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New oscillation results for second-order neutral delay dynamic equations

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Abstract

This paper is concerned with oscillatory behavior of a certain class of second-order neutral delay dynamic equations

$$(r(t)[x(t) + p(t)x(\tau(t))]^\Delta)^\Delta + q(t)x(\delta(t)) = 0,$$

on a time scale \mathbb{T} with $\sup \mathbb{T} = \infty$, where $0 \leq p(t) \leq p_0 < \infty$. Some new results are presented that not only complement and improve those related results in the literature, but also improve some known results for a second-order delay dynamic equation without a neutral term. Further, the main results improve some related results for second-order neutral differential equations.

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1 Introduction

In this paper, we are concerned with oscillation of a class of second-order neutral delay dynamic equations,

$$(r(t)[x(t) + p(t)x(\tau(t))]^\Delta)^\Delta + q(t)x(\delta(t)) = 0, \quad (1.1)$$

where $t \in [t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$, and

(H₁) $r, p, q \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $r(t) > 0$, $0 \leq p(t) \leq p_0 < \infty$, $q(t) > 0$;

(H₂) $\delta \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $\delta(t) \leq t$, $\lim_{t \rightarrow \infty} \delta(t) = \infty$, $\tau \circ \delta = \delta \circ \tau$;

(H₃) $\tau \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $\tau(t) \leq t$, $\tau^\Delta(t) \geq \tau_0 > 0$, $\tau([t_0, \infty)_{\mathbb{T}}) = [\tau(t_0), \infty)_{\mathbb{T}}$, where τ_0 is a constant.

Throughout this paper, we assume that solutions of (1.1) exist for any $t \in [t_0, \infty)_{\mathbb{T}}$. A solution x of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, we call it nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions oscillate.

A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} . Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$. For some concepts related to the notion of time scales, see [1, 2].

Recently, there has been an increasing interest in obtaining sufficient conditions for oscillatory or nonoscillatory behavior of different classes of differential equations and dynamic equations on time scales; we refer the reader to the papers [3–38]. In the following, we present some details that motivate the contents of this paper. Regarding oscillation of second-order neutral differential equations, Grammatikopoulos *et al.* [16] established that the condition

$$\int_{t_0}^{\infty} q(s)[1 - p(s - \delta)] ds = \infty$$

ensures oscillation of the linear neutral differential equation

$$(x(t) + p(t)x(t - \tau))'' + q(t)x(t - \delta) = 0.$$

Later, Grace and Lalli [15] obtained that the conditions

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \left(\theta(s)q(s)[1 - p(s - \delta)] - \frac{(\theta'(s))^2 r(s - \delta)}{4\theta(s)} \right) ds = \infty$$

for some positive function $\theta \in C^1([t_0, \infty), \mathbb{R})$ ensure oscillation of the linear neutral differential equation

$$(r(t)(x(t) + p(t)x(t - \tau)))' + q(t)x(t - \delta) = 0.$$

Baculiková and Džurina [8] established that the conditions

$$1 < p_1 \leq p(t) \leq p_2 < \infty, \quad \tau'(t) \geq \tau_0 > 0, \quad \tau \circ \delta = \delta \circ \tau, \quad \delta(t) < \tau(t) < t,$$

and

$$\int_{t_0}^{\infty} q(s) ds = \infty$$

ensure oscillation of the linear neutral differential equation

$$(r(t)[x(t) + p(t)x(\tau(t))])' + q(t)x(\delta(t)) = 0. \tag{1.3}$$

Recently, Zhong *et al.* [38] improved this result, and they obtained that the conditions (1.2), $p \geq 0$, $p \neq 1$, and

$$\int_{t_0}^{\infty} \left(\theta(s)q(s) \frac{1 - \varepsilon}{1 + p(1 + \varepsilon)} - \frac{(\theta'(s))^2 r(\delta(s))}{4\theta(s)\delta'(s)} \right) ds = \infty$$

for some constant $\varepsilon \in (0, 1)$ and for some positive function $\theta \in C^1([t_0, \infty), \mathbb{R})$ guarantee oscillation of the linear neutral differential equation

$$(r(t)(x(t) + px(t - \tau)))' + q(t)x(\delta(t)) = 0.$$

Hasanbulli and Rogovchenko [20] used the standard integral averaging technique to obtain some new oscillation criteria for the second-order neutral delay differential equation

$$(r(t)(x(t) + p(t)x(t - \tau)))' + q(t)f(x(t), x(\sigma(t))) = 0.$$

For oscillation of second-order dynamic equations on time scales, Erbe *et al.* [13] established a sufficient condition which ensures that the solution x of the delay dynamic equation

$$(r(t)x^\Delta(t))^\Delta + q(t)x(\tau(t)) = 0 \tag{1.4}$$

is either oscillatory or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$ under the condition

$$\int_{t_0}^\infty \frac{1}{r(t)} \int_{t_0}^t q(s) \Delta s \Delta t = \infty.$$

