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C^1 regularity of the stable subspaces with a general nonuniform dichotomy

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Abstract

For nonautonomous linear difference equations, we establish C^1 regularity of the stable subspaces under sufficiently C^1-parameterized perturbations. We consider the general case of nonuniform dichotomies, which corresponds to the existence of what we call nonuniform (μ, ν)-dichotomies.

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1. Introduction

We consider nonautonomous linear difference equations

$$v_{m+1} = A_m v_m + B_m(\lambda) v_m \tag{1.1}$$

in a Banach space, where λ is a parameter in some open subset Y of a Banach space (the parameter space), and $\lambda \to B_m(\lambda)$ is of class C^1 for each $m \in J = \mathbb{N}$. Assuming that the unperturbed dynamics

$$v_{m+1} = A_m v_m \tag{1.2}$$

admits a very general nonuniform dichotomy (see Section 2 for the definition), and that

$$\sup_{m \in J, \lambda \in Y} \|B_m(\lambda)\| \quad \text{and} \quad \sup_{m \in J, \lambda \in Y} \|B'_m(\lambda)\|$$

are sufficiently small, we establish the optimal C^1 regularity of the stable subspaces on the parameter λ for Equation (1.1).

The classical notion of (uniform) exponential dichotomy, essentially introduced by Perron in [1], plays an important role in a large part of the theory of differential equations and dynamical systems. We refer the reader to the books [2-5] for details and references. Inspired both in the classical notion of exponential dichotomy and in the notion of nonuni-formly hyperbolic trajectory introduced by Pesin in [6,7], Barreira and Valls [8-11] have introduced the notion of nonuniform exponential dichotomies and have developed the corresponding theory in a systematic way for the continuous and discrete dynamics during the last few years. See also the book [12] for details. As mentioned in [12], in finite-dimensional spaces essentially any linear differential equation with nonzero Lyapunov exponents admits a nonuniform exponential dichotomy.

The works of Barreira and Valls can be regarded as a nice contribution to the nonuniform hyperbolicity theory [13].

There are some works concerning the smooth dependence of the stable and unstable sub-spaces on the parameter. For example, in the case of continuous time, that is, for linear differential equations

$$v' = [A(t) + B(t, \lambda)]v,$$

Johnson and Sell [14] considered exponential dichotomies in $\mathbb{R}$ (in a finite dimensional space), and proved that for Ck perturbations, if the perturbation and its derivatives in λ are bounded and equicontinuous in the parameter, then the projections are of class Ck in λ. In the case of discrete time, Barreira and Valls established the optimal C^1 dependence of the stable and unstable subspaces on the parameter in [15] for the uniform exponential dichotomies and in [16] for the nonuniform exponential dichotomies.

In our study, we establish the optimal C^1 dependence of the stable subspaces on the parameter for very general nonuniform dichotomies (which was first introduced by Bento and Silva in [17]) for (1.1). Such dichotomies include for example the classical notion of uniform exponential dichotomies, as well as the notions of nonuniform exponential dichotomies and nonuniform polynomial dichotomies. The proof in this article follows essentially the ideas in [16], with some essential difficulties because we consider the new dichotomies. We also note that we can establish the optimal C^1 dependence of the unstable subspaces on the parameter using the similar discussion as in [16], and we omit the detail for short.

2. Setup

Let $\mathcal{B}(X)$ be the set of bounded linear operations in the Banach space X. Let $(A_m)_{m\in J}$ be a sequence of invertible operators in $\mathcal{B}(X)$. For each $m, n \in J$, we set

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}...A_n, & \text{if } m > n, \\ \text{Id}, & \text{if } m = n, \\ A_m^{-1}...A_{n-1}^{-1}, & \text{if } m < n. \end{cases}$$

In order to introduce the notion of nonuniform dichotomy, it is convenient to consider the notion of growth rate. We say that an increasing function $\mu : J \to (0, +\infty)$ is a growth rate if

$$\mu(0) = 1 \quad \text{and} \quad \lim_{n\to+\infty} \mu(n) = +\infty.$$

Given two growth rates μ and ν, we say that the sequence $(A_m)_{m\in J}$ (or the cocycle $\mathcal{A}(m,n)$) admits a nonuniform (μ, ν) dichotomy if there exist projections $P_m \in \mathcal{B}(X)$ for each $m \in J$ such that

$$\mathcal{A}(m,n)P_n = P_m\mathcal{A}(m,n), \quad m, n, \in J$$

and there exist constants α, $D > 0$ and $\varepsilon > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le D\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \nu^{\varepsilon}(n), \tag{2.1}$$

and

$$\left\| \mathcal{A}(m,n)^{-1} Q_m \right\| \le D \left(\frac{\mu(m)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon}(m), \tag{2.2}$$

for each $m \geq n$, where Q_m = Id - P_m is the complementary projection of P_m.

