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The curl theorem of a triangular integral

Kiyohisa Tokunaga

Correspondence: kiyohisa@bene.fit.ac.jp
Graduate School of Engineering, Fukuoka Institute of Technology, Wajiro, Higashi-ku, Fukuoka 811-0295, Japan

Abstract

As the foundation of double integral, we propose a triangular integral, which is an antisymmetric double integral by single limit of double dependent sums of triangularly divided areas. Extending integrand from scalar function to tensor one, we derive the curl theorem based on this triangular double integral. It is derived by substituting the total differentials in the transformation lemma, which is based on this triangular double integral. We may thus infer that this triangular integral is the inverse operation of the total differential.

1 Introduction

The variational principle of the 2D theory is conventionally given as

$$\delta \iint_D \mathcal{L} dx\, dy = 0, \tag{1.1}$$

where the integrand $\mathcal{L} = \mathcal{L}(X, Y, X_x, X_y, Y_x, Y_y)$ is a scalar functional and D is a domain. Here, $X = X(x, y)$, $Y = Y(x, y)$, $X_x \equiv \frac{\partial X}{\partial x}, X_y \equiv \frac{\partial X}{\partial y}$, $Y_x \equiv \frac{\partial Y}{\partial x}$ and $Y_y \equiv \frac{\partial Y}{\partial y}$. The double integral in (1.1) is conventionally defined as

$$\iint_D \mathcal{L} dx\, dy = \lim_{n\to\infty} \sum_{i=1}^{n} \lim_{k\to\infty} \sum_{j=1}^{k} \mathcal{L}(x_i, y_j) \Delta x_i \Delta y_j, \tag{1.2}$$

where $\Delta x_i \equiv x_i - x_{i-1}$ and $\Delta y_j \equiv y_j - y_{j-1}$. According to the conventional method of the perpendicularly combined form of the Riemann and the Lebesgue integrals [1,2], the area of double integral demands double limits at infinities, $k \to \infty$ and $n \to \infty$, of double independent sums, $j = 1,2, \dots, k$ and $i = 1,2, \dots, n$, of rectangularly divided areas as shown in (1.2). Based on this definition of the conventional rectangular double integral (1.2), the curl theorem on the 2D plane is formulated as

$$\oint_{\partial D} (X\, dx + Y\, dy) = \iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dx\, dy, \tag{1.3}$$

where ∂D is an integral path.

Meanwhile, the total differential is widely used even in the exterior derivative [3]. However, it is not known how to derive the curl theorem (1.3) by substituting the total differentials in an integral formula based on the conventional rectangular double integral method. Extending integrand from scalar function to tensor one, we may derive the curl theorem by substituting the total differentials in an integral formula. It depends on how to define a new kind of double integral. We extend the variational principle (1.1) to

$$\delta \int \int_D L_{\alpha\beta} dx^\alpha dx'^\beta = 0, \tag{1.4}$$

where the integrand $L_{\alpha\beta} = L_{\alpha\beta}(X_\mu, X_{\mu,\nu})$ is a tensor functional and indices are summed over $\alpha, \beta = 1, 2$. Here, $X_\mu = X_\mu(x^\lambda)$ and $X_{\mu,\nu} \equiv \frac{\partial X_\mu}{\partial x^\nu}(\lambda, \mu, \nu = 1, 2)$. A new type of double integral in (1.4) is defined as

$$\iint_D L_{\alpha\beta} dx^\alpha dx'^\beta = \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{k} L_{\alpha\beta}\left(\left(x^\lambda\right)_{k,j}\right) \Delta(x^\alpha)_k \Delta(x^\beta)_j, \tag{1.5}$$

where $\Delta(x^\alpha)_k \equiv (x^\alpha)_k - (x^\alpha)_{k-1}$, $\Delta(x^\beta)_j \equiv (x^\beta)_j - (x^\beta)_{j-1}$ and indices are summed over α,β = 1,2. It makes possible to introduce a new kind of triangular double integral by the following two properties:

1. replacing rectangular area by triangular one;
2. replacing double limits of independent double sums by single limit of dependent double sums.

We propose an antisymmetric triangular double integral. It demands only single limit at infinity $n \to \infty$ of double dependent sums, $j = 1, 2, \ldots, k$ and $k = 1, 2, \ldots, n$, of triangularly divided areas as shown in Definition 1. We succeed to define a new kind of triangular integral method, which may derive the curl theorem by substituting the total differentials in an integral formula. In this article, we formulate the curl theorem based on a new kind of integral formula (1.5). We name it as *the curl theorem of a triangular integral* on the 2D plane as shown in the main theorem (4.6). In detail, we derive (4.1) by substituting the total differentials (4.3) and (4.4) in (4.2). The curl theorem of a triangular integral on the 2D plane (4.6) is finally derived from (4.1) and (4.5) under the condition of a closed curve (4.8). We may thus infer that this triangular integral is the inverse operation of the total differential.

There are three advantages of this theory. One is the conceptual coherence despite of its complicated procedure of calculation in the derivation of the curl theorem (see Section 4.1). Another one is that this theory is applicable for finite element method in the case of $1 < n < \infty$ (see (2.12) and (2.18)). The other one is applicability to the integral in the variational principle of multiple variables in the case that the integrand is extended to tensor (see (1.4) and (1.5)).

This article is structured as follows. In Section 2, a triangularly divided area is introduced. This triangular double integral is defined by single limit of double sums of triangularly divided areas. In Section 3, the combination and the transformation lemmata are derived. In Section 4, the curl theorem on the 2D plane is derived by substituting the total differentials in the transformation lemma. In Section 5, the curl theorems of a triangular integral in the 3D space and the 4D hyper-space are presented.

2 Single limit of double sums

A triangular integral as the foundation of double integral is proposed in Section 2.1 and its example is shown in Section 2.2.

2.1 Sum of triangular areas

The sum of triangular areas is introduced as follows.

First of all, the respective triangular area is introduced. Let $f(x, y) = 0$ be a piecewise smooth curve of equation on the xy-plane, expressed in terms of the Cartesian coordinates $(x, y) \in \mathbb{R}^2$. Assume there are two fixed points of $A(x_A, y_A)$ and $B(x_B, y_B)$. Suppose there is a sequence of points $\{P_k(x_k, y_k) \mid k = 0, 1, 2, \ldots, n\}$ on $f(x, y) = 0$, where the initial and the terminal points are respectively $P_0(x_0, y_0) = A(x_A, y_A)$ and $P_n(x_n, y_n) = B(x_B, y_B)$. The respective triangular area ΔS_k $(k = 1, 2, \ldots, n)$, which is surrounded by three vertices of $O(0, 0)$, $P_{k-1}(x_{k-1}, y_{k-1})$ and $P_k(x_k, y_k)$, is introduced as in Figure 1 by

$$\Delta S_k = \frac{1}{2}(x_k y_{k-1} - y_k x_{k-1}) \qquad (k = 1, 2, \ldots, n). \tag{2.1}$$

The increments of x_k and y_k, i.e., Δx_k and Δy_k, are respectively denoted as

$$\Delta x_k \equiv x_k - x_{k-1} \qquad (k = 1, 2, \ldots, n), \tag{2.2}$$

$$\Delta y_k \equiv y_k - y_{k-1} \qquad (k = 1, 2, \ldots, n). \tag{2.3}$$

Substituting Δx_k and Δy_k in ΔS_k of (2.1), it is modified to be

$$\Delta S_k = \frac{1}{2}(y_k \Delta x_k - x_k \Delta y_k) \qquad (k = 1, 2, \ldots, n). \tag{2.4}$$

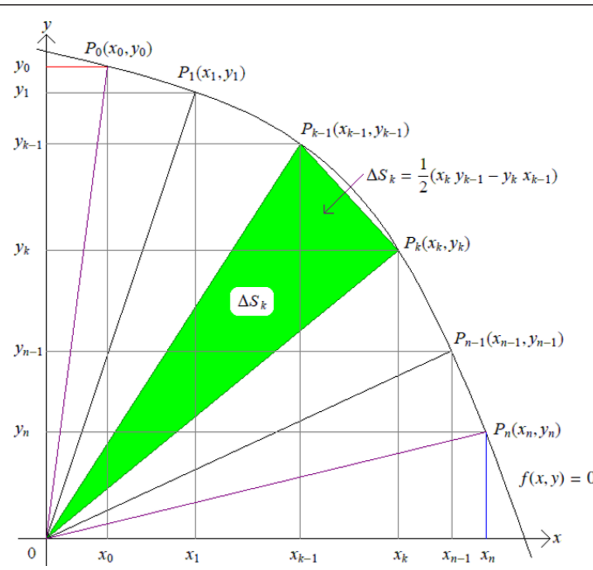

Figure 1 Sum of triangularly divided areas. The respective triangular area ΔS_k, which is surrounded by three vertices of $O(0, 0)$, $P_{k-1}(x_{k-1}, y_{k-1})$ and $P_k(x_k, y_k)$, is introduced as

$$\Delta S_k = \frac{1}{2}(x_k\, y_{k-1} - y_k\, x_{k-1}) \qquad (k = 1, 2, \ldots, n).$$

Next, we introduce *triangular sum* S_n ($n = 1, 2, 3, \ldots$) as a sum of n triangular areas ΔS_k, i.e.,

$$S_n = \sum_{k=1}^{n} \Delta S_k. \tag{2.5}$$

Substituting (2.4) in (2.5), it is modified to be

$$S_n = \frac{1}{2}\sum_{k=1}^{n} (y_k \Delta x_k - x_k \Delta y_k). \tag{2.6}$$

Furthermore, the sum of triangular areas is modified to be the sum of a triangular area and double sums as follows. Using the notations of Δx_k in (2.2) and Δy_k in (2.3), x_k and y_k are respectively expressed as

$$x_k = x_0 + \sum_{j=1}^{k} \Delta x_j \qquad (k = 1, 2, \ldots, n), \tag{2.7}$$

$$y_k = y_0 + \sum_{j=1}^{k} \Delta y_j \qquad (k = 1, 2, \ldots, n). \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.6), it is finally modified to be

$$S_n = \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{k} \left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) + \frac{1}{2}(x_{\mathrm{B}} y_{\mathrm{A}} - y_{\mathrm{B}} x_{\mathrm{A}}). \tag{2.9}$$

The cases of $n = 1, 2, 3$ and $n \to \infty$ of

$$\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{k} \left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) \tag{2.10}$$

in (2.9) are shown in the following. Let the coordinates of point Q be (x_n, y_0).

