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Stably asymptotic average shadowing property and dominated splitting

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Abstract

Let f be a diffeomorphism of a closed n-dimensional C^∞ manifold. In this article, we show that C^1-generically, if f has the C^1-stably asymptotic average shadowing property on a closed set then it admits a dominated splitting.

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1 Introduction

The notion of the pseudo-orbits very often appears in several branches of the modern theory of dynamical system. For instance, the pseudo-orbit property (shadowing property) usually plays an important role in stability theory In this article, we consider the asymptotic average shadowing property, which was introduced in Gu [1], is a special version of the shadowing property We find a relation between the stably asymptotic average shadowing property (on manifold) and the dominated splitting structure on the vector bundle. In differentiable dynamical system, dominated splitting on the vector bundle is a nature generalization of hyperbolicity and is investigated by many mathematicians [2-11].

Here we denote M a closed n-dimensional smooth manifold, and let Diff(M) be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $||\cdot||$ on the tangent bundle TM. Let $f\in$ Diff (M). A sequence $\{x_i\}_{i=-\infty}^{\infty}$ in M is called an *asymptotic average pseudo orbit* of f if

$$\lim_{n\to\infty}\frac{1}{2n}\sum_{i=-n}^{n-1}d(f(x_i),x_{i+1})=0.$$

An asymptotic average pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ is said to be *asymptotically shadowed in average by the point z* if

$$\lim_{n\to\infty}\frac{1}{2n}\sum_{i=-n}^{n-1}d(f^i(z),(x_i))=0.$$

Given an invariant set Λ of f, we say f has the asymptotic average shadowing property on Λ if for any asymptotic pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$, there exist a point $z\in\Lambda$ which asymptotically shadows $\{x_i\}_{i\in\mathbb{Z}}$.

Let $f \in$ Diff(M), and let Λ be a closed f-invariant set. We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that $\cap_{n\in\mathbb{N}} f^n(U) = \Lambda$. Now we can introduce a notion of C^1-stably the asymptotic average shadowing property on a locally maximal invariant set.

Definition 1.1 *Let Λ be a locally maximal invariant set of $f \in$ Diff(M). We say that f has the C^1-stably asymptotic average shadowing property on Λ, (or Λ is C^1-stably asymptotic average shadowable with respect to f) if there are a compact neighborhood U of f and a C^1-neighborhood $\mathcal{U}(f)$ of f such that $\Lambda = \Lambda_f(U) = \cap_{n\in\mathbb{Z}} f^n(U)$ (locally maximal), and for any $g \in \mathcal{U}(f)$, $g|_{\Lambda_g(U)}$ has the asymptotic average shadowing property, where $\Lambda_g(U) = \cap_{n\in\mathbb{Z}} g^n(U)$ is the continuation of Λ.*

Let $\Lambda \subset M$ be an f-invariant closed set. We say that Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\left\|D_x f^n\left|E_{(x)}\right\| . \left\|D_x f^{-n}\right|_{F(f^n(x))}\right\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$.

The following remark gives an equivalent definition of dominated splitting.

Remark 1.2 *Let Λ be a closed f-invariant set. A splitting $T_\Lambda M = E \oplus F$ is called a l-dominated splitting for a positive integer l if E and F are Df-invariant and*

$$\left\|Df^l\left|_{E(x)}\right\| / m(Df^l\left|_{F(x)}\right.) \le \frac{1}{2},$$

for all $x \in \Lambda$, where $m(A) = \inf\{||Av||: ||v|| = 1\}$ denotes the minimum norm of a linear map A.

Now we can state main results of this article.

Theorem 1.3 *Let Λ be a closed set of $f \in$ Diff(M). Then C^1-generically, if f has the C^1-stably asymptotic average shadowing property on Λ then it admits a dominated splitting.*

Theorem 1.4 *Let Λ be a transitive set. If f has the C^1-stably asymptotic average shadowing property on Λ then it admits a dominated splitting.*

2 Proof of theorems

Theorems 1.3 and 1.4 are all base on the following proposition:

Proposition 2.1 *Let Λ be a closed locally manximal invariant set of f, if f has the C^1-stably asymptotic average shadowing property on Λ, and there exist a sequence g_n goes to f and periodic orbits P_n of g_n which converges to Λ in Hausdorff limits, then Λ admits a dominated splitting.*

Firstly, we give the notation of *pre-sink* (*resp. pre-source*) which prevent the stably asymptotic average shadowing property. A periodic point p of f is called a *pre-sink* (*resp. pre-source*) if $Df^{\pi(p)}(p)$ has a multiplicity one eigenvalue with modulus 1 and the other eigenvalues has norm strictly less than 1 (resp. bigger than 1).