When $\mu(m) = \nu(m) = e^{\rho(m)}$, we recover the notion of ρ-*nonuniform exponential dichotomy*, while we recover the notion of *nonuniform polynomial dichotomy* when $\mu(m) = \nu(m) = 1 + m$.

For example, if μ and ν are arbitrary growth rates and ε, $\alpha > 0$, consider a sequence of linear operators $A_n : \mathbb{R}^2 \to \mathbb{R}^2$ given by diagonal matrices

$$A_n = \begin{pmatrix} a_n & 0 \\ 0 & b_n \end{pmatrix},$$

where

$$\begin{cases} a_m = \left(\dfrac{\mu(m+1)}{\mu(m)} \right)^{-\alpha} e^{\frac{\varepsilon}{2} \log \nu(m+1)(\cos(m+1)-1) - \frac{\varepsilon}{2} \log \nu(m)(\cos m - 1)}, \\ b_m = \left(\dfrac{\mu(m+1)}{\mu(m)} \right)^{\alpha} e^{-\frac{\varepsilon}{2} \log \nu(m+1)(\cos(m+1)-1) + \frac{\varepsilon}{2} \log \nu(m)(\cos m - 1)}, \end{cases}$$

for any $m \in J$. Then $(A_m)_{m \in J}$ admits a nonuniform (μ, ν) dichotomy with the projections $P_m : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $P_m(x, y) = (x, 0)$, and we have

$$\left\| \mathcal{A}(m,n) P_n \right\| \le \left(\frac{\mu(m)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon}(n),$$

and

$$\left\| \mathcal{A}(m,n)^{-1} Q_m \right\| \le \left(\frac{\mu(m)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon}(m)$$

for each $m \geq n$.

In this article, for each $n \in J$, we define the stable and unstable subspaces by

$$E_n = P_n(X) \quad \text{and} \quad F_n = Q_n(X).$$

3. Main results

We establish the existence of stable subspaces E_n^{λ} on J for each $\lambda \in Y$, such that the maps $\lambda \mapsto E_n^{\lambda}$ are of class C^1. As the same in [10], we look for each space E_n^{λ} as a graph over E_n. More precisely, we look for linear operators $\Phi_{n,\lambda}$: $E_n \to F_n$ such that

$$E_n^{\lambda} = \operatorname{graph}(\mathrm{Id}_{En} + \Phi_{n,\lambda}), \quad n \in J, \quad \lambda \in Y.$$

Given a constant $\kappa < 1$, let $\mathcal{X}$ be the space of families $\Phi = (\Phi_{n,\lambda})_{n \in J, \lambda \in Y}$ of linear operators $\Phi_{n,\lambda}$:$E_n \to F_n$ such that

$$\|\Phi\| := \sup \left\{ \left\| \Phi_{n,\lambda} \right\| \nu^{\varepsilon}(|n|) : (n,\lambda) \in J \times Y \right\} \le \kappa$$

and

$$C_{\lambda\mu}(\Phi) := \sup \left\{ \left\| \Phi_{n,\lambda} - \Phi_{n,\mu} \right\| \nu^{\varepsilon}(|n|) : n \in J \right\} \le \kappa \left\| \lambda - \mu \right\|$$

for each $\lambda, \mu \in Y$. Equipping $\mathcal{X}$ with the distance

$$\|\Phi - \Psi\| = \sup\left\{\left\|\Phi_{n,\lambda} - \Psi_{n,\lambda}\right\| \nu^{\varepsilon}(|n|) : (n,\lambda) \in J \times Y\right\},$$

it becomes a complete metric space. Given $\Phi \in \mathcal{X}$ and $\lambda \in Y$, for each $n \in J$, we consider the vector space

$$E_n^{\lambda} = \text{graph}(\text{Id}_{En} + \Phi_{n,\lambda}) = \left\{(\xi, \Phi_{n,\lambda}\xi) : \xi \in E_n\right\}.$$