Zhang [36] obtained some oscillation results for (1.4) in the case where

$$\int_{t_0}^\infty \frac{1}{r(t)} \int_{t_0}^t q(s) \int_s^\infty \frac{\Delta u}{r(u)} \Delta s \Delta t = \infty. \tag{1.5}$$

Agarwal *et al.* [4], Saker [29], and Tripathy [33] considered the equation

$$(r(t)(x(t) + p(t)x(t - \tau)))^\Delta + q(t)x(t - \delta) = 0, \tag{1.6}$$

and established some oscillation results for (1.6) provided that $0 \leq p(t) \leq 1$ and

$$\int_{t_0}^\infty \frac{\Delta t}{r(t)} = \infty. \tag{1.7}$$

In particular, Tripathy [33] obtained some oscillation criteria for (1.6) when (1.7) holds and $0 \leq p(t) \leq p_0 < \infty$, and established that the condition

$$\int_{t_0}^\infty \min\{q(s), q(s - \tau)\} \Delta s = \infty$$

ensures oscillation of (1.6).

The question regarding the study of oscillatory properties of (1.1) (including the case when $\mathbb{T} = \mathbb{R}$) has been solved by some recent papers; see [4, 8, 17, 19, 25, 29, 32, 33] *etc.* Based on the conditions $\tau(t) \leq t$ and $\sigma(t) \leq t$, they established some results. The ideas can be divided into two aspects, *i.e.*, comparison methods and the Riccati transformation. In order to compare our results in Section 2 with those related subjects in [4, 8, 17, 19, 25, 29, 32, 33], we list their results as follows.

Theorem 1.1 (See [8]) *Let (1.2) hold and τ^{-1} be the inverse function of τ . Assume (H_1) - (H_3) for $\mathbb{T} = \mathbb{R}$ and $\delta(t) \leq \tau(t) \leq t$. If*

$$\liminf_{t \rightarrow \infty} \int_{\tau^{-1}(\delta(t))}^t \left(\min\{q(s), q(\tau(s))\} \int_{t_0}^{\delta(s)} \frac{du}{r(u)} \right) ds > \frac{\tau_0 + p_0}{\tau_0 e},$$

then (1.3) is oscillatory.

Theorem 1.2 (See [17, 32]) *Let (1.2) hold. Assume (H_1) - (H_3) for $\mathbb{T} = \mathbb{R}$ and $t \geq \delta(t) \geq \tau(t)$. If there exists a positive function $\alpha \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\alpha(s) \min\{q(s), q(\tau(s))\} - \left(1 + \frac{p_0}{\tau_0}\right) \frac{r(\tau(s))(\alpha'(s))^2}{4\tau_0\alpha(s)} \right] ds = \infty,$$

then (1.3) is oscillatory.

Theorem 1.3 (See [4, 29]) *Let (1.7) hold. Assume (H_1) - (H_3) for $\tau(t) = t - \tau$, $\delta(t) = t - \delta$, and $p_0 = 1$. If there exists a positive function $\alpha \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that*

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\alpha(s)q(s)(1 - p(s - \delta)) - \frac{r(s - \delta)(\alpha^\Delta(s))^2}{4\alpha(s)} \right] \Delta s = \infty,$$

then (1.1) is oscillatory.

Theorem 1.4 (See [19, 25]) *Let (1.7) hold. Assume (H_1) - (H_3) and $t \geq \delta(t) \geq \tau(t)$. If there exists a positive function $\alpha \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that*

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\alpha(s) \min\{q(s), q(\tau(s))\} - \left(1 + \frac{p_0}{\tau_0}\right) \frac{r(\tau(s))(\alpha^\Delta(s))^2}{4\tau_0\alpha(s)} \right] \Delta s = \infty,$$

then (1.1) is oscillatory.

Theorem 1.5 (See [33]) *Let (1.7) hold. Assume (H_1) - (H_3) for $\tau(t) = t - \tau$, $\delta(t) = t - \delta$, and $\delta \geq \tau > 0$. If there exists a positive function $\alpha \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that*

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\alpha(s) \min\{q(s), q(s - \tau)\} - \frac{(1 + p_0)r(s - \tau)(\alpha^\Delta(s))^2}{4\alpha(s)} \right] \Delta s = \infty,$$

then (1.1) is oscillatory.

The natural question now is: *Can one obtain new oscillation criteria for (1.1) that improve the results in [4, 19, 25, 29, 33]?* The aim of this paper is to give an affirmative answer to this question. As a special case when $\mathbb{T} = \mathbb{R}$, the obtained results improve those by [8, 15, 17, 32, 38]. As a special case when $p(t) = 0$, the obtained results improve those reported in [13, 36].

