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Frobenius-Euler polynomials and umbral calculus in the p-adic case

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Abstract
In this paper, we study some p-adic Frobenius-Euler measure related to umbral calculus in the p-adic case. Finally, we derive some identities of Frobenius-Euler polynomials from our study.
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1 Introduction

Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively.

For $f \in \mathbb{N}$ with $(f,p) = 1$, let

$$X = \varprojlim_N \mathbb{Z}/fp^N\mathbb{Z};$$
$$a + fp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{fp^N}\}, \quad 0 \le a \le fp^N - 1,$$
$$X^* = \bigcup_{0<a<fp,(a,p)=1} (a + fp^N\mathbb{Z}_p), \quad N \in \mathbb{N} \text{ (see [1–7])}.$$

Note that the natural map $\mathbb{Z}/fp^N\mathbb{Z} \to \mathbb{Z}/p^N\mathbb{Z}$ induces

$$\pi : X \to \mathbb{Z}_p.$$

If g is a function on $\mathbb{Z}_p$, we denote by the same g the function $g \circ \pi$ on X. Namely, we can consider g as a function on X.

For $k \ge 0$ and $\lambda \in \mathbb{C}_p$ with $|1-\lambda|_p > 1$, the Frobenius-Euler measure on X is defined by

$$\mu_\lambda(x + fp^N\mathbb{Z}_p) = \frac{\lambda^{fp^N - x}}{1 - \lambda^{fp^N}} \quad \text{(see [5, 8])}, \tag{1.1}$$

where the p-adic absolute value on $\mathbb{C}_p$ is normalized by $|p|_p = \frac{1}{p}$.

As is well known, the Frobenius-Euler polynomials are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t - \lambda}\right)e^{xt} = e^{H(x|\lambda)t} = \sum_{n=0}^{\infty} H_n(x|\lambda)\frac{t^n}{n!} \quad \text{(see [5, 7, 9])}, \tag{1.2}$$

with the usual convention about replacing $H^n(x|\lambda)$ by $H_n(x|\lambda)$. In the special case, $x = 0$, $H_n(0|\lambda) = H_n(\lambda)$ are called the nth Frobenius-Euler numbers

$$H_n(x|\lambda) = \left(H(\lambda) + x\right)^n = \sum_{l=0}^{\infty} \binom{n}{l} H_l(\lambda) x^{n-l} \quad \text{(see [6, 9, 10])}. \tag{1.3}$$

Thus, by (1.2) and (1.3), we easily get

$$\left(H(\lambda) + 1\right)^n - \lambda H_n(\lambda) = (1-\lambda)\delta_{0,n} \quad \text{(see [1–19])}, \tag{1.4}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $r \in \mathbb{N}$, the Frobenius-Euler polynomials of order r are defined by the generating function

$$\begin{aligned} \left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} &= \underbrace{\left(\frac{1-\lambda}{e^t-\lambda}\right) \times \cdots \times \left(\frac{1-\lambda}{e^t-\lambda}\right)}_{r\text{-times}} e^{xt} \\ &= \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!} \quad \text{(see [5, 9])}. \end{aligned} \tag{1.5}$$

In the special case, $x = 0$, $H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)$ are called the nth Frobenius-Euler numbers of order r. The nth Frobenius-Euler polynomials can be represented by (1.1) as follows:

$$\begin{aligned} \frac{\lambda H_n(x|\lambda)}{1-\lambda} &= \int_X (x+y)^n \, d\mu_\lambda(y) = \int_{\mathbb{Z}_p} (x+y)^n \, d\mu_\lambda(y) \\ &= \lim_{N\to\infty} \frac{1}{1-\lambda^{p^N}} \sum_{y=0}^{p^N-1} (x+y)^n \lambda^{p^N-y} \quad \text{(see [6, 7])}. \end{aligned} \tag{1.6}$$

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}_p$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbb{C}_p \right\}. \tag{1.7}$$

Let $\mathbb{P} = \mathbb{C}_p[x]$ and $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$.