1. In the case of $n = 1$, i.e., for two points of $P_0(x_0, y_0)$ and $P_1(x_1, y_1) = P_n(x_n, y_n)$, it holds

$$\frac{1}{2}\sum_{k=1}^{1}\sum_{j=1}^{k} \left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) = 0. \tag{2.11}$$

2. In the case of $n = 2$, i.e., for three points of $P_0(x_0, y_0)$, $P_1(x_1, y_1)$ and $P_2(x_2, y_2) = P_n(x_n, y_n)$, as shown in Figure 2, the area of $P_0P_1P_2$ is

$$\frac{1}{2}\sum_{k=1}^{2}\sum_{j=1}^{k} \left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) = \frac{1}{2}(\Delta x_2 \Delta y_1 - \Delta y_2 \Delta x_1). \tag{2.12}$$

Introducing $R_{2.0}$ as the area of triangle P_0QP_n, we obtain

$$\begin{aligned} R_{2.0} &= \frac{1}{2}(x_2 - x_0)(y_0 - y_2) \\ &= -\frac{1}{2}(\Delta x_1 + \Delta x_2)(\Delta y_1 + \Delta y_2). \end{aligned} \tag{2.13}$$

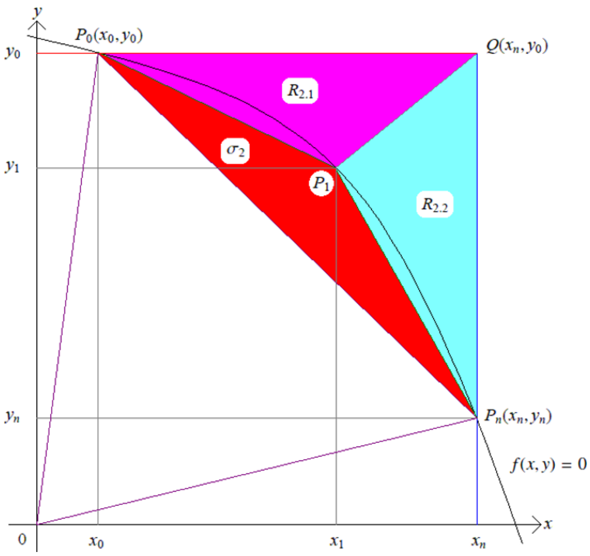

Figure 2 Triangularly divided areas in the case of $n = 2$. The area of triangle $P_0P_1P_2$ is

$$\sigma_2 = R_{2.0} - R_{2.1} - R_{2.2}$$
$$= \frac{1}{2}\left(\Delta x_2\, \Delta y_1 - \Delta y_2\, \Delta x_1\right).$$

Introducing $R_{2.1}$ as the area of triangle P_0P_1Q, we obtain

$$\begin{aligned} R_{2.1} &= \frac{1}{2}(x_2 - x_0)(y_0 - y_1) \\ &= -\frac{1}{2}(\Delta x_1 + \Delta x_2)\Delta y_1. \end{aligned} \tag{2.14}$$

Introducing $R_{2.2}$ as the area of triangle P_1P_2Q, we obtain

$$\begin{aligned} R_{2.2} &= \frac{1}{2}(x_2 - x_1)(y_0 - y_2) \\ &= -\frac{1}{2}\Delta x_2(\Delta y_1 + \Delta y_2). \end{aligned} \tag{2.15}$$

We introduce σ_2 as the area of triangle $P_0P_1P_2$ as

$$\sigma_2 = R_{2.0} - R_{2.1} - R_{2.2}. \tag{2.16}$$

Substituting (2.13), (2.14) and (2.15) in (2.16), it is modified to be

$$\sigma_2 = \frac{1}{2}(\Delta x_2 \Delta y_1 - \Delta y_2 \Delta x_1). \tag{2.17}$$

We thus see the coincidence of (2.12) and (2.17).

3. In the case of $n = 3$, i.e., for four points of $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ and $P_3(x_3, y_3) = P_n(x_n, y_n)$, as shown in Figure 3, the area of $P_0P_1P_2P_3$ is

$$\frac{1}{2}\sum_{k=1}^{3}\sum_{j=1}^{k}(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j) = \frac{1}{2}(\Delta x_2 \Delta y_1 - \Delta y_2 \Delta x_1) + \frac{1}{2}(\Delta x_3 \Delta y_1 - \Delta y_3 \Delta x_1) + \frac{1}{2}(\Delta x_3 \Delta y_2 - \Delta y_3 \Delta x_2). \tag{2.18}$$

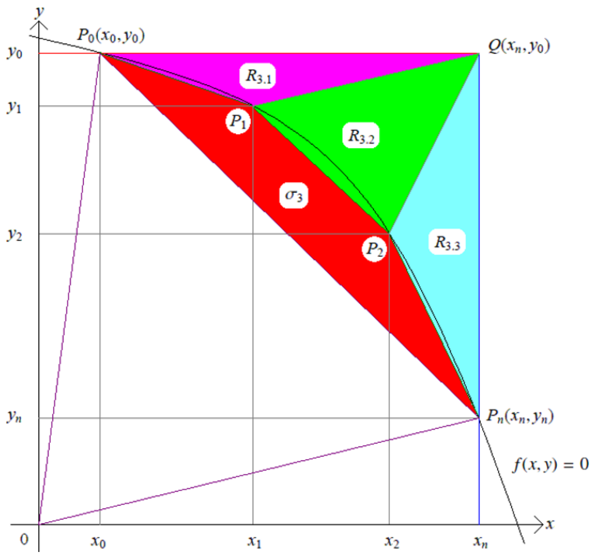

Figure 3 Triangularly divided areas in the case of *n* = 3. The area of $P_0P_1P_2P_3$ is

$$\begin{aligned}\sigma_3 &= R_{3.0} - R_{3.1} - R_{3.2} - R_{3.3}\\ &= \frac{1}{2}\left(\Delta x_2\Delta y_1 - \Delta y_2\Delta x_1\right) + \frac{1}{2}\left(\Delta x_3\Delta y_1 - \Delta y_3\Delta x_1\right) + \frac{1}{2}\left(\Delta x_3\Delta y_2 - \Delta y_3\Delta x_2\right).\end{aligned}$$

Introducing $R_{3.0}$ as the area of triangle P_0QP_n, we obtain

$$\begin{aligned}R_{3.0} &= \frac{1}{2}(x_3 - x_0)(y_0 - y_3)\\ &= -\frac{1}{2}(\Delta x_1 + \Delta x_2 + \Delta x_3)(\Delta y_1 + \Delta y_2 + \Delta y_3).\end{aligned} \tag{2.19}$$

Introducing $R_{3.1}$ as the area of triangle P_0P_1Q, we obtain

$$\begin{aligned}R_{3.1} &= \frac{1}{2}(x_3 - x_0)(y_0 - y_1)\\ &= -\frac{1}{2}(\Delta x_1 + \Delta x_2 + \Delta x_3)\Delta y_1.\end{aligned} \tag{2.20}$$

Introducing $R_{3.2}$ as the area of triangle P_1P_2Q, we obtain

$$\begin{aligned}R_{3.2} &= (x_3 - x_1)(y_0 - y_2) - \frac{1}{2}(x_3 - x_1)(y_0 - y_1) - \frac{1}{2}(x_2 - x_1)(y_1 - y_2) - \frac{1}{2}(x_3 - x_2)(y_0 - y_2)\\ &= -\frac{1}{2}\Delta x_2\Delta y_1 - \frac{1}{2}\Delta x_2\Delta y_2 - \frac{1}{2}\Delta x_3\Delta y_2.\end{aligned} \tag{2.21}$$

Introducing $R_{3.3}$ as the area of triangle P_2P_3Q, we obtain

$$\begin{aligned}R_{3.3} &= \frac{1}{2}(x_3 - x_2)(y_0 - y_3)\\ &= -\frac{1}{2}\Delta x_3(\Delta y_1 + \Delta y_2 + \Delta y_3).\end{aligned} \tag{2.22}$$

We introduce σ_3 as the area of $P_0P_1P_2P_3$ as

$$\sigma_3 = R_{3.0} - R_{3.1} - R_{3.2} - R_{3.3}. \tag{2.23}$$

Substituting (2.19), (2.20), (2.21) and (2.22) in (2.23), it is modified to be

$$\sigma_3 = \frac{1}{2}(\Delta x_2 \Delta y_1 - \Delta y_2 \Delta x_1) + \frac{1}{2}(\Delta x_3 \Delta y_1 - \Delta y_3 \Delta x_1) + \frac{1}{2}(\Delta x_3 \Delta y_2 - \Delta y_3 \Delta x_2). \quad (2.24)$$

We thus see the coincidence of (2.18) and (2.24).