2 Main results

In this section, we establish the main results. All functional inequalities considered in this section are assumed to hold eventually, that is, they are satisfied for all t large enough. For

our further references, let us denote

$$Q(t) := \min\{q(t), q(\tau(t))\} \quad \text{and} \quad z(t) := x(t) + p(t)x(\tau(t)).$$

Theorem 2.1 *Assume (H₁)-(H₃) and (1.7). If there exist functions $\eta, a \in C_{\text{rd}}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $\eta(t) > 0$, $a(t) \geq 0$, and*

$$\limsup_{t \rightarrow \infty} \int_{t_2}^t \left(\eta^\sigma(s) Q(s) \frac{\int_{t_1}^{\delta(s)} \frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)} \frac{\Delta v}{r(v)}} - E(s) \right) \Delta s = \infty \tag{2.1}$$

for all sufficiently large t_1 and for some $t_2 \geq t_1$, where

$$\begin{aligned} E(s) &:= -\eta^\sigma(s) \left[r(s)a^2(s) \frac{\int_{t_1}^s \frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)} \frac{\Delta v}{r(v)}} - (r(s)a(s))^\Delta \right] + \frac{B^2(s)}{4A(s)} \\ &\quad - \frac{p_0}{\tau_0} \left[\eta^\sigma(s) \left[\tau_0 r(\tau(s))a^2(s) \frac{\int_{t_1}^{\tau(s)} \frac{\Delta v}{r(v)}}{\int_{t_1}^{\tau\sigma(s)} \frac{\Delta v}{r(v)}} - (r(\tau(s))a(s))^\Delta \right] - \frac{D^2(s)}{4C(s)} \right], \\ A(s) &:= \frac{\eta^\sigma(s)}{r(s)\eta^2(s)} \frac{\int_{t_1}^s \frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)} \frac{\Delta v}{r(v)}}, \quad B(s) := \frac{\eta^\Delta(s)}{\eta(s)} + \frac{2\eta^\sigma(s)a(s)}{\eta(s)} \frac{\int_{t_1}^s \frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)} \frac{\Delta v}{r(v)}}, \\ C(s) &:= \frac{\tau_0 \eta^\sigma(s)}{r(\tau(s))\eta^2(s)} \frac{\int_{t_1}^{\tau(s)} \frac{\Delta v}{r(v)}}{\int_{t_1}^{\tau\sigma(s)} \frac{\Delta v}{r(v)}}, \quad D(s) := \frac{\eta^\Delta(s)}{\eta(s)} + \frac{2\tau_0 \eta^\sigma(s)a(s)}{\eta(s)} \frac{\int_{t_1}^{\tau(s)} \frac{\Delta v}{r(v)}}{\int_{t_1}^{\tau\sigma(s)} \frac{\Delta v}{r(v)}} \end{aligned}$$

then (1.1) is oscillatory.

Proof Assume that (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, suppose that it is an eventually positive solution. From (1.1) and [1, Theorem 1.93], we obtain

$$\frac{p_0}{\tau^\Delta(t)} (r(\tau(t))z^\Delta(\tau(t)))^\Delta + p_0 q(\tau(t))x(\delta(\tau(t))) = 0. \tag{2.2}$$

Combining (1.1) and (2.2), we are led to

$$(r(t)z^\Delta(t))^\Delta + \frac{p_0}{\tau_0} (r(\tau(t))z^\Delta(\tau(t)))^\Delta + Q(t)z(\delta(t)) \leq 0. \tag{2.3}$$

From (1.1) and (1.7), we have

$$z^\Delta(t) > 0 \quad \text{and} \quad (r(t)z^\Delta(t))^\Delta < 0$$

for $t \in [t_1, \infty)_{\mathbb{T}}$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ is large enough. Define the function ω by

$$\omega(t) := \eta(t) \left[\frac{r(t)z^\Delta(t)}{z(t)} + r(t)a(t) \right], \quad t \geq t_1. \tag{2.4}$$

Hence, we have $\omega(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and

$$\omega^\Delta = \eta^\Delta \left[\frac{rz^\Delta}{z} + ra \right] + \eta^\sigma \left[\frac{rz^\Delta}{z} + ra \right]^\Delta$$

$$\begin{aligned}
 &= \frac{\eta^\Delta}{\eta} \omega + \eta^\sigma (ra)^\Delta + \eta^\sigma \left[\frac{rz^\Delta}{z} \right]^\Delta \\
 &= \frac{\eta^\Delta}{\eta} \omega + \eta^\sigma (ra)^\Delta + \eta^\sigma \frac{(rz^\Delta)^\Delta z - r(z^\Delta)^2}{zz^\sigma} \\
 &= \frac{\eta^\Delta}{\eta} \omega + \eta^\sigma (ra)^\Delta + \eta^\sigma \frac{(rz^\Delta)^\Delta}{z^\sigma} - \eta^\sigma \frac{r(z^\Delta)^2}{zz^\sigma} \\
 &= \frac{\eta^\Delta}{\eta} \omega + \eta^\sigma (ra)^\Delta + \eta^\sigma \frac{(rz^\Delta)^\Delta}{z^\sigma} - \eta^\sigma r \left(\frac{z^\Delta}{z} \right)^2 \frac{z}{z^\sigma}.
 \end{aligned} \tag{2.5}$$