The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F} \quad \text{(see [11, 15])} \tag{1.8}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t) | x^n \rangle = a_n \quad \text{for all } n \geq 0. \tag{1.9}$$

From (1.8) and (1.9), we have

$$\langle t^n | x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0). \tag{1.10}$$

Here, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional (see [11, 15]). We will call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra (see [11, 15]).

The order $o(f(t))$ of power series $f(t)$ ($\neq 0$) is the smallest integer k for which a_k does not vanish (see [11, 15]). A series $f(t)$ for which $o(f(t)) = 1$ is called a delta series. If a series $f(t)$ has $o(f(t)) = 0$, then $f(t)$ is called an invertible series (see [11, 15]). Let $f(t), g(t) \in \mathcal{F}$. Then we easily see that $\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$. From (1.10), we note that

$$\langle e^{yt}|x^n\rangle = y^n, \qquad \langle e^{yt}|p(x)\rangle = p(y), \tag{1.11}$$

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \quad f(t) \in \mathcal{F}, \tag{1.12}$$

and

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k, \quad p(x) \in \mathbb{P} \text{ (see [15])}. \tag{1.13}$$

For $f_1(t), f_2(t), \dots, f_m(t) \in \mathcal{F}$, we have

$$\langle f_1(t)f_2(t)\cdots f_m(t)|x^n\rangle = \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1,\dots,i_m} \langle f_1(t)|x^{i_1}\rangle \cdots \langle f_m(t)|x^{i_m}\rangle. \tag{1.14}$$

By (1.13), we get

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{n} \langle t^l|p(x)\rangle \binom{l}{k} \frac{k!}{l!} x^{l-k} \tag{1.15}$$

and

$$p^{(k)}(0) = \langle t^k|p(x)\rangle = \langle 1|p^{(k)}(x)\rangle.$$

Thus, by (1.15), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{(see [11, 15])}. \tag{1.16}$$

By (1.16), we easily see that

$$e^{yt} p(x) = p(x+y) \quad \text{(see [15])}. \tag{1.17}$$

Let $S_n(x)$ denote a polynomial of degree n. Suppose that $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials satisfying $\langle g(t)f(t)^k|S_n(x)\rangle = n!\delta_{n,k}$ for all $n, k \geq 0$. The sequence $S_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (g(t), t)$, then $S_n(x)$ is called the *Appell sequence* for $g(t)$ (see [15]).

For $p(x) \in \mathbb{P}$, we have

$$\langle f(t)|xp(x)\rangle = \langle \partial_t f(t)|p(x)\rangle = \langle f'(t)|p(x)\rangle, \tag{1.18}$$

$$\langle e^{yt} - 1|p(x)\rangle = p(y) - p(0). \tag{1.19}$$

If $S_n(x) \sim (g(t), f(t))$, then we have

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|S_k(x)\rangle}{k!} g(t)f(t)^k, \quad h(t) \in \mathcal{F}, \tag{1.20}$$

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t)f(t)^k|p(x)\rangle}{k!} S_k(x), \quad p(x) \in \mathbb{P}, \tag{1.21}$$

$$f(t)S_n(x) = nS_{n-1}(x), \tag{1.22}$$

and

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k \quad \text{for all } y \in \mathbb{C}_p, \tag{1.23}$$

where $\bar{f}(t)$ is compositional inverse of $f(t)$ (see [11, 15]). In [9], Kim and Kim have studied some identities of Frobenius-Euler polynomials arising from umbral calculus. In this paper, we study some p-adic Frobenius-Euler integral on $\mathbb{Z}_p$ related to umbral calculus in the p-adic case. Finally, we derive some new and interesting identities of Frobenius-Euler polynomials from our study.