4. In the case of $n \to \infty$, i.e., for a segment of $\overline{AB}$ and a piecewise smooth curve of equation $f(x, y) = 0$, as shown in Figure 4, the area σ surrounded by the segment and the curve is

$$\sigma = \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{k} \left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right). \quad (2.25)$$

Definition 1. (Definition of a triangular integral) The triangular double integral of $\frac{1}{2} \int \int_{f(x,y)\le 0}^{[A,B]} (dx\, dy' - dy\, dx')$ is defined in the case of a piecewise smooth curve of equation $f(x, y) = 0$ by the formula,

$$\frac{1}{2} \int \int_{f(x,y)\le 0}^{[A,B]} (dx\, dy' - dy\, dx') = \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{k} \left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right), \quad (2.26)$$

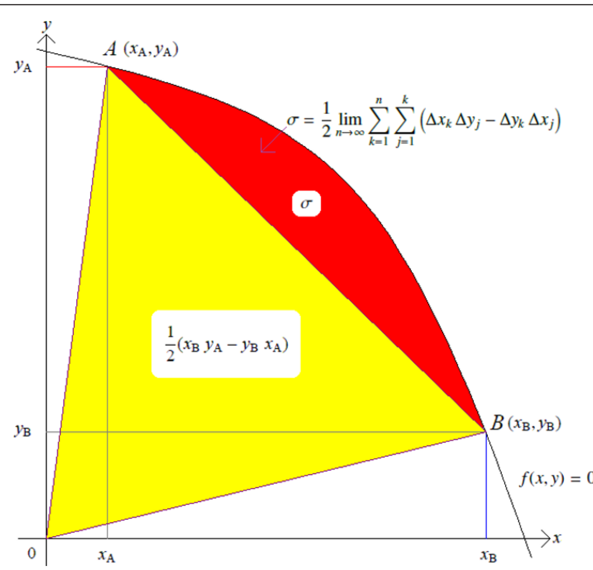

Figure 4 Single limit at infinity $n \to \infty$ of double sums of triangularly divided areas. The area surrounded by $f(x, y) = 0$ and $\overline{AB}$ is

$$\sigma = \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{k} \left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) = \frac{1}{2} \iint_{f(x,y)\le 0}^{[A,B]} (dx\, dy' - dy\, dx').$$

The area of the triangle *OAB* is $\frac{1}{2}\left(x_B\, y_A - y_B\, x_A\right)$.

where *dx'* and *dy'* respectively correspond to increments of Δx_j and Δy_j, while *dx* and *dy* respectively correspond to those of Δx_k and Δy_k.

Suppose S is the limit of S_n at infinity $n \to \infty$, i.e.,

$$S = \lim_{n\to\infty} S_n. \tag{2.27}$$

Theorem 1. *The area S of OAB surrounded by* $\overline{OA}$, $\overline{OB}$ *and the graph of a piecewise smooth curve of equation* $f(x, y) = 0$ *is expressed as*

$$S = \frac{1}{2}\int\int_{f(x,y)\le 0}^{[A,B]} (dx\,dy' - dy\,dx') + \frac{1}{2}(x_B y_A - y_B x_A), \tag{2.28}$$

where $O(0,0)$, $A(x_A, y_A)$, $B(x_B, y_B)$, $f(x_A, y_A) = 0$ *and* $f(x_B, y_B) = 0$.

Proof. As $n \to \infty$ in (2.9), we obtain S as

$$S = \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) + \frac{1}{2}(x_B y_A - y_B x_A) \tag{2.29}$$

as shown in Figure 4. Q.E.D.

Remark. The antisymmetric double integral here introduced demands only single limit of double sums of triangularly divided areas.

2.2 Two kinds of solutions of an example of a parabola

An example that the integrand is constant is shown in Example 1. An example that the integrand is not constant is shown in Section 4.2. Two kinds of solutions of the problem of a parabola are shown in the following. The first and the second solutions are respectively given by the arithmetic and the geometric sequences.

Example 1. An area surrounded by a parabola of

$$y = -x^2 + 9 \tag{2.30}$$

and a segment of $\overline{AB}$, where $A(x_A, y_A) = (1,8)$ and $B(x_B, y_B) = (2,5)$.

1. Integration by the arithmetic sequence (The first kind of solution)

The arithmetic sequences x_j and x_k are respectively

$$x_j = 1 + \frac{j}{n} \qquad (j = 0, 1, 2, \ldots, k), \tag{2.31}$$

$$x_k = 1 + \frac{k}{n} \qquad (k = 0, 1, 2, \ldots, n). \tag{2.32}$$

Using (2.2), the increments of the arithmetic sequences x_j and x_k, i.e., $\Delta x = \Delta x_j = \Delta x_k$, are

$$\Delta x = \frac{1}{n}. \tag{2.33}$$

Substituting (2.30) in (2.3), the increments of the arithmetic sequences y_j and y_k, i.e., Δy_j and Δy_k, are respectively

$$\Delta y_j = -\frac{2}{n} - \frac{2j-1}{n^2} \qquad (j = 0, 1, 2, \ldots, k), \tag{2.34}$$

$$\Delta y_k = -\frac{2}{n} - \frac{2k-1}{n^2} \qquad (k = 0, 1, 2, \ldots, n). \tag{2.35}$$

Thus, the antisymmetric double increment of the arithmetic sequence (2.31) and (2.32) is

$$\frac{1}{2}(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j) = \frac{k-j}{n^3} \qquad (j = 1, 2, \ldots, k), (k = 1, 2, \ldots, n). \tag{2.36}$$

The double dependent sums of (2.36) for j = 1,2, . . . , k and k = 1,2,. . . , n is

$$\begin{aligned} \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) &= \frac{1}{n^3}\sum_{k=1}^{n}\sum_{j=1}^{k}(k-j) \\ &= \frac{1}{6}\left(1 - \frac{1}{n^2}\right). \end{aligned} \tag{2.37}$$

As $n \to \infty$ in (2.37), we obtain

$$\begin{aligned} \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) &= \frac{1}{6}\lim_{n\to\infty}\left(1 - \frac{1}{n^2}\right) \\ &= \frac{1}{6}. \end{aligned} \tag{2.38}$$

2. Integration by the geometric sequence (The second kind of solution)

The geometric sequences x_j and x_k are respectively

$$x_j = 2^{\frac{j}{n}} \qquad (j = 0, 1, 2, \ldots, k), \tag{2.39}$$

$$x_k = 2^{\frac{k}{n}} \qquad (k = 0, 1, 2, \ldots, n). \tag{2.40}$$

The increments of the geometric sequences x_j, x_k, y_j and y_k, i.e., Δx_j, Δx_k, Δy_j and Δy_k, are respectively

$$\Delta x_j = 2^{\frac{j}{n}}(1 - 2^{-\frac{1}{n}}) \qquad (j = 0, 1, 2, \ldots, k), \tag{2.41}$$

$$\Delta y_j = 2^{\frac{2j}{n}}\left(2^{-\frac{2}{n}} - 1\right) \qquad (j = 0, 1, 2, \ldots, k), \tag{2.42}$$

$$\Delta x_k = 2^{\frac{k}{n}}(1 - 2^{-\frac{1}{n}}) \qquad (k = 0, 1, 2, \ldots, n), \tag{2.43}$$

$$\Delta y_k = 2^{\frac{2k}{n}}(2^{-\frac{2}{n}} - 1) \qquad (k = 0, 1, 2, \ldots, n). \tag{2.44}$$

Thus, the antisymmetric double increment of the geometric sequence (2.39) and (2.40) is

$$\frac{1}{2}\left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) = \frac{1}{2}\left(1 - 2^{-\frac{1}{n}}\right)\left(2^{-\frac{2}{n}} - 1\right)\left(2^{\frac{k}{n}}2^{\frac{2j}{n}} - 2^{\frac{2k}{n}}2^{\frac{j}{n}}\right) \qquad (j = 1, 2, \ldots, k), (k = 1, 2, \ldots, n). \tag{2.45}$$

The double dependent sums of (2.45) for $j = 1,2, \ldots, k$ and $k = 1, 2, \ldots, n$ is

$$\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) = \frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(1 - 2^{-\frac{1}{n}}\right)\left(2^{-\frac{2}{n}} - 1\right)\left(2^{\frac{k}{n}}2^{\frac{2j}{n}} - 2^{\frac{2k}{n}}2^{\frac{j}{n}}\right). \tag{2.46}$$

As $n \to \infty$ in (2.46), we obtain

$$\begin{aligned}\frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\Delta x_k \Delta y_j - \Delta y_k \Delta x_j\right) &= \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(1 - 2^{-\frac{1}{n}}\right)\left(2^{-\frac{2}{n}} - 1\right)\left(2^{\frac{k}{n}}2^{\frac{2j}{n}} - 2^{\frac{2k}{n}}2^{\frac{j}{n}}\right) \\ &= \frac{1}{6}.\end{aligned} \tag{2.47}$$

3 The combination lemma and the transformation lemma

The combination lemma is shown in Section 3.1 and the transformation lemma is shown in Section 3.2.