Moreover, for each $m, n \in J$, we set

$$\mathcal{A}_{\lambda}(m,n) = \begin{cases} C_{m-1}...C_n, & \text{if } m > n, \\ \text{Id}, & \text{if } m = n, \\ C_m^{-1}...C_{n-1}^{-1}, & \text{if } m < n, \end{cases}$$

where $C_k = A_k + B_k(\lambda)$ for each $k \in J$.

Now we state the main result of this article.

Theorem 3.1. *Assume that the sequence $(A_m)_{m\in J}$ admits a nonuniform (μ, ν) dichotomy, and for each $m \in J, B_m : Y \to \mathcal{B}(X)$ are C^1 functions satisfying*

$$\left\|B_m(\lambda)\right\| \le \delta \nu^{-\beta}(m+1) \quad \text{and} \quad \left\|B_m'(\lambda)\right\| \le \delta \nu^{-\beta}(m+1). \tag{3.1}$$

Suppose further that

$$\vartheta = \sum_{n=1}^{\infty} \left(\frac{\mu(n)}{\mu(n+1)}\right)^{-\alpha} \nu^{3\varepsilon-\beta}(n+1) < \infty. \tag{3.2}$$

Then for δ sufficiently small there exists a unique $\Phi \in \mathcal{X}$ such that

$$E_m^{\lambda} = \mathcal{A}_{\lambda}(m,n)E_n^{\lambda} \tag{3.3}$$

for each $m, n \in J$. Moreover,

(1) *for each $n \in J$, $m \ge n$ and $\lambda \in Y$ we have*

$$\left\|\mathcal{A}_{\lambda}(m,n)\left|E_n^{\lambda}\right.\right\| \le D'\left(\frac{\mu(m)}{\mu(n)}\right)^{\alpha} \nu^{\varepsilon}(n) \tag{3.4}$$

for some constant $D' > 0$;

(2) *The map $\lambda \mapsto \Phi_{n,\lambda}$ is of class C^1 for each $n \in J$.*

Proof. Given $n \in J$ and $(\xi, \eta) \in E_n \times F_n$, the vector

$$(x_m, y_m) = \mathcal{A}_{\lambda}(m,n)(\xi,\eta) \in E_m \times F_m$$

satisfies

$$x_m = \mathcal{A}(m,n)\xi + \sum_{l=n}^{m-1} P_m \mathcal{A}(m,l+1)B_l(\lambda)(x_l, y_l) \tag{3.5}$$

and

$$y_m = \mathcal{A}(m,n)\eta + \sum_{l=n}^{m-1} Q_m \mathcal{A}(m,l+1)B_l(\lambda)(x_l, y_l) \tag{3.6}$$

for each $m \geq n$.

Due to the required invariance in (3.3), given $(x_n, y_n) \in E_n^\lambda$ we must have $y_m = \Phi_{m,\lambda} x_m$ for each m, and thus Equations (3.5)-(3.6) are equivalent to

$$x_m = \mathcal{A}(m,n)x_n + \sum_{l=n}^{m-1} P_m \mathcal{A}(m, l+1)B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda})x_l \tag{3.7}$$

and

$$\Phi_{m,\lambda}x_m = \mathcal{A}(m,n)\Phi_{n,\lambda}x_n + \sum_{l=n}^{m-1} Q_m \mathcal{A}(m, l+1)B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda})x_l \tag{3.8}$$

for each $m \geq n$.

Now we introduce linear operators related to (3.7). Given $\Phi \in \mathcal{X}$, $n \in J$ and $\lambda \in Y$, we consider the linear operators $W_{m,\lambda}^n = W_{m,\Phi,\lambda}^n : E_n \to E_m$ determined recursively by the identities

$$W_{m,\lambda}^n = P_m \mathcal{A}(m,n) + \sum_{l=n}^{m-1} P_m \mathcal{A}(m, l+1)B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda})W_{l,\lambda}^n \tag{3.9}$$

for $m > n$, setting $W_{n,\lambda}^n = \mathrm{Id}_{E_n}$. We note that for $x_n = \xi \in E_n$, the sequence

$$x_m = W_{m,\lambda}^n x_n = W_{m,\lambda}^n \xi \tag{3.10}$$

is the solution of Equation (3.5) with $y_l = \Phi_{l,\lambda}x_l$ for each $l \geq n$. Equivalently, it is a solution of Equation (3.7).