2 Frobenius-Euler polynomials associated with umbral calculus

Let

$$g(t;\lambda) = \frac{e^t - \lambda}{1 - \lambda} \in \mathcal{F}. \tag{2.1}$$

Then we see that $g(t;\lambda)$ is an invertible series. From (1.2), we have

$$\sum_{k=0}^{\infty} H_k(x|\lambda) \frac{t^k}{k!} = \frac{1}{g(t;\lambda)} e^{xt}. \tag{2.2}$$

Hence, by (2.2), we get

$$\left(\frac{1-\lambda}{e^t - \lambda}\right) x^n = \frac{1}{g(t;\lambda)} x^n = H_n(x|\lambda). \tag{2.3}$$

By (2.2) and (2.3), we get

$$H_n(x|\lambda) \sim \big(g(t;\lambda), t\big).$$

From (1.6), we have

$$\int_{\mathbb{Z}_p} e^{(x+y)t}\, d\mu_{\lambda}(y) = \frac{\lambda}{e^t - \lambda} e^{xt}, \tag{2.4}$$

and

$$\int_{\mathbb{Z}_p} e^{(x+y+1)t}\,d\mu_\lambda(y) - \lambda\int_{\mathbb{Z}_p} e^{(x+y)t}\,d\mu_\lambda(y) = \lambda e^{xt}. \tag{2.5}$$

By (2.5), we get

$$\int_{\mathbb{Z}_p} (x+y+1)^n\,d\mu_\lambda(y) - \lambda\int_{\mathbb{Z}_p} (x+y)^n\,d\mu_\lambda(y) = \lambda x^n. \tag{2.6}$$

From (1.6) and (2.6), we have

$$\frac{\lambda}{1-\lambda}H_n(x+1|\lambda) - \frac{\lambda^2}{1-\lambda}H_n(x|\lambda) = \lambda x^n. \tag{2.7}$$

From (2.2), we can easily derive

$$H_{n+1}(x|\lambda) = \left(x - \frac{g'(t;\lambda)}{g(t;\lambda)}\right)H_n(x|\lambda). \tag{2.8}$$

By (2.8), we get

$$g(t;\lambda)H_{n+1}(x|\lambda) = g(t;\lambda)xH_n(x|\lambda) - g'(t;\lambda)H_n(x|\lambda). \tag{2.9}$$

Thus, from (2.9), we have

$$\left(e^t-\lambda\right)H_{n+1}(x|\lambda) = \left(e^t-\lambda\right)xH_n(x|\lambda) - e^tH_n(x|\lambda). \tag{2.10}$$

By (2.10), we get

$$H_{n+1}(x+1|\lambda) - \lambda H_{n+1}(x|\lambda) = x\left(H_n(x+1|\lambda) - \lambda H_n(x|\lambda)\right). \tag{2.11}$$

From (2.11), we note that

$$\begin{aligned} H_n(x+1|\lambda) - \lambda H_n(x|\lambda) &= x\left(H_{n-1}(x+1|\lambda) - \lambda H_{n-1}(x|\lambda)\right) \\ &= x^2\left(H_{n-2}(x+1|\lambda) - \lambda H_{n-2}(x|\lambda)\right) = \cdots \\ &= x^n\left(H_0(x+1|\lambda) - \lambda H_0(x|\lambda)\right) = x^n(1-\lambda). \end{aligned} \tag{2.12}$$

Let us consider the linear functional $f(t)$ such that

$$\langle f(t)|p(x)\rangle = \int_{\mathbb{Z}_p} p(u)\,d\mu_\lambda(u) \tag{2.13}$$

for all polynomials $p(x)$ can be determined from (1.12) to be

$$f(t) = \sum_{k=0}^{\infty}\frac{\langle f(t)|x^k\rangle}{k!}t^k = \sum_{k=0}^{\infty}\int_{\mathbb{Z}_p} u^k\,d\mu_\lambda(u)\frac{t^k}{k!} = \int_{\mathbb{Z}_p} e^{ut}\,d\mu_\lambda(u). \tag{2.14}$$

By (2.4) and (2.14), we get

$$f(t) = \int_{\mathbb{Z}_p} e^{ut}\, d\mu_\lambda(u) = \frac{\lambda}{e^t - \lambda}. \tag{2.15}$$

Therefore, by (2.15), we obtain the following theorem.

