{(2)}(\phi_0), r^{(2)}(\phi_1), r^{(3)}(\phi_0), r(3)(\phi), ..., r^{(n)}(\phi_0), r^{(n)}(\phi_1), ...$ where $r^{(n+1)}(x) = r(r^n(x))$.

In particular, if l (r, resp.) is a constant map in Proposition 2.2 (or Proposition 2.2', resp.), say $l(\phi_0) = l(\phi_1) = a$ for some $x \in X$, then $<\phi_n> = <\phi_0, \phi_1, a, a, ...>$.

Theorem 2.3. *Let* $<\phi_n>_L$ *and* $<\phi_n>_R$ *be the left-*-Fibonacci and the right-*-Fibonacci sequences generated by* ϕ_0 *and* ϕ_1. *Then* $<\phi_n>_L = <\phi_n>_R$ *if and only if* $\phi_n * (\phi_{n-1} * \phi_n) = (\phi_n * \phi_{n-1}) * \phi_n$ *for any* $n \geq 1$.

Proof. If $<\phi_n>_L = <\phi_n>_R$, then $\phi_0 * \phi_1 = \phi_2 = \phi_1 * \phi_0$ and hence $(\phi_1 * \phi_0) * \phi_1 = \phi_2 * \phi_1 = \phi_3 = \phi_1 * \phi_2 = \phi_1 * (\phi_0 * \phi_1)$. Similarly $(\phi_2 * \phi_1) * \phi_2 = \phi_3 * \phi_2 = \phi_4 = \phi_2 * \phi_3 = \phi_2 * (\phi_1 * \phi_2)$. By induction on n, we obtain $\phi_n * (\phi_{n-1} * \phi_n) = (\phi_n * \phi_{n-1}) * \phi_n$ for any $n \geq 1$.

If we assume that $\phi_n * (\phi_{n-1} * \phi_n) = (\phi_n * \phi_{n-1}) * \phi_n$ for any $n \geq 1$, then $\phi_n * \phi_{n+1} = \phi_n * (\phi_{n-1} * \phi_n) = (\phi_n * \phi_{n-1}) * \phi_n = \phi_{n+1} * \phi_n$ for any $n \geq 1$. □

A groupoid $(X, *)$ is said to be *flexible* if $(x * y) * x = x * (y * x)$ for any $x, y \in X$.

Proposition 2.4. *Let* $X := \mathbf{R}$ *be the set of all real numbers and let* $A, B \in \mathbf{R}$. *Then any groupoid* $(X, *)$ *of the types* $x*y := A + B(x + y)$ *or* $x*y := Bx + (1 - B)y$ *for any* $x, y \in X$ *is flexible.*

Proof. Define a binary operation "*" on X by $x * y := A + Bx + Cy$ for any $x, y \in X$, where $A, B, C \in X$. Assume that $(X, *)$ is flexible. Then $(x * y) * x = x * (y * x)$ for any $x, y \in X$. It follows that $A + B(x + y) + Cx = A + Bx + C(y * x)$. It follows that

$$AB + B(B - 1)x = AC + C(C - 1)x \tag{1}$$

for any $x \in X$. If we let $x := 0$ in (1), then we obtain $AB = AC$. If $A \neq 0$, then $B = C$ and $x * y = A + B(x + y)$. If $A = 0$, then it follows from (1) that

$$B(B - 1)x = C(C - 1)x \tag{2}$$

for any $x \in X$. If we let $x := 1$ in (2), then we obtain $B(B - 1) = C(C - 1)$ and hence

$$C = \frac{1 \pm \sqrt{1 + 4B(B - 1)}}{2} = \begin{cases} B \\ 1 - B, \end{cases}$$

i.e., $x * y = Bx + By$ or $x * y = Bx + (1 - B)y$. This proves the proposition. □

Proposition 2.5. *Let* $(X, *)$ *be a flexible groupoid. Then* $(X, *)$ *is commutative if and only if* $<\phi_0, \phi_1, \ldots>_L = <\phi_0, \phi_1, \ldots>_R$ *for any* $\phi_0, \phi_1 \in X$.