Using (3.10) we can rewrite (3.8) in the form

$$\Phi_{m,\lambda}W_{m,\lambda}^n = \mathcal{A}(m,n)\Phi_{n,\lambda} + \sum_{l=n}^{m-1} Q_m \mathcal{A}(m, l+1)B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda})W_{l,\lambda}^n \tag{3.11}$$

Lemma 3.2. *Given δ sufficiently small, for each $\Phi \in \mathcal{X}$ and $\lambda \in Y$, the following properties are equivalent:*

(1) (3.11) *holds for every $n \in J$ and $m \geq n$;*
(2) *for every $n \in J$ and $m \geq n$ we have*

$$\Phi_{n,\lambda} = -\sum_{l=n}^{\infty} Q_n \mathcal{A}(l+1,n)^{-1}B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda})W_{l,\lambda}^n \tag{3.12}$$

Proof of the lemma. We first show that the series in (3.12) are well defined. Using (2.2) and (3.1), we obtain

$$\begin{aligned}
&\sum_{l=n}^{\infty} \left\| Q_n \mathcal{A}(l+1,n)^{-1}B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda})W_{l,\lambda}^n \right\| \nu^{\varepsilon}(n) \\
&\leq \delta D(1+\kappa) \sum_{l=n}^{\infty} \left(\frac{\mu(l+1)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \left\| W_{l,\lambda}^n \right\| \nu^{\varepsilon}(n) \\
&\leq 2\delta D \sum_{l=n}^{\infty} \left(\frac{\mu(l+1)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \left\| W_{l,\lambda}^n \right\| \nu^{\varepsilon}(n).
\end{aligned} \tag{3.13}$$

By (3.9), for each $m \geq n$ we have

$$\begin{aligned} \left\| W^n_{m,\lambda} \right\| &\leq D\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \nu^{\varepsilon}(n) \\ &\quad + \delta D(1+\kappa) \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \left\| W^n_{l,\lambda} \right\| . \end{aligned} \tag{3.14}$$

Setting

$$\Upsilon = \sup_{m\geq n} \left\{ \left(\frac{\mu(m)}{\mu(n)}\right)^{\alpha} \left\| W^n_{m,\lambda} \right\| \right\}.$$

Then we have

$$\begin{aligned} \Upsilon &\leq D\nu^{\varepsilon}(n) + \delta D(1+\kappa)\Upsilon \sum_{l=n}^{m-1} \left(\frac{\mu(l)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \\ &\leq D\nu^{\varepsilon}(n) + 2\delta D\vartheta\Upsilon. \end{aligned} \tag{3.15}$$

Taking δ sufficiently small such that $2\delta\vartheta D < 1/2$ (independently of n) we obtain

$$\Upsilon \leq 2D\nu^{\varepsilon}(n),$$

and therefore,

$$\left\| W^n_{m,\lambda} \right\| \leq 2D\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \nu^{\varepsilon}(n). \tag{3.16}$$

Combined (3.13) and (3.16), we have

$$\begin{aligned} &\sum_{l=n}^{\infty} \left\| Q_n \mathcal{A}(l+1,n)^{-1} B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W^n_{l,\lambda} \right\| \nu^{\varepsilon}(n) \\ &\leq 4\delta D^2 \sum_{l=n}^{\infty} \left(\frac{\mu(l)\mu(l+1)}{\mu^2(n)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1)\nu^{2\varepsilon}(n) \\ &\leq 4\delta D^2 \sum_{l=n}^{\infty} \nu^{3\varepsilon-\beta}(l+1) \\ &\leq \kappa \end{aligned} \tag{3.17}$$

provided that δ sufficiently small.