Let $y = f(x)$ be a differentiable function on the xy-plane. Assume there are two fixed points of $A = (x_A, y_A)$ and $B = (x_B, y_B)$ on $y = f(x)$, where $y_A = f(x_A)$ and $y_B = f(x_B)$. An ordinary integral is denoted in accordance with the notation of double integral denoted in Definition 1. New notation for an ordinary integral of $y = f(x)$ in $x \in [x_A, x_B]$ is

$$\int_{y=f(x)}^{[A,B]} y\,dx \equiv \int_{x_A}^{x_B} f(x)dx. \tag{3.1}$$

3.1 The combination lemma on the 2D plane

The combination lemma shows that the sum of an integral along x-axis and that along y-axis between $[A, B]$ is equal to the subtraction of two rectangles, $x_B y_B$ and $x_A y_A$.

Lemma 1. *(The combination lemma) Assume $y = f(x)$ is a differentiable function on the xy-plane, and $x = f^{1}(y)$ is also a differentiable one, where f^{1} is the inverse function of f. Between $A(x_A, y_A) = (x_0, y_0)$ and $B(x_B, y_B) = (x_n, y_n)$, it holds*

$$\int_{y=f(x)}^{[A,B]} y\,dx + \int_{x=f^{-1}(y)}^{[A,B]} x\,dy = x_B y_B - x_A y_A, \tag{3.2}$$

where $y_A = f(x_A)$ and $y_B = f(x_B)$.

We name (3.2) as the *combination lemma*.

Proof. It is divided into the following cases.

1. For a monotonically increasing function $y = f(x)$ in $x \in [x_A, x_B]$ or $x = f^{1}(y)$ in $y \in [y_A, y_B]$, it is indicated that the first term of the left-hand side of (3.2) is an integral of $y = f(x)$ along x-axis i.e.,

$$\int_{y=f(x)}^{[A,B]} y\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} y_k \Delta x_k \tag{3.3}$$

and the second term of the left-hand side of (3.2) is an integral of $x = f^{1}(y)$ along y-axis, i.e.,

$$\int_{x=f^{-1}(y)}^{[A,B]} x\,dy = \lim_{n\to\infty}\sum_{k=1}^{n} x_k \Delta y_k \tag{3.4}$$

as shown in Figure 5.

2. For a monotonically decreasing function $y = f(x)$ in $x \in [x_A, x_B]$ or $x = f^{1}(y)$ in $y \in [y_B, y_A]$, it holds

$$\int_{x=f^{-1}(y)}^{[B,A]} x\,dy - x_A\left(y_A - y_B\right) = \int_{y=f(x)}^{[A,B]} y\,dx - y_B\left(x_B - x_A\right) \tag{3.5}$$

as shown in Figure 6. We thus obtain (3.2). Q.E.D.

3.2 The transformation lemma on the 2D plane

The transformation lemma transforms an integrand to the second integral variable. It transforms a single integral to an antisymmetric double integral of which integrand is constant.

Proposition 1. *(Integral along x-axis) For a piecewise smooth curve of equation f(x, y) = 0, it holds*

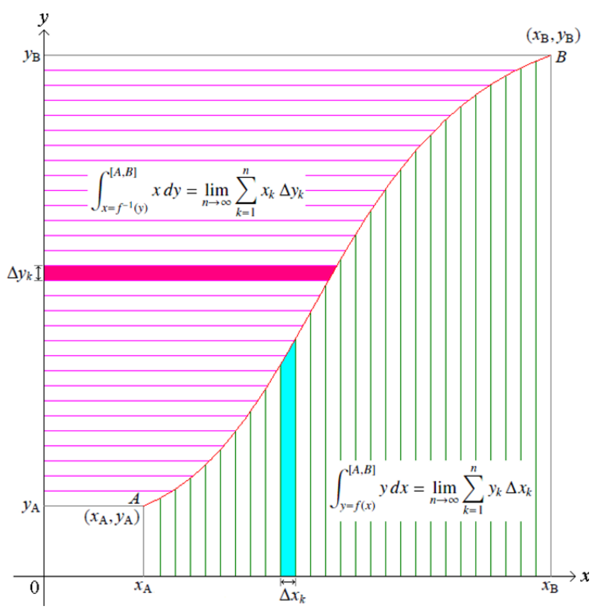

Figure 5 Two kinds of integrals for a monotonically increasing function. An integral along *x*-axis is

$$\int_{y=f(x)}^{[A,B]} y\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} y_k \Delta x_k$$

and that along *y*-axis is

$$\int_{x=f^{-1}(y)}^{[A,B]} x\,dy = \lim_{n\to\infty}\sum_{k=1}^{n} x_k \Delta y_k.$$

It holds

$$\int_{y=f(x)}^{[A,B]} y\,dx + \int_{x=f^{-1}(y)}^{[A,B]} x\,dy = x_B y_B - x_A y_A,$$

where $y_A = f(x_A)$ and $y_B = f(x_B)$.

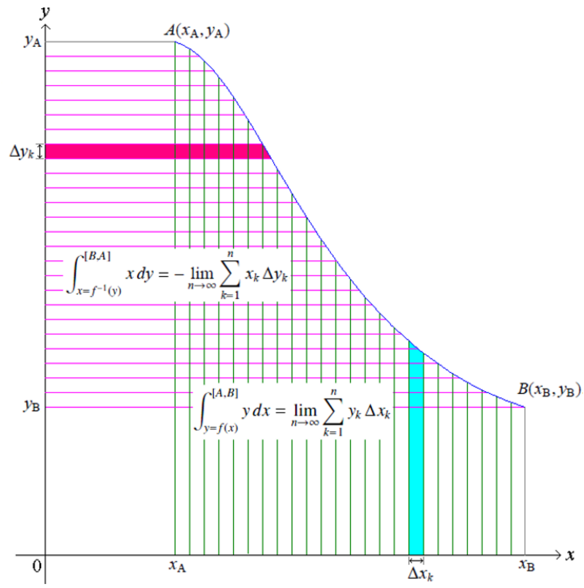

Figure 6 Two kinds of integrals for a monotonically decreasing function. An integral along *x*-axis is

$$\int_{y=f(x)}^{[A,B]} y\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} y_k \Delta x_k$$

and that along *y*-axis is

$$\int_{x=f^{-1}(y)}^{[B,A]} x\,dy = -\lim_{n\to\infty} \sum_{k=1}^{n} x_k \Delta y_k.$$

It holds

$$\int_{x=f^{-1}(y)}^{[B,A]} x\,dy - x_A(y_A - y_B) = \int_{y=f(x)}^{[A,B]} y\,dx - y_B(x_B - x_A),$$

where $y_A = f(x_A)$ and $y_B = f(x_B)$.

$$\int_{f(x,y)=0}^{[A,B]} y\,dx = \int \int_{f(x,y)\leq 0}^{[A,B]} dx\,dy' + x_B y_A - x_A y_A. \tag{3.6}$$

Proof. Substituting (2.8) into (3.3), it is modified to be

$$\int_{f(x,y)=0}^{[A,B]} y\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{k} \Delta y_j \Delta x_k + y_A (x_B - x_A). \tag{3.7}$$

Q.E.D.

Proposition 2. *(Integral along y-axis) For a piecewise smooth curve of equation f(x, y) =* 0, *it holds*

$$\int_{f(x,y)=0}^{[B,A]} x\,dy = -\int \int_{f(x,y)\leq 0}^{[A,B]} dy\,dx' + x_A y_A - x_A y_B. \tag{3.8}$$

Proof. Substituting (2.7) into (3.4), it is modified to be

$$\int_{f(x,y)=0}^{[B,A]} x\,dy = -\lim_{n\to\infty} \sum_{k=1}^{n} \sum_{j=1}^{k} \Delta x_j \Delta y_k + x_A \left(y_A - y_B\right). \tag{3.9}$$

Q.E.D.

Lemma 2. *(The transformation lemma) For a piecewise smooth curve of equation f(x, y) = 0, it holds*

$$\int_{f(x,y)=0}^{[A,B]} y\,dx = \frac{1}{2}\int\int_{f(x,y)\le 0}^{[A,B]} \left(dx\,dy' - dy\,dx'\right) + \frac{1}{2}\left(x_B - x_A\right)\left(y_A + y_B\right). \tag{3.10}$$

Proof. (Combining integral along x-axis with that along y-one) Combining (3.6) with (3.8), we obtain

$$\int_{f(x,y)=0}^{[A,B]} y\,dx + \int_{f(x,y)=0}^{[B,A]} x\,dy = \int\int_{f(x,y)\le 0}^{[A,B]} \left(dx\,dy' - dy\,dx'\right) + x_B\,y_A - x_A\,y_B. \tag{3.11}$$

Substituting (3.2) into (3.11), we obtain the lemma. Q.E.D.

4 The curl theorem of a triangular integral on the 2D plane

The curl theorem of the conventional rectangular integral (1.3) is modified to be that of a new kind of triangular integral (4.6) in Section 4.1 and its example is shown in Section 4.2.

4.1 Proof of the curl theorem on the 2D plane

We present two lemmata in the following before the proof of Theorem 2.

Lemma 3. *Let X = X(x, y) be a partially differentiable function with respect to x and y. It holds*

$$\begin{aligned}\int_{f(x,y)=0}^{[A,B]} X\,dx = {} & \frac{1}{2}\int\int_{f(x,y)\le 0}^{[A,B]} \left(\frac{\partial X'}{\partial y'}dx\,dy' - \frac{\partial X}{\partial y}dy\,dx'\right) + \frac{1}{2}\int\int_{f(x,y)\le 0}^{[A,B]} \left(\frac{\partial X'}{\partial x'} - \frac{\partial X}{\partial x}\right)dx\,dx' \\ & + \frac{1}{2}(x_B - x_A)(X_A + X_B)\end{aligned} \tag{4.1}$$

for a piecewise smooth curve of equation f(x, y) = 0 between $A(x_A, y_A)$ *and* $B(x_B, y_B)$*, where* $X_A = X(x_A, y_B)$ *and* $X_B = X(x_B, y_B)$.