By virtue of $(r(t)z^\Delta(t))^\Delta < 0$, we have

$$z(t) \geq z^\Delta(t)r(t) \int_{t_1}^t \frac{\Delta s}{r(s)},$$

and so

$$\left(\frac{z(t)}{\int_{t_1}^t \frac{\Delta s}{r(s)}} \right)^\Delta \leq 0, \tag{2.6}$$

which implies that

$$\frac{z(t)}{z^\sigma(t)} \geq \frac{\int_{t_1}^t \frac{\Delta s}{r(s)}}{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r(s)}}. \tag{2.7}$$

On the other hand, we have by (2.4) that

$$\left(\frac{z^\Delta}{z} \right)^2 = \left[\frac{\omega}{r\eta} - a \right]^2 = \left[\frac{\omega}{r\eta} \right]^2 + a^2 - 2 \frac{\omega a}{r\eta}. \tag{2.8}$$

Putting (2.7) and (2.8) into (2.5), we have

$$\begin{aligned}
 \omega^\Delta &\leq \eta^\sigma \frac{(rz^\Delta)^\Delta}{z^\sigma} - \eta^\sigma \left[ra^2 \frac{\int_{t_1}^t \frac{\Delta s}{r(s)}}{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r(s)}} - (ra)^\Delta \right] \\
 &\quad + \left[\frac{\eta^\Delta}{\eta} + \frac{2\eta^\sigma a}{\eta} \frac{\int_{t_1}^t \frac{\Delta s}{r(s)}}{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r(s)}} \right] \omega - \eta^\sigma \frac{\int_{t_1}^t \frac{\Delta s}{r(s)}}{r\eta^2 \int_{t_1}^{\sigma(t)} \frac{\Delta s}{r(s)}} \omega^2 \\
 &\leq \eta^\sigma \frac{(rz^\Delta)^\Delta}{z^\sigma} - \eta^\sigma \left[ra^2 \frac{\int_{t_1}^t \frac{\Delta s}{r(s)}}{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r(s)}} - (ra)^\Delta \right] + \frac{B^2}{4A}.
 \end{aligned} \tag{2.9}$$

Now, define the function u by

$$u(t) := \eta(t) \left[\frac{r(\tau(t))z^\Delta(\tau(t))}{z(\tau(t))} + r(\tau(t))a(t) \right], \quad t \geq t_1. \tag{2.10}$$

Hence, we have by [1, Theorem 1.93] and (H_3) that

$$u^\Delta = \eta^\Delta \left[\frac{(r \circ \tau)(z^\Delta \circ \tau)}{z \circ \tau} + (r \circ \tau)a \right] + \eta^\sigma \left[\frac{(r \circ \tau)(z^\Delta \circ \tau)}{z \circ \tau} + (r \circ \tau)a \right]^\Delta$$

$$\begin{aligned}
 &= \frac{\eta^\Delta}{\eta} u + \eta^\sigma ((r \circ \tau) a)^\Delta + \eta^\sigma \left[\frac{(r \circ \tau)(z^\Delta \circ \tau)}{z \circ \tau} \right]^\Delta \\
 &= \frac{\eta^\Delta}{\eta} u + \eta^\sigma ((r \circ \tau) a)^\Delta + \eta^\sigma \frac{((r \circ \tau)(z^\Delta \circ \tau))^\Delta (z \circ \tau) - (r \circ \tau)(z^\Delta \circ \tau)(z^\Delta \circ \tau)^\Delta}{(z \circ \tau)(z \circ \tau^\sigma)} \\
 &= \frac{\eta^\Delta}{\eta} u + \eta^\sigma ((r \circ \tau) a)^\Delta + \eta^\sigma \frac{((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z \circ \tau^\sigma} - \eta^\sigma \tau^\Delta (r \circ \tau) \left(\frac{z^\Delta \circ \tau}{z \circ \tau} \right)^2 \frac{z \circ \tau}{z \circ \tau^\sigma} \\
 &\leq \frac{\eta^\Delta}{\eta} u + \eta^\sigma ((r \circ \tau) a)^\Delta + \eta^\sigma \frac{((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z \circ \tau^\sigma} \\
 &\quad - \tau_0 \eta^\sigma (r \circ \tau) \left(\frac{z^\Delta \circ \tau}{z \circ \tau} \right)^2 \frac{z \circ \tau}{z \circ \tau^\sigma}. \tag{2.11}
 \end{aligned}$$

Note that (2.6) implies that

$$\frac{z(\tau(t))}{z(\tau^\sigma(t))} \geq \frac{\int_{t_1}^{\tau(t)} \frac{\Delta s}{r(s)}}{\int_{t_1}^{\tau^\sigma(t)} \frac{\Delta s}{r(s)}}. \tag{2.12}$$

On the other hand, we have by (2.10) that

$$\left(\frac{z^\Delta \circ \tau}{z \circ \tau} \right)^2 = \left[\frac{u}{(r \circ \tau)\eta} - a \right]^2 = \left[\frac{u}{(r \circ \tau)\eta} \right]^2 + a^2 - 2 \frac{ua}{(r \circ \tau)\eta}. \tag{2.13}$$