Now we assume that identity (3.11) holds. It is equivalent to

$$\begin{aligned} \Phi_{n,\lambda} &= Q_n \mathcal{A}(m,n)^{-1} \Phi_{m,\lambda} W^n_{m,\lambda} \\ &\quad - \sum_{l=n}^{m-1} Q_n \mathcal{A}(l+1,n)^{-1} B_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W^n_{l,\lambda}. \end{aligned} \tag{3.18}$$

Using (3.16), for each $m \geq n$ we have

$$\left\| Q_n \mathcal{A}(m,n)^{-1} \Phi_{m,\lambda} W^n_{m,\lambda} \right\| \leq 2\kappa D^2 \left(\frac{\mu(m)}{\mu(n)}\right)^{-2\alpha} \nu^{\varepsilon}(n)$$

Since $\alpha > 0$, letting $m \to +\infty$ in (3.18) we obtain identity (3.12).

Conversely, let us assume that identity (3.12) holds. Then

$$\begin{aligned}&\mathcal{A}(m,n)\Phi_{n,\lambda}+\sum_{l=n}^{m-1}Q_m\mathcal{A}(m,l+1)B_l(\lambda)(\mathrm{Id}_{E_l}+\Phi_{l,\lambda})W_{l,\lambda}^n\\ &=-\sum_{l=n}^{\infty}Q_m\mathcal{A}(l+1,m)^{-1}B_l(\lambda)(\mathrm{Id}_{E_l}+\Phi_{l,\lambda})W_{l,\lambda}^n\\ &\quad+\sum_{l=n}^{m-1}Q_m\mathcal{A}(l+1,m)^{-1}B_l(\lambda)(\mathrm{Id}_{E_l}+\Phi_{l,\lambda})W_{l,\lambda}^n\\ &=-\sum_{l=m}^{\infty}Q_m\mathcal{A}(l+1,m)^{-1}B_l(\lambda)(\mathrm{Id}_{E_l}+\Phi_{l,\lambda})W_{l,\lambda}^n\end{aligned}$$

for each $m \geq n$. Since $W_{l,\lambda}^n = W_{l,\lambda}^m W_{m,\lambda}^n$, it follows from (3.12) with n replace by m that (3.11) holds for each $m \geq n$.

We define linear operators $A(\Phi)_{n,\lambda} : E_n \to F_n$ each $\Phi \in \mathcal{X}$, $n \in J$, and $\lambda \lfloor Y$ by

$$A(\Phi)_{n,\lambda}=-\sum_{l=n}^{\infty}Q_n\mathcal{A}(l+1,n)^{-1}B_l(\lambda)(\mathrm{Id}_{E_l}+\Phi_{l,\lambda})W_{l,\lambda}^n.$$

Lemma 3.3. *For δ sufficiently small, A is well defined and $A(\mathcal{X}) \subset \mathcal{X}$.*

Proof of the lemma. By (3.17) the operator A is well-defined and

$$\|A(\Phi)\| \leq \kappa$$

for δ sufficiently small. Furthermore, writing

$$W_{l,\lambda}^n = W_{l,\lambda}, \quad W_{l,\mu}^n = W_{l,\mu},$$

we have

$$\begin{aligned}b_l &:= \|B_l(\lambda)(\mathrm{Id}_{E_l}+\Phi_{l,\lambda})W_{l,\lambda}-B_l(\mu)(\mathrm{Id}_{E_l}+\Phi_{l,\mu})W_{l,\mu}\|\\ &\leq \|B_l(\lambda)-B_l(u)\|\cdot\|W_{l,\lambda}\|\cdot(1+\|\Phi_{l,\lambda}\|)\\ &\quad+\|B_l(\mu)\|\cdot\|W_{l,\lambda}-W_{l,\mu}\|\cdot(1+\|\Phi_{l,\lambda}\|)+\|B_l(\mu)\|\,\|W_{l,\mu}\|\,\|\Phi_{l,\lambda}-\Phi_{l,\mu}\|\\ &\leq 2\delta D(1+\kappa)\,\|\lambda-\mu\|\,\nu^{-\beta}(l+1)\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon}(n)\\ &\quad+\delta(1+\kappa)\nu^{-\beta}(l+1)\,\|W_{l,\lambda}-W_{l,\mu}\|\\ &\quad+2\delta D\kappa\nu^{-\beta}(l+1)\,\|\lambda-\mu\|\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon}(n)\nu^{-\varepsilon}(l)\\ &\leq 6\delta D\,\|\lambda-\mu\|\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha}\nu^{-\beta}(l+1)\nu^{\varepsilon}(n)+2\delta\nu^{-\beta}(l+1)\,\|W_{l,\lambda}-W_{l,\mu}\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|W_{m,\lambda}-W_{m,\mu}\| &\leq \sum_{l=n}^{m-1}\|P_m\mathcal{A}(m,l+1)\|\cdot b_l\\ &\leq 6\delta D^2\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon}(n)\,\|\lambda-\mu\|\sum_{l=n}^{m-1}\left(\frac{\mu(l)}{\mu(l+1)}\right)^{-\alpha}\nu^{\varepsilon-\beta}(l+1)\\ &\quad+2\delta D\sum_{l=n}^{m-1}\left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha}\nu^{\varepsilon-\beta}(l+1)\,\|W_{l,\lambda}-W_{l,\mu}\|\\ &\leq 6\delta D^2\vartheta\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon}(n)\,\|\lambda-\mu\|\\ &\quad+2\delta D\sum_{l=n}^{m-1}\left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha}\nu^{\varepsilon-\beta}(l+1)\,\|W_{l,\lambda}-W_{l,\mu}\|.\end{aligned}$$