Proof. Rewriting the transformation lemma (3.10) for $X = X(x, y)$ and $f(x, y) = 0$, it is expressed as

$$\int_{f(x,y)=0}^{[A,B]} X\,dx = \frac{1}{2}\int\int_{f(x,y)\le 0}^{[A,B]} \left(dx\,dX' - dX\,dx'\right) + \frac{1}{2}(x_B - x_A)(X_A + X_B). \tag{4.2}$$

Substituting two kinds of the total differentials of X, i.e., dX and dX',

$$dX = \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy, \tag{4.3}$$

$$dX' = \frac{\partial X'}{\partial x'}dx' + \frac{\partial X'}{\partial y'}dy' \tag{4.4}$$

in (4.2), it is modified to be (4.1). Q.E.D.

Remark. Precise modifications in the proof of Lemma 3 are shown in Appendix 1.

Lemma 4. *Let Y = Y(x, y) be a partially differentiable function with respect to x and y. It holds*

$$\int_{f(x,y)=0}^{[A,B]} Y\,dy = \frac{1}{2}\int\int_{f(x,y)\le 0}^{[A,B]}\left(\frac{\partial Y'}{\partial x'}dy\,dx' - \frac{\partial Y}{\partial x}dx\,dy'\right) + \frac{1}{2}\int\int_{f(x,y)\le 0}^{[A,B]}\left(\frac{\partial Y'}{\partial y'} - \frac{\partial Y}{\partial y}\right)dy\,dy' + \frac{1}{2}\left(y_{\mathrm{B}} - y_{\mathrm{A}}\right)\left(Y_{\mathrm{A}} + Y_{\mathrm{B}}\right) \tag{4.5}$$

for a piecewise smooth curve of equation $f(x, y) = 0$ *between* $A(x_{\mathrm{A}}, y_{\mathrm{A}})$ *and* $B(x_{\mathrm{B}}, y_{\mathrm{B}})$, *where* $Y_{\mathrm{A}} = Y(x_{\mathrm{A}}, y_{\mathrm{A}})$ *and* $Y_{\mathrm{B}} = Y(x_{\mathrm{B}}, y_{\mathrm{B}})$.

Proof. In the similar procedure as Lemma 3, we obtain (4.5). Q.E.D.

Theorem 2. *(The curl theorem of a triangular integral on the 2D plane) Let* ∂D *be a piecewise smooth curve of equation* $f(x, y) = 0$ *on the xy-plane, which is expressed in terms of the Cartesian coordinates* $(x, y) \in \mathbb{R}^2$. *Let D be the region inside and on* ∂D. *Let* $X_1 = X = X(x, y)$ *and* $X_2 = Y = Y(x, y)$ *be partially differentiable functions with respect to* $x^1 = x$ *and* $x^2 = y$ *in D. It holds*

$$\oint_{\partial D} X_\alpha dx^\alpha = \frac{1}{2}\iint_D \left(\frac{\partial X'_\alpha}{\partial x'^\beta} - \frac{\partial X_\beta}{\partial x^\alpha}\right)dx^\alpha dx'^\beta, \tag{4.6}$$

where indices are summed over $\alpha, \beta = 1,2$.

Proof. Combining (4.1) with (4.5), we obtain

$$\begin{aligned}\int_{f(x,y)=0}^{[A,B]}(X\,dx + Y\,dy) = {} & \frac{1}{2}\int\int_{f(x,y)\le 0}^{[A,B]}\left[\left(\frac{\partial X'}{\partial y'} - \frac{\partial Y}{\partial x}\right)dx\,dy' + \left(\frac{\partial Y'}{\partial x'} - \frac{\partial X}{\partial y}\right)dy\,dx'\right] \\ & + \frac{1}{2}\int\int_{f(x,y)\le 0}^{[A,B]}\left[\left(\frac{\partial X'}{\partial x'} - \frac{\partial X}{\partial x}\right)dx\,dx' + \left(\frac{\partial Y'}{\partial y'} - \frac{\partial Y}{\partial y}\right)dy\,dy'\right] \\ & + \frac{1}{2}(x_{\mathrm{B}} - x_{\mathrm{A}})(X_{\mathrm{A}} + X_{\mathrm{B}}) + \frac{1}{2}(y_{\mathrm{B}} - y_{\mathrm{A}})(Y_{\mathrm{A}} + Y_{\mathrm{B}}).\end{aligned} \tag{4.7}$$

For an integral on a closed curve, the initial point $A(x_{\mathrm{A}}, y_{\mathrm{A}})$ coincides with the terminal one $B(x_{\mathrm{B}}, y_{\mathrm{B}})$, i.e., $A(x_{\mathrm{A}}, y_{\mathrm{A}}) = B(x_{\mathrm{B}}, y_{\mathrm{B}})$. It then holds

$$\frac{1}{2}(x_{\mathrm{B}} - x_{\mathrm{A}})(X_{\mathrm{A}} + X_{\mathrm{B}}) + \frac{1}{2}(y_{\mathrm{B}} - y_{\mathrm{A}})(Y_{\mathrm{A}} + Y_{\mathrm{B}}) = 0 \tag{4.8}$$

regardless of values of X_{A}, X_{B}, Y_{A} and Y_{B}. We thus obtain (4.6). Q.E.D.

Remark. $\frac{\partial X}{\partial x}$ and $\frac{\partial X'}{\partial x'}$ must be rigorously distinguished as integrands. See (5.20) and (5.19) in detail. An inequality holds for $\frac{\partial X}{\partial x}$ and $\frac{\partial X'}{\partial x'}$ in an integral form, i.e.,

$$\iint_D \left(\frac{\partial X}{\partial x} - \frac{\partial X'}{\partial x'}\right)dx\,dx' = \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\frac{\Delta X[\Delta x_k]}{\Delta x_k} - \frac{\Delta X[\Delta x_j]}{\Delta x_j}\right)\Delta x_k \Delta x_j \ne 0. \tag{4.9}$$

Here, $\Delta X[\Delta x_k]$ is denoted as $\Delta X[\Delta x_k] \equiv X(x_k, y_k)\text{-}X(x_{k-1}, y_k)$. See an example in (5.23). However, they coincide as explicit forms of derivatives. An equality holds for $\frac{\partial X}{\partial x}$ and $\frac{\partial X'}{\partial x'}$ in arbitrary derivative form, i.e.,

$$\frac{\partial X(x, y)}{\partial x} - \frac{\partial X'(x, y)}{\partial x'} = \lim_{\Delta x_k \to 0}\frac{\Delta X[\Delta x_k]}{\Delta x_k} - \lim_{\Delta x_j \to 0}\frac{\Delta X[\Delta x_j]}{\Delta x_j} = 0. \tag{4.10}$$

Similar formulae hold for $\frac{\partial X}{\partial y}$ and $\frac{\partial X'}{\partial y'}$, for $\frac{\partial Y}{\partial x}$ and $\frac{\partial Y'}{\partial x'}$ and for $\frac{\partial Y}{\partial y}$ and $\frac{\partial Y'}{\partial y'}$.

Corollary 1. *In the case of which it holds*

$$\oint_{\partial D} X_\alpha dx^\alpha = -\frac{1}{2}\int\int_D E_{\alpha\beta}\, dx^\alpha\, dx'^\beta, \qquad (4.11)$$

where indices are summed over α, β = 1, 2, *a sufficient condition to hold (4.11) is*

$$-E_{\alpha\beta} = \frac{\partial X'_\alpha}{\partial x'^\beta} - \frac{\partial X_\beta}{\partial x^\alpha} \qquad (\alpha, \beta = 1, 2). \qquad (4.12)$$

Proof. Using (4.6), (4.11) is modified to be

$$\frac{1}{2}\iint_D \left(\frac{\partial X'_\alpha}{\partial x'^\beta} - \frac{\partial X_\beta}{\partial x^\alpha}\right) dx^\alpha dx'^\beta = -\frac{1}{2}\iint_D E_{\alpha\beta} dx^\alpha dx'^\beta, \qquad (4.13)$$

where indices are summed over α, β = 1, 2. A sufficient condition to hold (4.13) is (4.12). Q.E.D.

Corollary 2. *In the case of (4.11), we obtain an antisymmetric property of* $E_{\alpha\beta}$ *as*

$$-E_{\alpha\beta} = E_{\beta\alpha} \qquad (\alpha, \beta = 1, 2). \qquad (4.14)$$

Proof. Interchanging α and β in (4.12), it also holds

$$E_{\beta\alpha} = \frac{\partial X_\alpha}{\partial x^\beta} - \frac{\partial X'_\beta}{\partial x'^\alpha} \quad (\alpha, \beta = 1, 2). \qquad (4.15)$$

Comparing (4. 12) with (4. 15), we obtain (4. 14). Q.E.D.