Putting (2.12) and (2.13) into (2.11), we have

$$\begin{aligned}
 u^\Delta &\leq \eta^\sigma \frac{((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z \circ \tau^\sigma} - \eta^\sigma \left[\tau_0 (r \circ \tau) a^2 \frac{\int_{t_1}^{\tau(t)} \frac{\Delta s}{r(s)}}{\int_{t_1}^{\tau^\sigma(t)} \frac{\Delta s}{r(s)}} - ((r \circ \tau) a)^\Delta \right] \\
 &\quad + \left[\frac{\eta^\Delta}{\eta} + \frac{2\tau_0 \eta^\sigma a}{\eta} \frac{\int_{t_1}^{\tau(t)} \frac{\Delta s}{r(s)}}{\int_{t_1}^{\tau^\sigma(t)} \frac{\Delta s}{r(s)}} \right] u - \frac{\tau_0 \eta^\sigma}{(r \circ \tau)\eta^2} \frac{\int_{t_1}^{\tau(t)} \frac{\Delta s}{r(s)}}{\int_{t_1}^{\tau^\sigma(t)} \frac{\Delta s}{r(s)}} u^2 \\
 &\leq \eta^\sigma \frac{((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z^\sigma} \frac{z^\sigma}{z \circ \tau^\sigma} - \eta^\sigma \left[\tau_0 (r \circ \tau) a^2 \frac{\int_{t_1}^{\tau(t)} \frac{\Delta s}{r(s)}}{\int_{t_1}^{\tau^\sigma(t)} \frac{\Delta s}{r(s)}} - ((r \circ \tau) a)^\Delta \right] + \frac{D^2}{4C} \\
 &\leq \eta^\sigma \frac{((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z^\sigma} - \eta^\sigma \left[\tau_0 (r \circ \tau) a^2 \frac{\int_{t_1}^{\tau(t)} \frac{\Delta s}{r(s)}}{\int_{t_1}^{\tau^\sigma(t)} \frac{\Delta s}{r(s)}} - ((r \circ \tau) a)^\Delta \right] + \frac{D^2}{4C}. \tag{2.14}
 \end{aligned}$$

Recalling (2.9) and (2.14), we have by (2.3) and (2.6) that

$$\begin{aligned}
 \omega^\Delta + \frac{p_0}{\tau_0} u^\Delta &\leq \eta^\sigma \frac{(rz^\Delta)^\Delta + \frac{p_0}{\tau_0} ((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z^\sigma} + E \\
 &\leq -\eta^\sigma Q \frac{z \circ \delta}{z^\sigma} + E \\
 &\leq -\eta^\sigma Q \frac{\int_{t_1}^{\delta(t)} \frac{\Delta s}{r(s)}}{\int_{t_1}^{\sigma(t)} \frac{\Delta s}{r(s)}} + E.
 \end{aligned}$$

Hence, we have

$$\int_{t_2}^t \left(\eta^\sigma(s) Q(s) \frac{\int_{t_1}^{\delta(s)} \frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)} \frac{\Delta v}{r(v)}} - E(s) \right) \Delta s \leq \omega(t_2) + \frac{p_0}{\tau_0} u(t_2),$$

which contradicts (2.1). The proof is complete. \square

Based on Theorem 2.1, we have the following corollary when $\eta(t) = t$ and $a(t) = 0$.

Corollary 2.2 *Assume (H_1) - (H_3) and (1.7). If*

$$\limsup_{t \rightarrow \infty} \int_{t_2}^t \left(\sigma(s) Q(s) \frac{\int_{t_1}^{\delta(s)} \frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)} \frac{\Delta v}{r(v)}} - \frac{1}{4\sigma(s)} \left(r(s) \frac{\int_{t_1}^{\sigma(s)} \frac{\Delta v}{r(v)}}{\int_{t_1}^s \frac{\Delta v}{r(v)}} + \frac{p_0}{\tau_0^2} r(\tau(s)) \frac{\int_{t_1}^{\tau^\sigma(s)} \frac{\Delta v}{r(v)}}{\int_{t_1}^{\tau(s)} \frac{\Delta v}{r(v)}} \right) \right) \Delta s = \infty$$

for all sufficiently large t_1 and for some $t_2 \geq t_1$, then (1.1) is oscillatory.

When $\mathbb{T} = \mathbb{R}$, we have from Corollary 2.2 the following result for the neutral differential equation (1.3).

Corollary 2.3 *Assume (H_1) - (H_3) for $\mathbb{T} = \mathbb{R}$ and (1.2). If*

$$\limsup_{t \rightarrow \infty} \int_{t_2}^t \left(s Q(s) \frac{\int_{t_1}^{\delta(s)} \frac{dv}{r(v)}}{\int_{t_1}^s \frac{dv}{r(v)}} - \frac{1}{4s} \left(r(s) + \frac{p_0}{\tau_0^2} r(\tau(s)) \right) \right) ds = \infty$$

for all sufficiently large t_1 and for some $t_2 \geq t_1$, then (1.3) is oscillatory.