Proof. Given $\phi_0, \phi_1 \in X$, $\phi_0 * \phi_1 = \phi_1 * \phi_0 = \phi_2$ since $(X, *)$ is commutative. Since $(X, *)$ is flexible, we obtain

$$\begin{aligned}\varphi_2 * \varphi_1 &= (\varphi_1 * \varphi_0) * \varphi_1\\ &= \varphi_1 * (\varphi_0 * \varphi_1)\\ &= \varphi_1 * \varphi_2 = \varphi_3\end{aligned}$$

By induction on n, we obtain

$$\begin{aligned}\varphi_n * \varphi_{n-1} &= (\varphi_{n-1} * \varphi_{n-2}) * \varphi_{n-1}\\ &= \varphi_{n-1} * (\varphi_{n-2} * \varphi_{n-1})\\ &= \varphi_{n-1} * \varphi_n\end{aligned}$$

Hence $<\phi_0, \phi_1, \ldots>_L = <\phi_0, \phi_1, \ldots>_R$. The converse is trivial, we omit the proof. □

Example 2.6. Let $X := \mathbf{R}$ be the set of all real numbers and let $x * y := -(x + y)$ for any $x, y \in X$. Consider aright-*-Fibonacci sequence $<\phi_n>_R$, where $\phi_0, \phi_1 \in X$. Since $\phi_2 = \phi_0 * \phi_1 = -(\phi_0 + \phi_1)$, $\phi_3 = \phi_1 * \phi_2 = -[\phi_1 + \phi_2] = -[\phi_1 - (\phi_0 + \phi_1)] = \phi_0$, $\phi_4 = \phi_2 * \phi_3 = \phi_1$, ..., we obtain $<\phi_n>_R = <\phi_0, \phi_1, -(\phi_0 + \phi_1), \phi_0, \phi_1, -(\phi_0 + \phi_1), \phi_0, \phi_1, -(\phi_0 + \phi_1), \phi_0, \phi_1, \ldots>$ and $\phi_{n+3} = \phi_n$ $(n = 0, 1, 2, \ldots)$. Since $(X, *)$ is commutative and flexible, by Proposition 2.5, $<\phi_n>_L = <\phi_n>_R$.

Proposition 2.7. *Let* $(X, *)$ *be a groupoid satisfying the following condition:*

$$(x * y) * x = x * (y * x) = y \tag{3}$$

for any $x, y \in X$. *Then* $<\phi_n>_L = <\phi_n>_R$ *if* $\phi_0 * \phi_1 = \phi_1 * \phi_0$.

Proof. Straightforward. □

Proposition 2.8. *The linear groupoid* $(\mathbf{R}, *)$, *with* $x * y := A - (x + y)$, $\forall x, y \in \mathbf{R}$, *where* $A \in \mathbf{R}$, *is the only linear groupoid satisfying the condition* (3).

Proof. By Proposition 2.4, we consider two cases: $x * y := A + B(x + y)$ or $x * y := Bx + (1- B)y$ where $A, B \in \mathbf{R}$. Assume that $x * y := A + B(x + y)$. Since $y = (x * y) * x$, we have

$$\begin{aligned}y &= (x * y) * x\\ &= A + B(x * y + x)\\ &= A + B(A + B(x + y) + x)\\ &= A(B + 1) + B(B + 1)x + B^2 y\end{aligned}$$

It follows that $B^2 = 1$, $B(B + 1) = 0$, $A(B + 1) = 0$. If $B = 1$, then $0 = B(B + 1) = 2$, a contradiction. If $B = -1$, then A is arbitrary. Hence $x * y = A - (x + y)$. Assume that $x * y := Bx + (1 - B)y$. Since $y = (x * y) * x$, we have

$$\begin{aligned}y &= (x * y) * x\\ &= [Bx + (1 - B)y] * x\\ &= B[Bx + (1 - B)y] + (1 - B)x\\ &= (B^2 - B + 1)x + B(1 - B)y\end{aligned}$$

It follows that $B^2 - B + 1 = 0$, $B(1 - B) = 1$ which leads to $B = \frac{1 \pm \sqrt{3}i}{2} \notin \mathbf{R}$, a contradiction.