4.2 An example of the curl theorem on the 2D plane

An example of Theorem 2 is shown in Example 2. The integral path of this problem is an ellipse of

$$x = a\cos\theta, \qquad (4.16)$$

$$y = b\sin\theta. \qquad (4.17)$$

Definition 2. (Definition of elliptical sequence) Using (4.16) and (4.17), we may respectively introduce elliptical sequences x_j, y_j, x_k and y_k as

$$x_j = a\cos\theta_j, \qquad y_j = b\sin\theta_j \qquad (j = 0, 1, 2, \ldots, k), \qquad (4.18)$$

$$x_k = a\cos\theta_k, \qquad y_k = b\sin\theta_k \qquad (k = 0, 1, 2, \ldots, n). \qquad (4.19)$$

The increments of x_j and y_j in (4.18), i.e., Δx_j and Δy_j, of the angular arithmetic sequence θ_j are respectively

$$\Delta x_j = a\left(\cos\theta_j - \cos\theta_{j-1}\right) \qquad (j = 0, 1, 2, \ldots, k), \qquad (4.20)$$

$$\Delta y_j = b\left(\sin\theta_j - \sin\theta_{j-1}\right) \qquad (j = 0, 1, 2, \ldots, k). \qquad (4.21)$$

The increments of x_k and y_k in (4.19), i.e., Δx_k and Δy_k, of the angular arithmetic sequence θ_k are respectively

$$\Delta x_k = a(\cos\theta_k - \cos\theta_{k-1}) \qquad (k = 0, 1, 2, \ldots, n), \qquad (4.22)$$

$$\Delta y_k = b(\sin\theta_k - \sin\theta_{k-1}) \qquad (k = 0, 1, 2, \ldots, n). \tag{4.23}$$

Example 2. We calculate both of the line integral and the area integral in the counterclockwise direction. The example is the curl theorem of a triangular integral on the 2D plane in the case of $X = X(x, y) = -x^2y$ and $Y = Y(x, y) = xy^2$, where the closed curve is an ellipse of (4.16) and (4.17).

The arithmetic sequences of θ_j and θ_k are respectively

$$\theta_j = j\frac{2\pi}{n} \qquad (j = 0, 1, 2, \ldots, k), \tag{4.24}$$

$$\theta_k = k\frac{2\pi}{n} \qquad (k = 0, 1, 2, \ldots, n)\,. \tag{4.25}$$

The increments of θ_j and θ_k, i.e., $\Delta\theta = \Delta\theta_j = \Delta\theta_k$, are

$$\Delta\theta = \frac{2\pi}{n}. \tag{4.26}$$

1. **Calculation by line integral**

The line integral is calculated to be

$$\begin{aligned}\oint_{\partial D}^{\circlearrowleft} \left(X\,dx + Y\,dy\right) &= \oint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}^{\circlearrowleft} \left(-x^2y\,dx + xy^2dy\right)\\ &= ab\left(a^2+b^2\right)\int_0^{2\pi} \sin^2\theta\,\cos^2\theta\,d\theta.\end{aligned} \tag{4.27}$$

The integral in (4.27) is executed in the formula of

$$\begin{aligned}\int_0^{2\pi} \sin^2\theta\cos^2\theta\,d\theta &= \lim_{n\to\infty}\sum_{k=1}^{n}\sin^2\theta_k\cos^2\theta_k\,\Delta\theta\\ &= \lim_{n\to\infty}\frac{2\pi}{n}\sum_{k=1}^{n}\sin^2\left(k\frac{2\pi}{n}\right)\cos^2\left(k\frac{2\pi}{n}\right)\\ &= \frac{\pi}{4}.\end{aligned} \tag{4.28}$$

Substituting (4.28) into (4.27), we finally obtain

$$\oint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}^{\circlearrowleft} (X\,dx + Y\,dy) = \frac{\pi}{4}ab\left(a^2+b^2\right). \tag{4.29}$$

2. **Calculation by area integral**

The area double integral in the region of $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ is calculated to be

$$\begin{aligned}&\frac{1}{2}\int\int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}^{\circlearrowleft}\left(\frac{\partial X'}{\partial y'} - \frac{\partial Y}{\partial x}\right)dx\,dy' + \frac{1}{2}\int\int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}^{\circlearrowleft}\left(\frac{\partial Y'}{\partial x'} - \frac{\partial X}{\partial y}\right)dy\,dx'\\ &+\frac{1}{2}\int\int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}^{\circlearrowleft}\left(\frac{\partial X'}{\partial x'} - \frac{\partial X}{\partial x}\right)dx\,dx' + \frac{1}{2}\int\int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}^{\circlearrowleft}\left(\frac{\partial Y'}{\partial y'} - \frac{\partial Y}{\partial y}\right)dy\,dy' = \frac{\pi}{4}ab\left(a^2+b^2\right).\end{aligned} \tag{4.30}$$

Calculations of the respective terms of (4.30) are shown in Appendix 2. We hence conclude that Theorem 2 holds for this problem.

Remark. It is very complicated to execute this triangular integral only by human's brain. Computer and formula manipulation software are recommended to verify these calculations in the case of $n \to \infty$. This theory is also applicable for numerical calculation in the case of $1 < n < \infty$.

5 The curl theorem of a triangular integral in the higher dimensions

The curl theorem of a triangular integral in the 3D space is derived in Section 5.1 and that in the 4D hyper-space is derived in Section 5.2.

5.1 The curl theorem in the 3D space

We extend the curl theorem of a triangular integral in the 3D space. We present three lemmata in the following before the proof of Theorem 3.

Lemma 5. *Let $X = X(x, y, z)$ be a partially differentiable function with respect to x, y and z. It holds*

$$\begin{aligned}\int_{g(x,y,z)=0}^{[A,B]} X\,dx = &\frac{1}{2}\iint_{g(x,y,z)\le 0}^{[A,B]} \left(\frac{\partial X'}{\partial y'}dx\,dy' - \frac{\partial X}{\partial y}dy\,dx' + \frac{\partial X'}{\partial z'}dx\,dz' - \frac{\partial X}{\partial z}dz\,dx'\right)\\ &+ \frac{1}{2}\iint_{g(x,y,z)\le 0}^{[A,B]} \left(\frac{\partial X'}{\partial x'} - \frac{\partial X}{\partial x}\right)dx\,dx' + \frac{1}{2}(x_{\mathrm{B}} - x_{\mathrm{A}})(X_{\mathrm{A}} + X_{\mathrm{B}})\end{aligned} \tag{5.1}$$

for a piecewise smooth curve of equation $g(x, y, z) = 0$ between $A(x_{\mathrm{A}}, y_{\mathrm{A}}, z_{\mathrm{A}})$ and $B(x_{\mathrm{B}}, y_{\mathrm{B}}, z_{\mathrm{B}})$, where $X_{\mathrm{A}} = X(x_{\mathrm{A}}, y_{\mathrm{A}}, z_{\mathrm{A}})$ and $X_{\mathrm{B}} = X(x_{\mathrm{B}}, y_{\mathrm{B}}, z_{\mathrm{B}})$.

Proof. Rewriting the transformation lemma (3.10) for $X = X(x, y, z)$ and $g(x, y, z) = 0$, it is expressed as

$$\int_{g(x,y,z)=0}^{[A,B]} X\,dx = \frac{1}{2}\iint_{g(x,y,z)\le 0}^{[A,B]}(dx\,dX' - dX\,dx') + \frac{1}{2}(x_{\mathrm{B}} - x_{\mathrm{A}})(X_{\mathrm{A}} + X_{\mathrm{B}}). \tag{5.2}$$

Substituting two kinds of the total differentials of X, i.e., dX and dX',

$$dX = \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy + \frac{\partial X}{\partial z}dz, \tag{5.3}$$

$$dX' = \frac{\partial X'}{\partial x'}dx' + \frac{\partial X'}{\partial y'}dy' + \frac{\partial X'}{\partial z'}dz' \tag{5.4}$$

in (5.2), it is modified to be (5.1). Q.E.D.

Lemma 6. *Let $Y = Y(x, y, z)$ be a partially differentiable function with respect to x, y and z. It holds*

$$\begin{aligned}\int_{g(x,y,z)=0}^{[A,B]} Y\,dy = &\frac{1}{2}\iint_{g(x,y,z)\le 0}^{[A,B]} \left(\frac{\partial Y'}{\partial x'}dy\,dx' - \frac{\partial Y}{\partial x}dx\,dy' + \frac{\partial Y'}{\partial z'}dy\,dz' - \frac{\partial Y}{\partial z}dz\,dy'\right)\\ &+ \frac{1}{2}\iint_{g(x,y,z)\le 0}^{[A,B]} \left(\frac{\partial Y'}{\partial y'} - \frac{\partial Y}{\partial y}\right)dy\,dy' + \frac{1}{2}(y_{\mathrm{B}} - y_{\mathrm{A}})(Y_{\mathrm{A}} + Y_{\mathrm{B}})\end{aligned} \tag{5.5}$$

for a piecewise smooth curve of equation $g(x, y, z) = 0$ between $A(x_{\mathrm{A}}, y_{\mathrm{A}}, z_{\mathrm{A}})$ and $B(x_{\mathrm{B}}, y_{\mathrm{B}}, z_{\mathrm{B}})$, where $Y_{\mathrm{A}} = Y(x_{\mathrm{A}}, y_{\mathrm{A}}, z_{\mathrm{A}})$ and $Y_{\mathrm{B}} = Y(x_{\mathrm{B}}, y_{\mathrm{B}}, z_{\mathrm{B}})$.

Proof. In the similar procedure as Lemma 5, we obtain (5.5). Q.E.D.

Lemma 7. *Let $Z = Z(x, y, z)$ be a partially differentiable function with respect to x, y and z. It holds*

$$\int_{g(x,y,z)=0}^{[A,B]} Z\,dz = \frac{1}{2}\iint_{g(x,y,z)\le 0}^{[A,B]} \left(\frac{\partial Z'}{\partial y'}dz\,dy' - \frac{\partial Z}{\partial y}dy\,dz' + \frac{\partial Z'}{\partial x'}dz\,dx' - \frac{\partial Z}{\partial x}dx\,dz'\right) + \frac{1}{2}\iint_{g(x,y,z)\le 0}^{[A,B]} \left(\frac{\partial Z'}{\partial z'} - \frac{\partial Z}{\partial z}\right) dz\,dz' + \frac{1}{2}(z_B - z_A)(Z_A + Z_B) \tag{5.6}$$

for a piecewise smooth curve of equation $g(x, y, z) = 0$ *between* $A(x_A, y_A, z_A)$ *and* $B(x_B, y_B, z_B)$, *where* $Z_A = Z(x_A, y_A, z_A)$ *and* $Z_B = Z(x_B, y_B, z_B)$.