Example 2.4 Consider the second-order neutral differential equation

$$\left(x(t) + \frac{1}{2}x(t-1) \right)'' + \frac{\gamma}{t^2}x(t) = 0 \quad \text{for } t \geq 1, \tag{2.15}$$

where $\gamma > 0$ is a constant, $r(t) = 1$, $p_0 = 1/2$, $\tau_0 = 1$, $q(t) = Q(t) = \gamma/t^2$, $\tau(t) = t - 1$, and $\delta(t) = t$. Note that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_2}^t \left(s Q(s) \frac{\int_{t_1}^{\delta(s)} \frac{dv}{r(v)}}{\int_{t_1}^s \frac{dv}{r(v)}} - \frac{1}{4s} \left(r(s) + \frac{p_0}{\tau_0^2} r(\tau(s)) \right) \right) ds \\ &= \left[\gamma - \frac{3}{8} \right] \limsup_{t \rightarrow \infty} \int_{t_2}^t \frac{ds}{s} = \infty, \quad \text{if } \gamma > \frac{3}{8}. \end{aligned}$$

Hence, by Corollary 2.3, (2.15) is oscillatory if $\gamma > 3/8$. Let $\theta(t) = t$ and $p = p(t) = 1/2$. Then

$$\int_{t_0}^{\infty} \left(\theta(s) q(s) [1 - p(s - \delta)] - \frac{(\theta'(s))^2 r(s - \delta)}{4\theta(s)} \right) ds = \infty, \quad \text{if } \gamma > 1/2$$

and

$$\int_{t_0}^{\infty} \left(\theta(s) q(s) \frac{1 - \varepsilon}{1 + p(1 + \varepsilon)} - \frac{(\theta'(s))^2 r(\delta(s))}{4\theta(s)\delta'(s)} \right) ds = \infty, \quad \text{if } \gamma > \frac{1 + \frac{1}{2}(1 + \varepsilon)}{4(1 - \varepsilon)}$$

for some constant $\varepsilon \in (0, 1)$. Since

$$\frac{1 + \frac{1}{2}(1 + \varepsilon)}{4(1 - \varepsilon)} > \frac{3}{8},$$

our result is better than [15, 38] in some cases.

Example 2.5 For $t \geq 1$, consider the second-order neutral delay differential equation

$$\left(t^{1/2} \left[x(t) + p_0 x\left(\frac{t}{2}\right) \right] \right)' + \frac{a}{t^{3/2}} x\left(\frac{t}{4}\right) = 0, \tag{2.16}$$

where $0 < p_0 < \infty$ and $a > 0$ is a constant. In [8], Baculiková and Džurina obtained that the condition

$$a > \frac{1 + 2p_0}{e \ln 2}$$

ensures oscillation of (2.16) (using Theorem 1.1). Letting $\eta(t) = t^{\frac{1}{2}}$, an application of Theorem 2.1 yields that the condition

$$a > \frac{1}{k_0} \left(\frac{1}{8} + \frac{\sqrt{2}p_0}{4} \right) \text{ for some constant } k_0 \in (0, 1)$$

guarantees oscillation of (2.16). For example, we can put $a > 1/6 + \sqrt{2}p_0/3$ (by letting $k_0 = 3/4$). Hence, our result improves that in [8] since

$$\frac{1 + 2p_0}{e \ln 2} > \frac{1}{6} + \frac{\sqrt{2}p_0}{3}.$$

Example 2.6 For $t \geq 1$, consider the second-order neutral delay differential equation

$$\left[x(t) + p_0 x\left(\frac{t}{2}\right) \right]'' + \frac{a}{t^2} x(t) = 0, \tag{2.17}$$

where $0 < p_0 < \infty$ and $a > 0$ is a constant. An application of Corollary 2.3 implies that the condition

$$a > \frac{1}{4} + p_0$$

guarantees oscillation of (2.17). However, applications of Theorem 1.2 and Theorem 1.4 (by letting $\alpha(t) = t$) yield that

$$a > \frac{1}{2} + p_0$$

ensures oscillation of (2.17). Hence, our result is new. Note that Theorem 1.3 and Theorem 1.5 cannot be applied in (2.17).

In the following, we give an oscillation criterion for (1.1) when

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} < \infty. \tag{2.18}$$

Theorem 2.7 *Assume (H₁)-(H₃) and (2.18). Suppose further that there exist two functions $\eta, a \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $\eta(t) > 0, a(t) \geq 0$, and (2.1) holds for all sufficiently large t_1 and for some $t_2 \geq t_1$. If there exists a positive function $b \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that*

$$b(t) - \frac{1}{r(t)R(t)} \geq 0, \quad b(t) - \frac{1}{r(\tau(t))R(\tau(t))} \geq 0, \tag{2.19}$$

and

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left(\eta^\sigma(s)Q(s) \frac{R^\sigma(s)}{R(\tau^\sigma(s))} - E_*(s) \right) \Delta s = \infty, \tag{2.20}$$