Theorem 2.1 *For $p(x) \in \mathbb{P}$, we have*

$$\left\langle \frac{\lambda}{e^t - \lambda} \middle| p(x) \right\rangle = \int_{\mathbb{Z}_p} p(u)\, d\mu_\lambda(u).$$

In particular,

$$\frac{\lambda}{1-\lambda} H_n(\lambda) = \left\langle \int_{\mathbb{Z}_p} e^{yt}\, d\mu_\lambda(y) \middle| x^n \right\rangle.$$

From (1.6), we have

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^n\, d\mu_\lambda(y) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(x+y)t}\, d\mu_\lambda(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} e^{yt}\, d\mu_\lambda(y) x^n \frac{t^n}{n!}. \tag{2.16}$$

By (1.6), (2.4) and (2.16), we get

$$\frac{\lambda}{1-\lambda} H_n(x|\lambda) = \int_{\mathbb{Z}_p} e^{yt}\, d\mu_\lambda(y) x^n = \frac{\lambda}{e^t - \lambda} x^n, \quad \text{for } n \geq 0. \tag{2.17}$$

Therefore, by (2.17), we obtain the following theorem.

Theorem 2.2 *For $p(x) \in \mathbb{P}$, we have*

$$\int_{\mathbb{Z}_p} p(x+y)\, d\mu_\lambda(y) = \int_{\mathbb{Z}_p} e^{yt}\, d\mu_\lambda(y) p(x) = \frac{\lambda}{e^t - \lambda} p(x).$$

In particular,

$$\frac{\lambda}{1-\lambda} H_n(x|\lambda) = \int_{\mathbb{Z}_p} e^{yt}\, d\mu_\lambda(y) x^n = \frac{\lambda}{e^t - \lambda} x^n \quad (n \geq 0).$$

By (1.6) and (2.16), we get

$$\frac{\lambda}{1-\lambda} H_n(x|\lambda) \sim \left(\frac{e^t - \lambda}{\lambda}, t \right). \tag{2.18}$$

From Appell identity and (2.18), we can derive the following identities:

$$H_n(x+y|\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_k(x|\lambda) y^{n-k}. \tag{2.19}$$

Let

$$g^r(t;\lambda) = \left(\frac{e^t - \lambda}{\lambda}\right)^r = \underbrace{\left(\frac{e^t - \lambda}{\lambda}\right) \times \cdots \times \left(\frac{e^t - \lambda}{\lambda}\right)}_{r\text{-times}} \in \mathcal{F}. \tag{2.20}$$

Then $g^r(t;\lambda)$ is an invertible functional in $\mathcal{F}$. By (1.5) and (2.20), we get

$$\frac{1}{g^r(t;\lambda)} e^{xt} = \frac{\lambda^r}{(1-\lambda)^r} \sum_{k=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!}. \tag{2.21}$$

Thus, from (2.21), we have

$$\frac{1}{g^r(t;\lambda)} x^n = \left(\frac{\lambda}{1-\lambda}\right)^r H_n^{(r)}(x|\lambda), \tag{2.22}$$

and

$$\left(\frac{\lambda}{1-\lambda}\right)^r t H_n^{(r)}(x|\lambda) = \frac{n}{g^r(t;\lambda)} x^{n-1} = n\left(\frac{\lambda}{1-\lambda}\right)^r H_{n-1}^{(r)}(x|\lambda). \tag{2.23}$$

By (2.22) and (2.23), we see that

$$\left(\frac{\lambda}{1-\lambda}\right)^r H_n^{(r)}(x|\lambda) \sim \big(g^r(t;\lambda), t\big). \tag{2.24}$$