Setting

$$\Upsilon_l = \left(\frac{\mu(l)}{\mu(n)}\right)^{\alpha} \left\| W_{l,\lambda} - W_{l,\mu} \right\| .$$

Then we have from the above inequality that

$$\Upsilon_m \leq 6\delta D^2 \vartheta \nu^{\varepsilon}(n) \|\lambda - \mu\| + 2\delta D \sum_{l=n}^{m-1} \Upsilon_l \left(\frac{\mu(l)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l).$$

Setting $\Upsilon = \sup\{\Upsilon_m : m \geq n\}$, we obtain

$$\Upsilon \leq 6\delta D^2 \vartheta \nu^{\varepsilon}(n) \|\lambda - \mu\| + 2\delta D \vartheta \Upsilon.$$

Taking δ sufficiently small such that $2\delta D\vartheta < 1/2$, we obtain

$$\Upsilon \leq 12\delta D^2 \vartheta \nu^{\varepsilon}(n) \|\lambda - \mu\| ,$$

and therefore,

$$\left\| W_{m,\lambda} - W_{m,\mu} \right\| \leq 12\delta D^2 \vartheta \left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \nu^{\varepsilon}(n) \|\lambda - \mu\| . \tag{3.19}$$

Therefore, it follows form (3.19) that

$$\begin{aligned} b_l &\leq 6\delta D \|\lambda - \mu\| \left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha} \nu^{-\beta}(l+1)\nu^{\varepsilon}(n) \\ &+ 2\delta\nu^{-\beta}(l+1) \cdot 12\delta D^2 \vartheta \left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha} \nu^{\varepsilon}(n) \|\lambda - \mu\| \\ &= K\delta \|\lambda - \mu\| \left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha} \nu^{-\beta}(l+1)\nu^{\varepsilon}(n) \end{aligned}$$

where $K = 6D + 24\delta D^2\vartheta > 0$.
Therefore, we obtain

$$\begin{aligned} &\left\| A(\Phi)_{n,\lambda} - A(\Phi)_{n,\mu} \right\| \nu^{\varepsilon}(n) \\ &\leq \sum_{l=n}^{\infty} \left\| Q_n \mathcal{A}(l+1,n)^{-1} \right\| b_l \nu^{\varepsilon}(n) \\ &\leq \delta KD \|\lambda - \mu\| \sum_{l=n}^{\infty} \left(\frac{\mu(l)\mu(l+1)}{\mu^2(n)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1)\nu^{2\varepsilon}(n) \\ &\leq \delta KD \|\lambda - \mu\| \sum_{l=n}^{\infty} \nu^{3\varepsilon-\beta}(l+1) \\ &\leq \delta KD\vartheta \|\lambda - \mu\| \end{aligned}$$

and provided that δ is sufficiently small, we obtain $C_{\lambda\mu}(A(\Phi)) \leq \kappa\|\lambda - \mu\|$. This shows that $A(\mathcal{X}) \subset \mathcal{X}$.