where

$$\begin{aligned} R(t) &:= \int_t^\infty \frac{\Delta s}{r(s)}, \\ E_*(s) &:= -\eta^\sigma(s)[r(s)b^2(s) - (r(s)b(s))^\Delta] + \frac{B_*^2(s)}{4A_*(s)} \\ &\quad - \frac{p_0}{\tau_0} \left[\eta^\sigma(s)[\tau_0 r(\tau(s))b^2(s) - (r(\tau(s))b(s))^\Delta] - \frac{D_*^2(s)}{4C_*(s)} \right], \\ A_*(s) &:= \frac{\eta^\sigma(s)}{r(s)\eta^2(s)}, \quad B_*(s) := \frac{\eta^\Delta(s)}{\eta(s)} + \frac{2\eta^\sigma(s)b(s)}{\eta(s)}, \\ C_*(s) &:= \frac{\tau_0 \eta^\sigma(s)}{r(\tau(s))\eta^2(s)}, \quad D_*(s) := \frac{\eta^\Delta(s)}{\eta(s)} + \frac{2\tau_0 \eta^\sigma(s)b(s)}{\eta(s)}, \end{aligned}$$

then (1.1) is oscillatory.

Proof Assume that (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, suppose that it is an eventually positive solution. By the proof of Theorem 2.1, we have (2.3). From (1.1), there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^\Delta(t) > 0$ or $z^\Delta(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. The proof of the case when $z^\Delta(t) > 0$ is the same as that of Theorem 2.1, and we can get a contradiction to (2.1). Now, we assume $z^\Delta(t) < 0$. Then we have

$$z^\Delta(s) \leq \frac{r(t)}{r(s)} z^\Delta(t), \quad s \geq t \geq t_1.$$

Integrating this from t to ∞ , we get

$$\frac{z^\Delta(t)}{z(t)} \geq -\frac{1}{r(t)R(t)}, \quad \frac{z^\Delta(\tau(t))}{z(\tau(t))} \geq -\frac{1}{r(\tau(t))R(\tau(t))},$$

and

$$\left(\frac{z}{R} \right)^\Delta \geq 0. \tag{2.21}$$

Define the function ω by

$$\omega(t) := \eta(t) \left[\frac{r(t)z^\Delta(t)}{z(t)} + r(t)b(t) \right], \quad t \geq t_1.$$

Then $\omega(t) \geq 0$,

$$\omega^\Delta = \frac{\eta^\Delta}{\eta} \omega + \eta^\sigma (rb)^\Delta + \eta^\sigma \frac{(rz^\Delta)^\Delta}{z^\sigma} - \eta^\sigma r \left(\frac{z^\Delta}{z} \right)^2 \frac{z}{z^\sigma}, \tag{2.22}$$

and

$$\left(\frac{z^\Delta}{z} \right)^2 = \left[\frac{\omega}{r\eta} - b \right]^2 = \left[\frac{\omega}{r\eta} \right]^2 + b^2 - 2 \frac{\omega b}{r\eta}. \tag{2.23}$$

Note that $z/z^\sigma \geq 1$. By virtue of (2.22) and (2.23), we have

$$\begin{aligned} \omega^\Delta &\leq \eta^\sigma \frac{(rz^\Delta)^\Delta}{z^\sigma} - \eta^\sigma [rb^2 - (rb)^\Delta] + \left[\frac{\eta^\Delta}{\eta} + \frac{2\eta^\sigma b}{\eta} \right] \omega - \frac{\eta^\sigma}{r\eta^2} \omega^2 \\ &\leq \eta^\sigma \frac{(rz^\Delta)^\Delta}{z \circ \tau^\sigma} - \eta^\sigma [rb^2 - (rb)^\Delta] + \frac{B_*^2}{4A_*}. \end{aligned} \tag{2.24}$$

Now, define the function u by

$$u(t) := \eta(t) \left[\frac{r(\tau(t))z^\Delta(\tau(t))}{z(\tau(t))} + r(\tau(t))b(t) \right], \quad t \geq t_1.$$

Hence, we have $u(t) \geq 0$,

$$\begin{aligned} u^\Delta &\leq \frac{\eta^\Delta}{\eta} u + \eta^\sigma ((r \circ \tau)b)^\Delta + \eta^\sigma \frac{((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z \circ \tau^\sigma} \\ &\quad - \tau_0 \eta^\sigma (r \circ \tau) \left(\frac{z^\Delta \circ \tau}{z \circ \tau} \right)^2 \frac{z \circ \tau}{z \circ \tau^\sigma}, \end{aligned} \tag{2.25}$$

and

$$\left(\frac{z^\Delta \circ \tau}{z \circ \tau} \right)^2 = \left[\frac{u}{(r \circ \tau)\eta} - b \right]^2 = \left[\frac{u}{(r \circ \tau)\eta} \right]^2 + b^2 - 2 \frac{ub}{(r \circ \tau)\eta}. \tag{2.26}$$