Proof. In the similar procedure as Lemma 5, we obtain (5.6). Q.E.D.

Theorem 3. *(The curl theorem of a triangular integral in the* 3*D space) Let D be a piecewise smooth surface in the xyz-space, which is expressed in terms of the Cartesian coordinates* $(x, y, z) \in \mathbb{R}^3$. *Let* ∂D *be the boundary of D. Let* $X_1 = X = X(x, y, z)$, $X_2 = Y = Y(x, y, z)$ *and* $X_3 = Z = Z(x, y, z)$ *be partially differentiable functions with respect to* $x^1 = x$, $x^2 = y$ *and* $x^3 = z$ *in D. It holds*

$$\oint_{\partial D} X_\alpha dx^\alpha = \frac{1}{2}\iint_D \left(\frac{\partial X'_\alpha}{\partial x'^\beta} - \frac{\partial X_\beta}{\partial x^\alpha}\right) dx^\alpha dx'^\beta, \tag{5.7}$$

where indices are summed over α, β = 1,2,3.

Proof. Combining (5.1) with (5.5) and (5.6), we obtain

$$\begin{aligned}
&\int_{g(x,y,z)=0}^{[A,B]} (X\,dx + Y\,dy + Z\,dz) \\
&= \frac{1}{2}\iint_{g(x,y,z)\le 0}^{[A,B]} \left[\left(\frac{\partial X'}{\partial y'} - \frac{\partial Y}{\partial x}\right) dx\,dy' + \left(\frac{\partial Y'}{\partial x'} - \frac{\partial X}{\partial y}\right) dy\,dx'\right. \\
&+ \left(\frac{\partial Y'}{\partial z'} - \frac{\partial Z}{\partial y}\right) dy\,dz' + \left(\frac{\partial Z'}{\partial y'} - \frac{\partial Y}{\partial z}\right) dz\,dy' + \left(\frac{\partial Z'}{\partial x'} - \frac{\partial X}{\partial z}\right) dz\,dx' + \left(\frac{\partial X'}{\partial z'} - \frac{\partial Z}{\partial x}\right) dx\,dz' \\
&+ \left.\left(\frac{\partial X'}{\partial x'} - \frac{\partial X}{\partial x}\right) dx\,dx' + \left(\frac{\partial Y'}{\partial y'} - \frac{\partial Y}{\partial y}\right) dy\,dy' + \left(\frac{\partial Z'}{\partial z'} - \frac{\partial Z}{\partial z}\right) dz\,dz'\right] \\
&+ \frac{1}{2}(x_B - x_A)(X_A + X_B) + \frac{1}{2}(y_B - y_A)(Y_A + Y_B) + \frac{1}{2}(z_B - z_A)(Z_A + Z_B)
\end{aligned} \tag{5.8}$$

for a piecewise smooth curve of equation $g(x, y, z) = 0$ between $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$, where $X_A = X(x_A, y_A, z_A)$, $X_B = X(x_B, y_B, z_B)$, $Y_A = Y(x_A, y_A, z_A)$, $Y_B = Y(x_B, y_B, z_B)$, $Z_A = Z(x_A, y_A, z_A)$ and $Z_B = Z(x_B, y_B, z_B)$. For an integral on a closed curve, the initial point $A(x_A, y_A, z_A)$ coincides with the terminal one $B(x_B, y_B, z_B)$, i.e., $A(x_A, y_A, z_A) = B(x_B, y_B, z_B)$. It then holds

$$\frac{1}{2}(x_B - x_A)(X_A + X_B) + \frac{1}{2}(y_B - y_A)(Y_A + Y_B) + \frac{1}{2}(z_B - z_A)(Z_A + Z_B) = 0 \tag{5.9}$$

regardless of the values of X_A, X_B, Y_A, Y_B, Z_A and Z_B. We thus obtain (5.7). Q.E.D.

Corollaries 1 and 2 also hold for α, β = 1,2,3.

5.2 The curl theorem in the 4D hyper-space

We extend the curl theorem of a triangular integral in the 4D hyper-space.

Theorem 4. *(The curl theorem of a triangular integral in the 4D hyper-space) Let D be a piecewise smooth surface in the txyz-hyper-space, which is expressed in terms of the Cartesian coordinates* $(t, x, y, z) \in \mathbb{R}^4$. *Let* ∂D *be the boundary of D. Let* $X_0 = T = T(t, x, y, z)$, $X_1 = X = X(t, x, y, z)$, $X_2 = Y = Y(t, x, y, z)$ *and* $X_3 = Z = Z(t, x, y, z)$ *be* *partially differentiable functions with respect to $x^0 = t$, $x^1 = x$, $x^2 = y$ and $x^3 = z$ in D. It holds*

$$\oint_{\partial D} X_\alpha dx^\alpha = \frac{1}{2}\iint_D \left(\frac{\partial X'_\alpha}{\partial x'^\beta} - \frac{\partial X_\beta}{\partial x^\alpha}\right) dx^\alpha dx'^\beta, \tag{5.10}$$

where indices are summed over α, β = 0,1,2,3.

Proof. In the 4D, it holds

$$\begin{aligned}
&\int_{h(t,x,y,z)=0}^{[A,B]} \left(T\,dt + X\,dx + Y\,dy + Z\,dz\right) \\
&= \frac{1}{2}\iint_{h(t,x,y,z)\le 0}^{[A,B]} \left[\left(\frac{\partial X'}{\partial y'} - \frac{\partial Y}{\partial x}\right) dx\,dy' + \left(\frac{\partial Y'}{\partial x'} - \frac{\partial X}{\partial y}\right) dy\,dx'\right. \\
&\quad + \left(\frac{\partial Y'}{\partial z'} - \frac{\partial Z}{\partial y}\right) dy\,dz' + \left(\frac{\partial Z'}{\partial y'} - \frac{\partial Y}{\partial z}\right) dz\,dy' + \left(\frac{\partial Z'}{\partial x'} - \frac{\partial X}{\partial z}\right) dz\,dx' + \left(\frac{\partial X'}{\partial z'} - \frac{\partial Z}{\partial x}\right) dx\,dz' \\
&\quad + \left(\frac{\partial X'}{\partial x'} - \frac{\partial X}{\partial x}\right) dx\,dx' + \left(\frac{\partial Y'}{\partial y'} - \frac{\partial Y}{\partial y}\right) dy\,dy' + \left(\frac{\partial Z'}{\partial z'} - \frac{\partial Z}{\partial z}\right) dz\,dz' + \left(\frac{\partial T'}{\partial t'} - \frac{\partial T}{\partial t}\right) dt\,dt' \\
&\quad + \left(\frac{\partial X'}{\partial t'} - \frac{\partial T}{\partial x}\right) dx\,dt' + \left(\frac{\partial Y'}{\partial t'} - \frac{\partial T}{\partial y}\right) dy\,dt' + \left(\frac{\partial Z'}{\partial t'} - \frac{\partial T}{\partial z}\right) dz\,dt' \\
&\quad \left. + \left(\frac{\partial T'}{\partial x'} - \frac{\partial X}{\partial t}\right) dt\,dx' + \left(\frac{\partial T'}{\partial y'} - \frac{\partial Y}{\partial t}\right) dt\,dy' + \left(\frac{\partial T'}{\partial z'} - \frac{\partial Z}{\partial t}\right) dt\,dz'\right] \\
&\quad + \frac{1}{2}(t_B - t_A)(T_A + T_B) + \frac{1}{2}(x_B - x_A)(X_A + X_B) + \frac{1}{2}(y_B - y_A)(Y_A + Y_B) + \frac{1}{2}(z_B - z_A)(Z_A + Z_B)
\end{aligned} \tag{5.11}$$

for a piecewise smooth curve of equation $h(t, x, y, z) = 0$ between $A(t_A, x_A, y_A, z_A)$ and $B(t_B, x_B, y_B, z_B)$, where $T_A = T(t_A, x_A, y_A, z_A)$, $T_B = T(t_B, x_B, y_B, z_B)$, $X_A = X(t_A, x_A, y_A, z_A)$, $X_B = X(t_B, x_B, y_B, z_B)$, $Y_A = Y(t_A, x_A, y_A, z_A)$, $Y_B = Y(t_B, x_B, y_B, z_B)$, $Z_A = Z(t_A, x_A, y_A, z_A)$ and $Z_B = Z(t_B, x_B, y_B, z_B)$. For an integral on a closed curve, the initial point $A(t_A, x_A, y_A, z_A)$ coincides with the terminal one $B(t_B, x_B, y_B, z_B)$, i.e., $A(t_A, x_A, y_A, z_A) = B(t_B, x_B, y_B, z_B)$. It then holds

$$\frac{1}{2}(t_B - t_A)(T_A + T_B) + \frac{1}{2}(x_B - x_A)(X_A + X_B) + \frac{1}{2}(y_B - y_A)(Y_A + Y_B) + \frac{1}{2}(z_B - z_A)(Z_A + Z_B) = 0 \tag{5.12}$$

regardless of the values of T_A, T_B, X_A, X_B, Y_A, Y_B, Z_A and Z_B. We thus obtain (5.10). Q.E.D.