Lemma 2.2 *Let Λ be a closed set of f. Suppose that $f_{|\Lambda}$ has the C^1-stably asymptotic average shadowing property. Let U and $\mathcal{U}(f)$ be given in the Definition* 1.1, *then for any $g \in \mathcal{U}(f)$, g has neither pre-sink nor pre-sources with the orbit staying in U.*

Proof. We prove the lemma by contradiction. Assume that there is $g \in \mathcal{U}(f)$ such that g has a pre-sink p with Orb(p) $\subset U$.

By the Franks' Lemma, we can linearize g at p with respect to the exponential coordinates $\exp_p$, i.e., after an arbitrarily small perturbation, we can get a diffeomorphism $g_1 \in \mathcal{U}(f)$ such that there is $\epsilon_1 > 0$ small enough with $B_{\varepsilon_1}(\mathrm{Orb}(p)) \subset U$ such that

$$g_1\Big|_{B_{\varepsilon_1}(g^i(p))} = \exp_{g^{i+1}(p)} \circ D_{g^i(p)}g \circ \exp^{-1}_{g^i(p)}\Big|_{B_{\varepsilon_1}(g^i(p))},$$

for any $0 \le i \le \pi(p) - 1$.

Since p is pre-sink of g, $D_pg^{\pi(p)}$ has a multiplicity one eigenvalue such that $|\lambda| = 1$ and other eigenvalues of $D_pg^{\pi(p)}$ have moduli less than 1. Denote by E^c_p the eigenspace corresponding to λ, and E^s_p the eigenspace corresponding to the eigenvalues with modulus less than 1. Thus $T_pM = E^c_p \oplus E^s_p$. If $\lambda \in \mathbb{R}$ then $\dim E^c_p = 1$, and if $\lambda \in \mathbb{C}$ then $\dim E^c_p = 2$.

At first, we consider the case $\dim E^c_p = 1$. For simplicity, we suppose that $\lambda = 1$, and $g_1^{\pi(p)}(p) = p$. The case of $\lambda = -1$ can be proved similarly. Since the eigenvalue $\lambda = 1$, there is a small arc $\mathcal{I}_p \subset B_{\varepsilon_1}(p) \cap \exp_p(E^c_p(\varepsilon_1))$ centered at p such that $g_1^{\pi(p)}|_{\mathcal{I}_p}$ is the identity map. Here $E^c_p(\varepsilon_1)$ is the ϵ_1-ball in E^c_p center at the origin O_p.

There exist $D > 0$ such that for any $z \in B_D(p)$, there exists $x \in \mathcal{I}_p$ such that $g_1^{n\pi(p)}(z) \to x$ as $n \to \infty$. Take two distinct points $a, b \in \mathcal{I}_p$ such that $d(a, b) = D/4$.

We construct an asymptotic average pseudo orbit of g_1 as follows.

$$\begin{aligned}
&x_{-i} = g_1^{-i}a,\\
&x_0 = a,\ x_1 = g_1(a), \ldots, x_{\pi(p)-1} = g_1^{\pi(p)-1}a,\ x_{\pi(p)} = b, \ldots,\\
&x_{(2^k-2)\pi(p)} = a, x_{(2^k-2)\pi(p)+1} = g_1(a), \ldots, x_{(2^k+2^{k-1}-2)\pi(p)-1} = g_1^{-1}a,\\
&x_{(2^k+2^{k-1}-2)\pi(p)} = b, \ldots, x_{(2^{k+1}-2)\pi(p)-1} = g_1^{-1}(b), \ldots.
\end{aligned}$$

One can easily check that $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is an asymptotic average pseudo orbit of g_1.