This proves the proposition. □

3 Flexibility in *Bin*(*X*)

Given groupoids $(X, *)$ and $(X, \bullet)$, we consider $(X, *)$ to be *wrapped around* $(X, \bullet)$ if for all $x, y, z \in X$, $(x \bullet y) * z = z * (y \bullet x)$. If $(X, *)$ and $(X, \bullet)$ are both commutative groupoids, then $(x \bullet y) * z = z * (x \bullet y) = z * (y \bullet x)$ and $(x * y) \bullet z = z \bullet (x * y)$ for all $x, y, z \in X$, i.e., $(X, *)$ and $(X, \bullet)$ are wrapped around each other.

Example 3.1. Let $X := \mathbf{R}$ be the set of all real numbers and let $x * y := x^2 y^2$, $x \bullet y := x - y$ for all $x, y \in X$. Then $(x \bullet y) * z = (x - y) * z = (x - y)^2 z^2 = z * (y \bullet x)$ for all $x, y, z \in X$, i.e., $(X, *)$ is wrapped around $(X, \bullet)$. On the other hand, $(x * y) \bullet z = x^2y^2 - z$ and $z \bullet (y * x) = z - y^2x^2$ so that $(X, \bullet)$ is not wrapped around $(X, *)$.

The notion of the semigroup $(Bin(X), \Box)$ was introduced by Kim and Neggers [3]. They showed that $(Bin(X), \Box)$ is a semigroup, i.e., the operation □ as defined in general is associative. Furthermore, the left-zero semigroup is an identity for this operation.

Proposition 3.2. *Let* $(X, *)$ *be wrapped around* $(X, \bullet)$. *If we define* $(X, \Box) := (X, *)\Box(X, \bullet)$, *i.e.,* $x \Box y := (x * y) \bullet (y * x)$ *for all* $x, y \in X$, *then* $(X, \Box)$ *is flexible.*

Proof. Given $x, y, z \in X$, since $(X, *)$ is wrapped around $(X, \bullet)$, we obtain $(X, *)$ be wrapped around $(X, \bullet)$

$$
\begin{aligned}
(x\Box y)\Box x &= [(x\Box y) * x] * [x * (x\Box y)] \\
&= [\{(x * y) \bullet (y * x)\} * x] \bullet [x * \{(x * y) \bullet (y * x)\}] \\
&= [x * \{(y * x) \bullet (x * y)\}] \bullet [\{(y * x) \bullet (x * y)\} * x] \\
&= [x * (y\Box x)] \bullet [(y\Box x) * x] \\
&= x\Box(y\Box x),
\end{aligned}
$$

proving the proposition. □

Example 3.3. Note that in the situation of Example 3.1, we have $x \Box y = (x * y) \bullet (y * x) = x^2y^2 - y^2x^2 = 0$ for all $x, y \in X$, i.e., $(X, \Box) = (X, *)\Box(X, \bullet)$ is a trivial groupoid $(X, \Box, t)$ where $t = 0$ and $x \Box y = 0$ for all $x, y \in X$.