Note that $z(\tau(t))/z(\tau^\sigma(t)) \geq 1$. By virtue of (2.25) and (2.26), we have

$$\begin{aligned} u^\Delta &\leq \eta^\sigma \frac{((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z \circ \tau^\sigma} - \eta^\sigma [\tau_0 (r \circ \tau)b^2 - ((r \circ \tau)b)^\Delta] \\ &\quad + \left[\frac{\eta^\Delta}{\eta} + \frac{2\tau_0 \eta^\sigma b}{\eta} \right] u - \frac{\tau_0 \eta^\sigma}{(r \circ \tau)\eta^2} u^2 \\ &\leq \eta^\sigma \frac{((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z \circ \tau^\sigma} - \eta^\sigma [\tau_0 (r \circ \tau)b^2 - ((r \circ \tau)b)^\Delta] + \frac{D_*^2}{4C_*}. \end{aligned} \tag{2.27}$$

Recalling (2.24) and (2.27), we have by (2.3) and (2.21) that

$$\omega^\Delta + \frac{p_0}{\tau_0} u^\Delta \leq \eta^\sigma \frac{(rz^\Delta)^\Delta + \frac{p_0}{\tau_0} ((r \circ \tau)(z^\Delta \circ \tau))^\Delta}{z^\sigma} \frac{z^\sigma}{z \circ \tau^\sigma} + E_* \leq -\eta^\sigma Q \frac{R^\sigma}{R \circ \tau^\sigma} + E_*.$$

Hence, we have

$$\int_{t_1}^t \left(\eta^\sigma(s) Q(s) \frac{R^\sigma(s)}{R(\tau^\sigma(s))} - E^*(s) \right) \Delta s \leq \omega(t_1) + \frac{p_0}{\tau_0} u(t_1),$$

which contradicts (2.20). The proof is complete. □

Example 2.8 Consider the second-order neutral differential equation

$$(t^2(x(t) + p_0x(t-1)))' + q_0x(t) = 0 \quad \text{for } t \geq 2, \tag{2.28}$$

where $r(t) = t^2$, $\tau_0 = 1$, $q(t) = Q(t) = q_0$, $\tau(t) = t - 1$, and $\delta(t) = t$. Note that $R(t) = t^{-1}$. Let $\eta(t) = 1/t$ and $a(t) = b(t) = 1/(t - 1)$. Then

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_2}^t \left(\eta^\sigma(s) Q(s) \frac{\int_{t_1}^{\delta(s)} \frac{\Delta v}{r(v)}}{\int_{t_1}^{\sigma(s)} \frac{\Delta v}{r(v)}} - E(s) \right) \Delta s \\ &= \limsup_{t \rightarrow \infty} \int_{t_2}^t \left[\frac{q_0}{s} + \frac{2}{(s-1)^2} - \frac{(s+1)^2}{4s(s-1)^2} - p_0 \frac{(s+1)^2}{4s^3} \right] ds \\ &= \infty, \quad \text{if } q_0 > \frac{1+p_0}{4} \end{aligned}$$

and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_0}^t \left(\eta^\sigma(s) Q(s) \frac{R^\sigma(s)}{R(\tau^\sigma(s))} - E^*(s) \right) \Delta s \\ &= \limsup_{t \rightarrow \infty} \int_2^t \left[\frac{q_0(s-1)}{s^2} + \frac{2}{(s-1)^2} - \frac{(s+1)^2}{4s(s-1)^2} - p_0 \frac{(s+1)^2}{4s^3} \right] ds \\ &= \infty, \quad \text{if } q_0 > \frac{1+p_0}{4}. \end{aligned}$$

Hence, by Theorem 2.7, (2.28) is oscillatory if $q_0 > (1 + p_0)/4$. When $p_0 = 0$, $q_0 > 1/4$ is a sharp condition for oscillation of the equation $(t^2x'(t))' + q_0x(t) = 0$. Note that the results of [13, 36] cannot give this result (see (1.5)), and hence our results improve those of [13, 36].

3 Discussions

In this paper, we have suggested some new oscillation criteria for second-order neutral delay dynamic equation (1.1) by employing the generalized Riccati substitution. To achieve these results, we are forced to require, similar as in [33], that $\tau^\Delta(t) \geq \tau_0 > 0$, $\tau \circ \delta = \delta \circ \tau$, and $\tau([t_0, \infty)_{\mathbb{T}}) = [\tau(t_0), \infty)_{\mathbb{T}}$. It would be interesting to seek other methods for further study of oscillatory properties or asymptotic problems of equation (1.1) in the case where $p(t) \geq 1$.

During the past three decades, there have been many classical results regarding oscillatory behavior of equation (1.1) in the case where $\mathbb{T} = \mathbb{R}$, some of which provided that $0 \leq p(t) < 1$; see, for example, [15, 16]. Examples given in this paper reveal some advantages even when one applies the obtained criteria to the case where $0 \leq p(t) < 1$.

These results show that the delay argument τ plays an important role in oscillation of second-order neutral delay dynamic equations; see the details in Example 2.6 and differences between Corollary 2.3 and Theorem 1.2, Theorem 1.4. Let us go through Example 2.6. One can easily see some superiorities in comparison to those related results, e.g., Theorem 1.2 and Theorem 1.4.

As a special case when $p(t) = 0$, the established results improve those of [13, 36] in some sense, which is shown by Example 2.8.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of the paper. All authors read and approved the final manuscript.

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