Since g_1 has the asymptotic average shadowing property on $\Lambda_{g_1}(U)$, we can find a point z such that the point z is shadows $\xi = \{x_i\}_{i\in\mathbb{Z}}$ in asymptotic average, i.e.,

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{i=-n}^{n-1} d(g_1^i(z), x_i) = 0.$$

It is easy to see that there is $n_0 > 0$ such that $g_1^{n_0}(z) \in B_D(p)$. Hence there exists a point $x \in \mathcal{I}_p$ such that $g_1^{n\pi(p)+n_0}(z) \to x$, as $n \to \infty$. From the choice of a, b and the fact that $g_1^{\pi(p)}|_{\mathcal{I}_p} = Id$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i-0}^{n-1} d(g_1^i(z), x_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(g_1^{i-n0}(x), x_i) > 0.$$

This is a contradiction.

Finally, we consider the case $\dim E^c_p = 2$. There is a disk $\mathcal{D}_p \subset B_{\varepsilon_1}(p) \cap \exp_p(E^c_p(\varepsilon_1))$ centered at p such that $g_1^{\pi(p)}|_{\mathcal{D}_p}$ is a rotation. Note that $\mathcal{D}$ consists of $g_1^{\pi(p)}$-invariant circles. We take a and b in different circles. Then by similar arguments as above, we get the contradiction. We omit the details and finish the proof here.

Let $GL(n)$ be the group of linear isomorphisms of $\mathbb{R}^n$. A sequence $\xi : \mathbb{Z} \to GL(n)$ is called *periodic* if there is $k > 0$ such that $\xi_{j+k} = \xi_j$ for $k \in \mathbb{Z}$. We call a finite subset $\mathcal{A} = \{\xi_i : 0 \le i \le k-1\} \subset GL(n)$ is a *periodic family* with period k. For a periodic family $\mathcal{A} = \{\xi_i : 0 \le i \le n-1\}$, we denote $\mathcal{C}_{\mathcal{A}} = \xi_{n-1} \circ \xi_{n-2} \circ \cdots \circ \xi_0$.

Definition 2.3 *We say that the periodic family $\mathcal{A} = \{\xi_i : 0 \le i \le n-1\}$ admits an l-dominated splitting, if there is a splitting $\mathbb{R}^n = E \oplus F$ which satisfies:*

(*a*) *E and F are $\mathcal{C}_{\mathcal{A}}$ invariant, i.e., $\mathcal{C}_{\mathcal{A}}(E) = E$ and $\mathcal{C}_{\mathcal{A}}(F) = F$,*
(*b*) *For any $k = 0,1,2,...,$*

$$\frac{\left\|\xi_{k+l-1} \circ \cdots \circ \xi_{k+1} \circ \xi_k|_{E_k}\right\|}{m(\xi_{k+l-1} \circ \cdots \circ \xi_{k1} \circ \xi_k|_{F_k})} \le \frac{1}{2},$$

where $E_k = \zeta_{k-1} \circ \zeta_{k-2} \circ ... \circ \zeta_0(E)$ and $F_k = \zeta_{k-1} \circ \zeta_{k-2} \circ ... \circ \zeta_0(F)$.

We know the following theorems for periodic family from [4] which is useful for our result.

Theorem 2.4 *Given any $\epsilon > 0$ and $K > 0$, there is positive integers $n_2 \ge 0$ and $l \ge 0$ which satisfies the following property: given any periodic family $\mathcal{A} = \{\xi_i : 0 \le i \le n-1\}$ which satisfies the period $n \ge n_2$ and $\max\left\{\|\xi_i\|, \left\|\xi_i^{-1}\right\|\right\} \le K$, for all $i = 0,1,...,n-1$, if $\mathcal{A}$ does not admits any l-dominated splitting, then one can find a periodic family $\mathcal{B} = \{\zeta_0, \zeta_1, ..., \zeta_{n-1}\}$ such that $\max\left\{\|\zeta_i - \xi_i\|, \left\|\zeta_i^{-1} - \xi_i^{-1}\right\|\right\} < \varepsilon$ for any $i = 0,1,...,n-1$, and $\det(\mathcal{C}_{\mathcal{A}}) = \det(\mathcal{C}_{\mathcal{B}})$ and the eigenvalues of $\mathcal{C}_{\mathcal{B}}$ are all real, and have same modulus.*

To prove Theorem 2.4, we need another lemma about uniformly contracting family. Let $\mathcal{A} = \{\xi_i : 0 \le i \le k-1\} \subset GL(n)$ be a periodic family. We say the sequence $\mathcal{A}$ is *uniformly contracting family* if there is a constant $\delta > 0$ such that for any δ-perturbation of $\mathcal{A}$ are sink, i.e., for any $\mathcal{B} = \{\xi_i : 0 \le i \le k-1\}$ with $||\zeta_i - \zeta_i|| < \delta$, all eigenvalue of $\mathcal{C}_{\mathcal{B}}$ have moduli less than 1. Similarly, we can define the *uniformly expanding periodic family*. The following theorem is well known.