Example 3.4. In Example 3.1, if we define $(X, \nabla) := (X, \bullet)\Box(X, *)$, i.e., $x \nabla y := (x \bullet y) * (y \bullet x)$ for all $x, y \in X$, then $x \nabla y = (x - y)^4$ and hence $(x \nabla y)\Delta x = ((x - y)^4 - x)^4 = (x - (y - x)^4)^4 = x\nabla(y \nabla x)$. Hence (X, ∇) is a flexible groupoid. Note that (X, ∇) is not a semigroup, since $0\nabla(0\nabla z) = z^{16} \neq z^4 = (0\nabla 0)\nabla z$. Obviously, $x \nabla y = y \nabla x$ for all $x, y \in X$. By applying Proposition 2.5, we obtain $<\phi_0, \phi_1, \ldots>_L = <\phi_0, \phi_1, \ldots>_R$ for any $\phi_0, \phi_1 \in (X, \nabla)$.

We obtain a Fibonacci-∇-sequence in the groupoid (X, ∇) discussed in Example 3.4 as follows:

Example 3.5. Consider a groupoid (X, ∇) in Example 3.4. Since $x \nabla y = (x - y)^4$, given $\phi_0, \phi_1 \in X$, we have $\phi_2 = \phi_0 \nabla \phi_1 = \phi_1 \nabla \phi_0 = (\phi_1 - \phi_0)^4$, and $\phi_3 = \phi_2 \nabla \phi_1 = (\phi_2 - \phi_1)^4 = [(\phi_1 - \phi_0)^4 - \phi_1]^4$. In this fashion, we have $\phi_4 = [[(\phi_1 - \phi_0)^4 - \phi_1]^4 - (\phi_1 - \phi_0)^4]^4$. In particular, if we let $\phi_0 = \phi_1 = 1$, then $\phi_2 = 0$, $\phi_3 = \phi_4 = 1$, $\phi_5 = 0$, $\phi_6 = \phi_7 = 1$, $\phi_8 = 0$, ... Hence $< 1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots >$ is a Fibonacci-∇-sequence in (X, ∇).

4 Limits of *-Fibonacci sequences

In this section, we discuss the limit of left(right)-*-Fibonacci sequences in a real groupoid (**R**, *).

Proposition 4.1. *Define a binary operation * on* **R** *by* $x * y := \frac{1}{2}(x + y)$*for any x, y ∈* **R**. *If* $<\phi_n>$*is a *-Fibonacci sequence on* (**R**, *), *then* $\lim_{n\to\infty}\varphi_n = \frac{1}{3}(\varphi_0 + 2\varphi_1)$.

Proof. Since $x * y = \frac{1}{2}(x + y) = y * x$ for any $x, y \in$ **R**, $<\phi_n>_L = <\phi_n>_R$ for any ϕ_0, ϕ_1 ∈ **R**.

It can be seen that $\varphi_2 = \frac{1}{2}(\varphi_0 + \varphi_1), \varphi_3 = \frac{1}{2^2}(\varphi_0 + 3\varphi_1)$. We let $\varphi_3 = \frac{1}{2^2}(A_3\varphi_0 + B_3\varphi_1)$. Since

$$\begin{aligned}\varphi_4 &= \varphi_2 * \varphi_3\\ &= \frac{1}{2}\left[\frac{\varphi_0 + \varphi_1}{2} + \frac{1}{4}(\varphi_0 + 3\varphi_1)\right]\\ &= \frac{1}{2^3}[3\varphi_0 + 5\varphi_1],\end{aligned}$$

we let it by $\varphi_4 = \frac{1}{2^3}[A_4\varphi_0 + B_4\varphi_1]$. In this fashion, if we let $\varphi_{n+2} := \frac{1}{2^{n+1}}[A_{n+2}\varphi_0 + B_{n+2}\varphi_1]$, we have

$$\begin{aligned}\varphi_{n+2} &= \frac{1}{2}[\varphi_n + \varphi_{n+1}]\\ &= \frac{1}{2}\left[\frac{A_n\varphi_0 + B_n\varphi_1}{2^{n-1}} + \frac{A_{n+1}\varphi_0 + B_{n+1}\varphi_1}{2^n}\right]\\ &= \frac{1}{2^{n+1}}[(2A_n + A_{n+1})\varphi_0 + (2B_n + B_{n+1})\varphi_1]\end{aligned}$$