From (2.4), we can derive the following identity:

$$\begin{aligned} &\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} e^{(x_1+x_2+\cdots+x_r+x)t}\, d\mu_\lambda(x_1) \cdots d\mu_\lambda(x_r) \\ &\quad = \left(\frac{\lambda}{e^t - \lambda}\right)^r e^{xt} = \left(\frac{\lambda}{1-\lambda}\right)^r \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!}. \end{aligned} \tag{2.25}$$

By (1.10) and (2.25), we get

$$\begin{aligned} &\left(\frac{\lambda}{1-\lambda}\right)^r H_n^{(r)}(x|\lambda) \\ &\quad = \left\langle \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} e^{(x_1+x_2+\cdots+x_r+x)t}\, d\mu_\lambda(x_1) \cdots d\mu_\lambda(x_r) \middle| x^n \right\rangle. \end{aligned} \tag{2.26}$$

From (1.14), we have

$$\begin{aligned} &\left\langle \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} e^{(x_1+x_2+\cdots+x_r)t}\, d\mu_\lambda(x_1) \cdots d\mu_\lambda(x_r) \middle| x^n \right\rangle \\ &\quad = \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1, \ldots, i_r} \left\langle \int_{\mathbb{Z}_p} e^{x_1 t}\, d\mu_\lambda(x_1) \middle| x^{i_1} \right\rangle \times \cdots \end{aligned}$$

$$\times \left\langle \int_{\mathbb{Z}_p} e^{x_r t}\, d\mu_\lambda(x_r) \middle| x^{i_r} \right\rangle$$
$$= \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1,\ldots,i_r} \left(\frac{\lambda}{1-\lambda}\right)^r H_{i_1}(x|\lambda)\cdots H_{i_r}(x|\lambda). \tag{2.27}$$

By (2.26) and (2.27), we get

$$H_n^{(r)}(x|\lambda) = \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1,\ldots,i_r} H_{i_1}(x|\lambda)\cdots H_{i_r}(x|\lambda),$$

where $\binom{n}{i_1,\ldots,i_r} = \frac{n!}{i_1!\cdots i_r!}$. From (2.25), we note that

$$g^r(t;\lambda) = \frac{1}{\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} e^{(x_1+x_2+\cdots+x_r)t}\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r)} = \left(\frac{e^t-\lambda}{\lambda}\right)^r. \tag{2.28}$$

Thus, by (2.28), we get

$$\begin{aligned}\frac{1}{g^r(t;\lambda)}e^{xt} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_r)t}\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r) e^{xt} \\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_r+x)t}\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r) \\ &= \left(\frac{\lambda}{1-\lambda}\right)^r \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}. \end{aligned} \tag{2.29}$$

By (2.29), we see that

$$\begin{aligned}&\left(\frac{\lambda}{1-\lambda}\right)^r H_n^{(r)}(x|\lambda) \\ &\quad= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} (x+x_1+\cdots+x_r)^n\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r) \\ &\quad= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} e^{(x_1+x_2+\cdots+x_r)t}\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r) x^n \\ &\quad= \frac{1}{g^r(t;\lambda)}x^n. \end{aligned} \tag{2.30}$$

Therefore, by (2.30), we obtain the following theorem.

Theorem 2.3 *For $p(x) \in \mathbb{P}$ and $r \in \mathbb{N}$, we have*

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} p(x_1+x_2+\cdots+x_r+x)\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r) = \left(\frac{\lambda}{e^t-\lambda}\right)^r p(x).$$

In particular,

$$\left(\frac{\lambda}{1-\lambda}\right)^r H_n^{(r)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_r)t}\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r) x^n.$$

Moreover,

$$\left(\frac{\lambda}{1-\lambda}\right)^r H_n^{(r)}(x|\lambda) \sim \left(\frac{1}{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} e^{(x_1+x_2+\cdots+x_r)t}\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r)}, t\right).$$