Theorem 2.5 [12]*For any $\delta > 0$ and $K > 0$, there are constants $C > 0, 0 < \lambda < 1$ and positive integer m such that if $\mathcal{A} = \{A_0, A_1, ..., A_{n-1}\}$ is a uniformly contracting periodic family which satisfies*

$$\max\left\{\|A_i\|, \left\|A_i^{-1}\right\|\right\} < K$$

for any $i = 0,1,...,n-1$ and $n > m$, then

$$\prod_{j=0}^{k-1}\left\|\prod_{i=0}^{m-1} A_{i+mj}\right\| \le C\lambda^k,$$

where $k = [n/m]$.

Now we return to our main proposition, the Proposition 2.1. Let P_n be given as in

Proposition 2.1. Choose $p_n \in P_n$, then we get a linear map sequence

$$\mathcal{A}_n = \{D_{p_n}f, D_{f(p_n)}f, \ldots, D_{f^{\pi(p_n)-1}(p_n)}f\}.$$

Lemma 2.6 [[10], Lemma 3.2.] *If* Λ *is not a periodic orbit and* $\mathcal{A}_n$ is given in above. *Then for any* $\epsilon > 0$ *there exists an* $n_0(\epsilon) > 0$ *such that for any* $n > n_0(\epsilon)$, $\mathcal{A}_n$ *is neither* ϵ*-uniformly contracting nor* ϵ*-uniformly expanding.*

Since the proof is essentially the same as that of [10], we omit the proof here. From the above lemma and main conclusion of [4], one can get the following lemma. The proof of the following can be found in [10].

Lemma 2.7 [[10], Lemma 3.3.] *Let* Λ, g_n *and* P_n *be given as in the assumption of Proposition* 2.1. *Then for any* $\epsilon > 0$ *there are* $n(\epsilon), l(\epsilon) > 0$ *such that for any* $n > n(\epsilon)$ *if* P_n *does not admit an* $l(\epsilon)$ *dominated splitting, then one can find* g'_n C^1 ϵ*-close* g_n *and preserving the orbit of* P_n *such that* P_n *is pre-sink or pre-source respecting* g'_n.

From the above lemmas and the next property of dominated splitting, we can get Proposition 2.1.

Lemma 2.8 [[3], Lemma 1.4.] *Let* g_n *converges to f and if* Λ_n *be a closed* g_n*-invariant set such that the Hausdorff limit of* Λ_n *equal to* Λ. *If* $\Lambda_{g_n}(U)$ *admits a l-dominated splitting respecting* g_n, *then* Λ *admits an l-dominated splitting respecting f.*

Now we can get our Theorems 1.3 and 1.4, Theorem 1.3 follows two results:

Lemma 2.9 [1,13]*Let* Λ *be a closed set of* $f \in \mathrm{Diff}(M)$. *If f has the asymptotic average shadowing property on* Λ *then* Λ *is a chain transitive set.*

The following Lemma is in [14].

Lemma 2.10 *There is a residual set* $\mathcal{G} \subset \mathrm{Diff}(M)$ *such that for any* $f \in \mathcal{G}$, *a compact f-invariant set* Λ *is a chain transitive set if and if* Λ *is a sequence* $\{P_n\}$ *of periodic orbits of f with the Hausdorff topology.*

Theorem 1.4 follows the result:

Lemma 2.11 [[11], Corollary 2.7.1.] *Let* Λ *be a transitive set. Then there are a sequence* $\{g_n\}$ *of diffeomorphism and a sequence* $\{P_n\}$ *of periodic orbits of* g_n *with period* $\pi(P_n) \to \infty$ *such that* $g_n \to f$ *in the* C^1*-topology and* $P_n \to_H \Lambda$ *as* $n \to \infty$, *where* $\to_H$ *is the Hausdorff limit, and* $\pi(P_n)$ *is the period of* P_n.

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Competing interests

The author declares that they have no competing interests.

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