It follows that

$$\begin{aligned}A_{n+2} &= 2A_n + A_{n+1}, A_3 = 1, A_4 = 3\\ B_{n+2} &= 2B_n + B_{n+1}, B_3 = 3, B_4 = 5\end{aligned}$$

so that the evolution for A_k is < 1, 3, 5, 11, 21, 43, 85, ... > = < A_3, A_4, A_5, ... > and the evolution for B_k is < 3, 5, 11, 21, 43, 85, ... > so that $B_k = A_{k+1}$. If we wish to solve explicitly for A_k, we note that the corresponding characteristic equation is $r^2 - r - 2 = 0$ with roots $r = 2$ or $r = -1$, i.e., $A_{k+3} = \alpha 2^k + \beta(-1)^k$, $\alpha + \beta = 1$ when $k = 0$, and $2\alpha - \beta = 3$ when $k = 1$ so that $\alpha = \frac{4}{3}, \beta = -\frac{1}{3}$. Hence $A_{k+3} = \frac{1}{3}[2^{k+2} + (-1)^{k+1}]$ and $B_{k+3} = A_{k+4} = \frac{1}{3}[2^{k+3} + (-1)^{k+2}]$. It follows that

$$\begin{aligned}\varphi_{n+3} &= \frac{1}{2^{k+2}}\left[\frac{1}{3}(2^{k+2} + (-1)^{k+1})\varphi_0 + \frac{1}{3}(2^{k+3} + (-1)^{k+2})\varphi_1\right]\\ &= \frac{1}{3}\left[\left\{1 + \frac{(-1)^{k+1}}{2^{k+2}}\right\}\varphi_0 + \left\{2 + \frac{(-1)^{k+2}}{2^{k+2}}\right\}\varphi_1\right]\end{aligned}$$

This shows that $\lim_{n\to\infty}\varphi_n = \frac{1}{3}(\varphi_0 + 2\varphi_1)$, proving the proposition.

Proposition 4.2. *Define a binary operation * on* **R** *by* $x * y := Ax + (1 - A)y$, $0 < A < 1$, *for any* $x, y \in$ **R**. *If* $<\phi_n>_L$ *is a left-*-Fibonacci sequence on* (**R**, *), *then* $\lim_{n\to\infty}\varphi_n = \frac{1}{2-A}[(1-A)\varphi_0 + \varphi_1]$.

Proof. Given $\phi_0, \phi_1 \in$ **R**, we consider $<\phi_n>_L$. Since $\phi_2 = \phi_1 * \phi_0 = A\phi_1 + (1 - A)\phi_0$ and $\phi_3 = \phi_2 * \phi_1 = (A^2 - A + 1)\phi_1 + A(1-A)\phi_0$, ..., if we assume that $\phi_n := A_n\phi_1 + B_n\phi_0 (n \geq 2)$, then

$$\begin{aligned}\varphi_{n+2} &= \varphi_{n+1} * \varphi_n \\ &= (A_{n+1}\varphi_1 + B_{n+1}\varphi_0) * (A_n\varphi_1 + B_n\varphi_0) \\ &= A(A_{n+1}\varphi_1 + B_{n+1}\varphi_0) + (1-A)(A_n\varphi_1 + B_n\varphi_0)\end{aligned}$$

It follows that

$$\begin{aligned}A_{n+2} &= AA_{n+1} + (1-A)A_n, \\ B_{n+2} &= AB_{n+1} + (1-A)B_n\end{aligned}$$