Corollaries 1 and 2 also hold for α, β = 0 1,2,3.

Appendix 1

Using Definition 1, the first term of the right-hand side of (4.2) in the proof of Lemma 3 is expressed as

$$\frac{1}{2}\iint_{f(x,y)\le 0}^{[A,B]} (dx\,dX' - dX\,dx') = \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}(\Delta x_k \Delta X_j - \Delta X_k \Delta x_j). \tag{5.13}$$

The total increments ΔX_j and ΔX_k in the right-hand side of (5.13) are respectively expressed as

$$\begin{aligned}
\Delta X_j &\equiv X(x_j, y_j) - X(x_{j-1}, y_{j-1}) \\
&= \frac{X(x_j, y_j) - X(x_{j-1}, y_j)}{x_j - x_{j-1}}\Delta x_j + \frac{X(x_{j-1}, y_j) - X(x_{j-1}, y_{j-1})}{y_j - y_{j-1}}\Delta y_j,
\end{aligned} \tag{5.14}$$

$$\begin{aligned}\Delta X_k &\equiv X(x_k, y_k) - X(x_{k-1}, y_{k-1}) \\ &= \frac{X(x_k, y_k) - X(x_{k-1}, y_k)}{x_k - x_{k-1}} \Delta x_k + \frac{X(x_{k-1}, y_k) - X(x_{k-1}, y_{k-1})}{y_k - y_{k-1}} \Delta y_k.\end{aligned} \tag{5.15}$$

Substituting (5.14) and (5.15) in the right-hand side of (5.13), it is modified to be

$$\begin{aligned}\frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}(\Delta x_k \Delta X_j - \Delta X_k \Delta x_j) &= \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{X(x_{j-1}, y_j) - X(x_{j-1}, y_{j-1})}{y_j - y_{j-1}}\Delta x_k \Delta y_j \\ &- \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{X(x_{k-1}, y_k) - X(x_{k-1}, y_{k-1})}{y_k - y_{k-1}}\Delta y_k \Delta x_j \\ &+ \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{X(x_j, y_j) - X(x_{j-1}, y_j)}{x_j - x_{j-1}}\Delta x_k \Delta x_j \\ &- \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{X(x_k, y_k) - X(x_{k-1}, y_k)}{x_k - x_{k-1}}\Delta x_k \Delta x_j.\end{aligned} \tag{5.16}$$

Each term of (5.16) is respectively expressed as

$$\frac{1}{2}\iint_{f(x,y)\le 0}^{[A,B]} \frac{\partial X'}{\partial y'} dx\, dy' = \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{X(x_{j-1}, y_j) - X(x_{j-1}, y_{j-1})}{y_j - y_{j-1}}\Delta x_k \Delta y_j, \tag{5.17}$$

$$\frac{1}{2}\iint_{f(x,y)\le 0}^{[A,B]} \frac{\partial X}{\partial y} dy\, dx' = \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{X(x_{k-1}, y_k) - X(x_{k-1}, y_{k-1})}{y_k - y_{k-1}}\Delta y_k \Delta x_j, \tag{5.18}$$

$$\frac{1}{2}\iint_{f(x,y)\le 0}^{[A,B]} \frac{\partial X'}{\partial x'} dx\, dx' = \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{X(x_j, y_j) - X(x_{j-1}, y_j)}{x_j - x_{j-1}}\Delta x_k \Delta x_j, \tag{5.19}$$

$$\frac{1}{2}\iint_{f(x,y)\le 0}^{[A,B]} \frac{\partial X}{\partial x} dx\, dx' = \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{X(x_k, y_k) - X(x_{k-1}, y_k)}{x_k - x_{k-1}}\Delta x_k \Delta x_j. \tag{5.20}$$

We thus obtain (4.1) in Lemma 3.

Appendix 2

Substituting (4.24) in (4.20) and (4.21), substituting (4.25) in (4.22) and (4.23), the respective terms of (4.30) are calculated as follows.

1. The first term of the left-hand side of (4.30) is

$$\begin{aligned}&\frac{1}{2}\int\int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}^{\circlearrowleft}\left(\frac{\partial X'}{\partial y'} - \frac{\partial Y}{\partial x}\right)dx\, dy' \\ &= -\frac{1}{2}\int\int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}^{\circlearrowleft}\left[(x')^2 + y^2\right]dx\, dy' \\ &= -\frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left[(x_j)^2 + (y_k)^2\right]\Delta x_k \Delta y_j \\ &= \frac{-ab}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(a^2\cos^2\theta_j + b^2\sin^2\theta_k\right)\left(\cos\theta_k - \cos\theta_{k-1}\right)\left(\sin\theta_j - \sin\theta_{j-1}\right) \\ &= \frac{-ab}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left[a^2\cos^2\left(j\frac{2\pi}{n}\right) + b^2\sin^2\left(k\frac{2\pi}{n}\right)\right] \\ &\quad\times\left\{\cos\left[k\frac{2\pi}{n}\right] - \cos\left[(k-1)\frac{2\pi}{n}\right]\right\}\left\{\sin\left[j\frac{2\pi}{n}\right] - \sin\left[(j-1)\frac{2\pi}{n}\right]\right\} \\ &= \frac{3}{8}\pi ab(a^2 + b^2).\end{aligned} \tag{5.21}$$

2. The second term of the lefthand side of (4.30) is

$$\begin{aligned}
&\frac{1}{2}\int\int^{\circlearrowleft}_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1}\left(\frac{\partial Y'}{\partial x'}-\frac{\partial X}{\partial y}\right)dy\,dx' \\
&=\frac{1}{2}\int\int^{\circlearrowleft}_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1}\left[(y')^2+x^2\right]dy\,dx' \\
&=\frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left[(y_j)^2+(x_k)^2\right]\Delta y_k\Delta x_j \\
&=\frac{ab}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(b^2\sin^2\theta_j+a^2\cos^2\theta_k\right)\left(\sin\theta_k-\sin\theta_{k-1}\right)\left(\cos\theta_j-\cos\theta_{j-1}\right) \\
&=\frac{ab}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left[b^2\sin^2\left(j\frac{2\pi}{n}\right)+a^2\cos^2\left(k\frac{2\pi}{n}\right)\right] \\
&\quad\times\left\{\sin\left[k\frac{2\pi}{n}\right]-\sin\left[(k-1)\frac{2\pi}{n}\right]\right\}\left\{\cos\left[j\frac{2\pi}{n}\right]-\cos\left[(j-1)\frac{2\pi}{n}\right]\right\} \\
&=\frac{3}{8}\pi ab(a^2+b^2).
\end{aligned}\tag{5.22}$$

3. The third term of the left-hand side of (4.30) is

$$\begin{aligned}
&\frac{1}{2}\int\int^{\circlearrowleft}_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1}\left(\frac{\partial X'}{\partial x'}-\frac{\partial X}{\partial x}\right)dx\,dx' \\
&=\int\int^{\circlearrowleft}_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1}(-x'y'+xy)dx\,dx' \\
&=\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}(-x_jy_j+x_ky_k)\Delta x_k\Delta x_j \\
&=a^3b\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(-\cos\theta_j\sin\theta_j+\cos\theta_k\sin\theta_k\right)\left(\cos\theta_k-\cos\theta_{k-1}\right)\left(\cos\theta_j-\cos\theta_{j-1}\right) \\
&=a^3b\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left[-\cos\left(j\frac{2\pi}{n}\right)\sin\left(j\frac{2\pi}{n}\right)+\cos\left(k\frac{2\pi}{n}\right)\sin\left(k\frac{2\pi}{n}\right)\right] \\
&\quad\times\left\{\cos\left[k\frac{2\pi}{n}\right]-\cos\left[(k-1)\frac{2\pi}{n}\right]\right\}\left\{\cos\left[j\frac{2\pi}{n}\right]-\cos\left[(j-1)\frac{2\pi}{n}\right]\right\} \\
&=-\frac{1}{2}\pi a^3b.
\end{aligned}\tag{5.23}$$

4. The fourth term of the left-hand side of (4.30) is

$$\begin{aligned}
&\frac{1}{2}\int\int^{\circlearrowleft}_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1}\left(\frac{\partial Y'}{\partial y'}-\frac{\partial Y}{\partial y}\right)dy\,dy' \\
&=\int\int^{\circlearrowleft}_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1}(x'y'-xy)dy\,dy' \\
&=\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}(x_jy_j-x_ky_k)\Delta y_k\Delta y_j \\
&=ab^3\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\cos\theta_j\sin\theta_j-\cos\theta_k\sin\theta_k\right)\left(\sin\theta_k-\sin\theta_{k-1}\right)\left(\sin\theta_j-\sin\theta_{j-1}\right) \\
&=ab^3\lim_{n\to\infty}\sum_{k=1}^{n}\sum_{j=1}^{k}\left[\cos\left(j\frac{2\pi}{n}\right)\sin\left(j\frac{2\pi}{n}\right)-\cos\left(k\frac{2\pi}{n}\right)\sin\left(k\frac{2\pi}{n}\right)\right] \\
&\quad\times\left\{\sin\left[k\frac{2\pi}{n}\right]-\sin\left[(k-1)\frac{2\pi}{n}\right]\right\}\left\{\sin\left[j\frac{2\pi}{n}\right]-\sin\left[(j-1)\frac{2\pi}{n}\right]\right\} \\
&=-\frac{1}{2}\pi ab^3.
\end{aligned}\tag{5.24}$$

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