Let us consider the function $f^*(t)$ in $\mathcal{F}$ such that

$$\langle f^*(t)|p(x)\rangle = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} p(x_1+x_2+\cdots+x_r)\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r) \tag{2.31}$$

for all polynomials $p(x)$ can be determined from (1.12) to be

$$\begin{aligned} f^*(t) &= \sum_{k=0}^{\infty} \frac{\langle f^*(t)|x^k\rangle}{k!} t^k \\ &= \sum_{k=0}^{\infty} \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} (x_1+\cdots+x_r)^k\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r)\frac{t^k}{k!} \\ &= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} e^{(x_1+\cdots+x_r)t}\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r). \end{aligned} \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

Theorem 2.4 *For $p(x) \in \mathbb{P}$, we have*

$$\begin{aligned} &\left\langle \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} e^{(x_1+x_2+\cdots+x_r)t}\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r) \Big| p(x) \right\rangle \\ &\quad= \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} p(x_1+x_2+\cdots+x_r)\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r). \end{aligned}$$

In particular,

$$\left\langle \left(\frac{\lambda}{e^t-\lambda}\right)^r \Big| p(x) \right\rangle = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} p(x_1+x_2+\cdots+x_r)\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r).$$

Indeed, the nth Frobenius-Euler number of order r is given by

$$\left(\frac{\lambda}{1-\lambda}\right)^r H_n^{(r)}(x|\lambda) = \left\langle \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{-times}} e^{(x_1+x_2+\cdots+x_r)t}\, d\mu_\lambda(x_1)\cdots d\mu_\lambda(x_r) \Big| x^n \right\rangle,$$

where $n \geq 0$.

Remark From (1.2) and (1.5), we note that

$$\begin{aligned}\frac{d}{d\lambda}\left(\frac{1-\lambda}{e^t-\lambda}\right) &= \frac{1-e^t}{(e^t-\lambda)^2} = \frac{1}{(1-\lambda)}\left(\frac{(1-\lambda)^2}{(e^t-\lambda)^2} - \frac{1-\lambda}{e^t-\lambda}\right)\\ &= \frac{1}{1-\lambda}\sum_{n=0}^{\infty}\left(H_n^{(2)}(\lambda) - H_n(\lambda)\right)\frac{t^n}{n!},\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\frac{d^2}{d\lambda^2}\left(\frac{1-\lambda}{e^t-\lambda}\right) &= 2!\frac{1-e^t}{(e^t-\lambda)^3} = \frac{2!}{(1-\lambda)^2}\left(\frac{(1-\lambda)^3}{(e^t-\lambda)^3} - \frac{(1-\lambda)^2}{(e^t-\lambda)^2}\right)\\ &= \frac{2!}{(1-\lambda)^2}\sum_{n=0}^{\infty}\left(H_n^{(3)}(\lambda) - H_n^{(2)}(\lambda)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.34}$$

Continuing this process, we obtain the following equation:

$$\begin{aligned}\frac{d^k}{d\lambda^k}\left(\frac{1-\lambda}{e^t-\lambda}\right) &= \frac{k!}{(1-\lambda)^k}\left(\frac{(1-\lambda)^{k+1}}{(e^t-\lambda)^{k+1}} - \frac{(1-\lambda)^k}{(e^t-\lambda)^k}\right)\\ &= \frac{k!}{(1-\lambda)^k}\sum_{n=0}^{\infty}\left(H_n^{(k+1)}(\lambda) - H_n^{(k)}(\lambda)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.35}$$

By (1.2), (1.5) and (2.35), we get

$$\frac{d^k}{d\lambda^k}H_n(\lambda) = \frac{k!}{(1-\lambda)^k}\left(H_n^{(k+1)}(\lambda) - H_n^{(k)}(\lambda)\right),$$

where k is a positive integer.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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