Hence we obtain the characteristic equation $r^2 - Ar - (1 - A) = 0$ with roots $r = 1$ or $r = A - 1$. Thus $A_{n+2} = \alpha(A - 1)^{n+2} + \beta$ for some $\alpha, \beta \in$ **R**. Since $A = A_2 = \alpha(A - 1)^2 + \beta$ and $A^2 - A + 1 = A_3 = \alpha(A - 1)^3 + \beta$, we obtain $\alpha = \frac{1}{A-2}$ and $\beta = \frac{1}{2-A}$ so that $A_{n+2} = \frac{(A-1)^{n+2}}{A-2}$. For B_{n+2} we obtain the same characteristic equation $r^2 - Ar - (1 - A) = 0$ with roots $r = 1$ or $r = A - 1$. Hence $B_{n+2} = \gamma(A - 1)^{n+2} + \delta$ for some $\gamma, \delta \in$ **R**. Since $1 - A = B_2 = \gamma(A - 1)^2 + \beta$, $A(1 - A) = B_3 = \gamma(A - 1)^3 + \delta$, we obtain $\gamma - \frac{1}{2-A}, \delta = \frac{1-A}{2-A}$ so that $B_{n+2} = \frac{1}{2-A}[(A-1)^{n+2} + (1-A)]$. Since $0 < A < 1$, $\lim_{n\to\infty}(A - 1)^{n+2} = 0$. It follows that $\lim_{n\to\infty}\varphi_{n+2} = \lim_{n\to\infty}(A_{n+2}\varphi_1 + B_{n+2}\varphi_0) = (\lim_{n\to\infty}A_{n+2})\varphi_1 + (\lim_{n\to\infty}B_{n+2})\varphi_0 = \frac{1}{2-A}[\varphi_1 + (1-A)\varphi_0]$, proving the proposition. □

Note that if $A = \frac{1}{2}$ in Proposition 4.2, then $\lim_{n\to\infty}\varphi_n = \frac{1}{3}(\varphi_0 + 2\varphi_1)$ as in Proposition 4.1. Note that (**R**, *) in Proposition 4.2 is neither a semigroup nor commutative and we may consider a right-*-Fibonacci sequence $<\phi_n>_R$ on (**R**, *).

5 Fibonacci sequences in a group

In this section, we discuss *-Fibonacci sequence in groups.

Example 5.1. Suppose that $X = S_4$ is a symmetric group of order 4 and suppose that $\phi_0 = (13)$, $\phi_1 = (12)$. We wish to determine $<\phi_n>_L$. Since $\phi_2 = \phi_1\phi_0 = (12)(13) = (123)$, $\phi_3 = \phi_2\,\phi_1 = (123)(13) = (12)$, $\phi_4 = \phi_3\phi_2 = (12)(123)$, ..., we obtain $<\phi_n>_L = <$ (13), (12), **(123)**, (**12**), (**13**), (**132**), (**23**), (**13**), (123), (12), (13), ... >, i.e., it is periodic of period 6.

Proposition 5.2. *Let* $(X, \bullet, e)$ *be a group and let* ϕ_0, ϕ_1 *be elements of X such that* $\phi_0 \bullet \phi_1 = \phi_1 \bullet \phi_0$. *If* $<\phi_n>_L$ *is a left-•-Fibonacci sequence in* $(X, \bullet, e)$ *generated by* ϕ_0 *and* ϕ_1, *then* $\varphi_{k+2} = \varphi_1^{F_{k+2}}\varphi_0^{F_{k+1}}$. *In particular, if* $\phi_0 = \phi_1$, *then* $\varphi_{k+2} = \varphi_1^{F_{k+3}}$ *where* F_k is the k^{th} *Fibonacci number.*