Now we note $\mathcal{F}$ be the space of sequences $U = (U_{n,\lambda})_{n\in \mathbb{N},\lambda\in Y}$ of linear operators $U_{n,\lambda} : E_n \to F_n$ indexed by Y such that $\lambda \mapsto U_{n,\lambda}$ is continuous for each $n \in J$, and

$$\|U\| = \sup_{(n,\lambda)\in J\times Y} \left\{ \left\| U_{n,\lambda} \right\| \right\} \leq 1 \tag{3.20}$$

Equipping $\mathcal{F}$ with this norm, it becomes a complete metric space, For each $(\Phi, U) \in \mathcal{X} \times \mathcal{F}$, $n \in J$, and $\lambda \in Y$, we also define linear operators $B(\Phi, U)_{n,\lambda}$ by

$$B(\Phi, U)_{n,\lambda} = -\sum_{l=n}^{\infty} Q_n \mathcal{A}(l+1, n)^{-1} G_{l,\lambda}, \tag{3.21}$$

where

$$G_{l,\lambda} = B_l(\lambda)(Z_{l,\lambda} + \Phi_{l,\lambda} Z_{l,\lambda} + U_{l,\lambda} W^n_{l,\lambda}) + B'_l(\lambda)(\mathrm{Id}_{E_l} + \Phi_{l,\lambda}) W_{l,\lambda} \tag{3.22}$$

and the linear operators $Z_{m,\lambda} = Z_{m,\Phi,U,\lambda} : E_n \to E_m$ are determined recursively by the identities

$$Z_{m,\lambda} = \sum_{l=n}^{m-1} P_m \mathcal{A}(m, l+1) G_{l,\lambda} \tag{3.23}$$

for $m > n$, setting $Z_{n,\lambda} = 0$. One observe that by the continuity of the functions $\Phi_{l,\lambda}$ and $U_{l,\lambda}$ on λ the functions $\lambda \mapsto W_{l,\lambda}$ and $\lambda \mapsto Z_{l,\lambda}$ are also continuous.

Lemma 3.4. *For δ sufficiently small, the operator B is well defined, and $B(\mathcal{X} \times \mathcal{F}) \subset \mathcal{F}$.*

Proof of the lemma. By (3.16) and (3.20) we have

$$\begin{aligned}
\|Z_{m,\lambda}\| &\leq \sum_{l=n}^{m-1} \|P_m \mathcal{A}(m, l+1)\| \left[\|B_l(\lambda)\| \left(1 + \|\Phi_{l,\lambda}\|\right) \|Z_{l,\lambda}\| \right. \\
&\quad \left. + \left(\|B_l(\lambda)\| \|U_{l,\lambda}\| + \|B'_l(\lambda)\| \left(1 + \|\Phi_{l,\lambda}\|\right)\right) \|W^n_{l,\lambda}\|\right] \\
&\leq 2\delta D \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \|Z_{l,\lambda}\| \\
&\quad + 6\delta D^2 \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha} \nu^{\varepsilon}(n) \\
&\leq 2\delta D \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \|Z_{l,\lambda}\| + 6\delta D^2 \vartheta \left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha}
\end{aligned}$$

Setting $\Upsilon_m = \left(\frac{\mu(m)}{\mu(n)}\right)^{\alpha} \|Z_{m,\lambda}\|$, we obtain

$$\begin{aligned}
\Upsilon_m &\leq 2\delta D \sum_{l=n}^{m-1} \left(\frac{\mu(l+1)}{\mu(n)}\right)^{\alpha} \nu^{\varepsilon-\beta}(l+1) \|Z_{l,\lambda}\| + 6\delta D^2 \vartheta \\
&\leq 2\delta D \vartheta \Upsilon_l + 6\delta D^2 \vartheta
\end{aligned}$$

and setting $\Upsilon = \sup\{\Upsilon_m : m \geq n\}$,

$$\Upsilon \leq 2\delta D \vartheta \Upsilon + 6\delta D^2 \vartheta.$$

Thus, taking δ sufficiently small such that $2\delta D \vartheta < \frac{1}{2}$, we have

$$\Upsilon \leq 12\delta D^2 \vartheta,$$

and therefore

$$\|Z_{m,\lambda}\| \leq 12\delta D^2 \vartheta \left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \tag{3.24}$$