Proof. Let ϕ_0, ϕ_1 be elements of X such that $\phi_0 \bullet \phi_1 = \phi_1 \bullet \phi_0$. Since $<\phi_n>_L$ is a left-$\bullet$-Fibonacci sequence in $(X, \bullet, e)$ generated by ϕ_0 and ϕ_1, we have $\varphi_3 = \varphi_1^2\varphi_0, \varphi_4 = \varphi_1^3\varphi_0^2, \varphi_5 = \varphi_1^5\varphi_0^5 = \varphi_1^{F_3}\varphi_0^{F_4}, \varphi_6 = \varphi_1^8\varphi_0^5 = \varphi_1^{F_6}\varphi_0^{F_4}$ and $\varphi_7 = \varphi_1^{F_7}\varphi_0^{F_6}$. If we assume that $\varphi_k = \varphi_1^{F_k}\varphi_0^{F_{k-1}}$ and $\varphi_{k+1} = \varphi_1^{F_{k+1}}\varphi_0^{F_k}$, then $\varphi_{k+2} = \varphi_{k+1}\varphi_k = \varphi_1^{k+1}\varphi_0^{F_k}\varphi_1^{F_k}\varphi_0^{F_{k-1}} = \varphi_1^{F_{k+2}}\varphi_0^{F_{k+1}}$. In particular, if $\phi_0 = \phi_1$, then $\varphi_{k+2} = \varphi_1^{F_{k+2}}\varphi_0^{F_{k+1}} = \varphi_1^{F_{k+3}}$. □

Proposition 5.3. *Let $(X, \bullet, e)$ be a group and let ϕ_0, ϕ_1 be elements of X such that $\phi_0 \bullet \phi_1 = \phi_1 \bullet \phi_0$. If $<\phi_n>_L$ is a full left-$\bullet$-Fibonacci sequence in $(X, \bullet, e)$ generated by ϕ_0 and ϕ_1, then $\varphi_{-(2k)} = \varphi_0^{F_{k+1}}\varphi_1^{-F_{2k}}$ and $\varphi_{-(2k+1)} = \varphi_0^{-F_{2(k+1)}}\varphi_1^{F_{2k+1}}$.*

Proof. Since $\phi_1 = \phi_0\ \phi_{-1}$, we have $\varphi_{-1} = \varphi_0^{-1}\varphi_1$. It follows from $\phi_0 = \phi_{-1}\ \phi_{-2}$ that $\varphi_{-2} = \varphi_0^2\varphi_1^{-1}$. In this fashion, since $\phi_0 \bullet \phi_1 = \phi_1 \bullet \phi_0$, we obtain $\varphi_{-3} = \varphi_0^{-F_4}\varphi_1^{F_3}$ and $\varphi_{-4} = \varphi_0^{F_5}\varphi_1^{-F_4}$. By induction, assume that $\varphi_{-(2k)} = \varphi_0^{F_{k+1}}\varphi_1^{-F_{2k}}$ and $\varphi_{-(2k+1)} = \varphi_0^{-F_{2(k+1)}}\varphi_1^{F_{2k+1}}$. Then we obtain

$$\begin{aligned}
\varphi_{-(2k+2)} &= [\varphi_{-(2k+1)}]^{-1}\varphi_{-2k} \\
&= \varphi_0^{F_{2k+2}}\varphi_1^{-F_{2k+1}}\varphi_0^{F_{2k+1}}\varphi_1^{-F_{2k}} \\
&= \varphi_0^{F_{2k+2}+F_{2k+1}}\varphi_1^{-(F_{2k+1}+F_{2k})} \\
&= \varphi_0^{F_{2k+3}}\varphi_1^{-F_{2k+2}},
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{-(2k+3)} &= [\varphi_{-(2k+2)}]^{-1}\varphi_{-(2k+1)} \\
&= \left[\varphi_0^{F_{2k+3}}\varphi_1^{-F_{2k+2}}\right]^{-1}\varphi_0^{-F_{2k+2}}\varphi_1^{F_{2k+1}} \\
&= \varphi_0^{F_{2k+2}+F_{2k+1}}\varphi_1^{-(F_{2k+2}+F_{2k+3})} \\
&= \varphi_0^{F_{2k+4}}\varphi_1^{-F_{2k+3}},
\end{aligned}$$

proving the proposition. □

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 31 January 2012 Accepted: 27 February 2012 Published: 27 February 2012

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doi:10.1186/1687-1847-2012-19
Cite this article as: Han *et al.*: Fibonacci sequences in groupoids. *Advances in Difference Equations* 2012 **2012**:19.