Setting

$$G = \sum_{l=n}^{\infty} \left\| Q_n \mathcal{A}(l+1,n)^{-1} \right\| \left\| G_{l,\lambda} \right\| , \tag{3.25}$$

it follows from (3.16) and (3.20) that

$$\begin{aligned} G &\le D \sum_{l=n}^{\infty} \left(\frac{\mu(l+1)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon}(l+1). \\ &\left\{ 2\delta \nu^{-\beta}(l+1) \left\| Z_{l,\lambda} \right\| + 6\delta D \nu^{-\beta}(l+1) \left(\frac{\mu(l)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon}(n) \right\} \\ &\le D \sum_{l=n}^{\infty} \left(\frac{\mu(l+1)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1). \\ &\left\{ 24\delta^2 D^2 \vartheta \left(\frac{\mu(l)}{\mu(n)} \right)^{-\alpha} + 6\delta D \left(\frac{\mu(l)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon}(n) \right\} \\ &\le 24\delta^2 D^3 \vartheta \sum_{l=n}^{\infty} \left(\frac{\mu(l+1)\mu(l)}{\mu^2(n)} \right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \\ &+ 6\delta D^2 \sum_{l=n}^{\infty} \left(\frac{\mu(l+1)\mu(l)}{\mu^2(n)} \right)^{-\alpha} \nu^{2\varepsilon-\beta}(l+1) \\ &\le 24\delta^2 D^3 \vartheta^3 + 6\delta D^2 \vartheta \le 1, \end{aligned} \tag{3.26}$$

provided that δ is sufficiently small. This shows that B is well defined for each n, and that $\|B(\Phi, U)\| \le 1$. Therefore, $B(\mathcal{X} \times \mathcal{F}) \subset \mathcal{F}$.

Now we define another map $S : \mathcal{X} \times \mathcal{F} \to \mathcal{X} \times \mathcal{F}$ by

$$S(\Phi, U) = (A(\Phi), B(\Phi, U)).$$

By Lemmas 3.3 and 3.4, it is clearly that the maps S is well defined and $S(\mathcal{X} \times \mathcal{F}) \subset \mathcal{X} \times \mathcal{F}$.

Lemma 3.5. *For δ sufficiently small, the operator S is a contraction.*

Proof of the lemma. Given Φ, $\Psi \in \mathcal{X}$, and set $W_{l,\Phi} = W^n_{l,\Phi,\lambda}, W_{l,\Psi} = W^n_{l,\Psi,\lambda}$, we obtain

$$\begin{aligned} &\left\| A(\Phi)_{n,\lambda} - A(\Psi)_{n,\lambda} \right\| \nu^{\varepsilon}(n) \\ &\le D \sum_{l=n}^{\infty} \left(\frac{\mu(l+1)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon}(l+1) \cdot \delta \nu^{-\beta}(l+1). \\ &\left\{ \left\| W_{l,\Phi} - W_{l,\Psi} \right\| + \left\| \Phi_{l,\lambda} W_{l,\Phi} - \Psi_{l,\lambda} W_{l,\Psi} \right\| \right\} \nu^{\varepsilon}(n) \\ &\le \delta D \sum_{l=n}^{\infty} \left(\frac{\mu(l+1)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1). \\ &\left\{ 2 \left\| W_{l,\Phi} - W_{l,\Psi} \right\| + \|\Phi - \Psi\| \left\| W_{l,\Psi} \right\| \nu^{-\varepsilon}(l) \right\} \nu^{\varepsilon}(n) \\ &\le 2\delta D \vartheta \left\| W_{l,\Phi} - W_{l,\Psi} \right\| \nu^{\varepsilon}(n) + 2\delta D^2 \vartheta \|\Phi - \Psi\| \left(\frac{\mu(l)}{\mu(n)} \right)^{-\alpha} \nu^{-\varepsilon}(l) \nu^{2\varepsilon}(n). \end{aligned} \tag{3.27}$$

By (3.16) we obtain

$$\begin{aligned}
&\left\| W_{m,\Phi} - W_{m,\Psi} \right\| \\
&\leq \delta D \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)} \right)^{-\alpha} \nu^{\varepsilon}(l+1) \cdot \nu^{-\beta}(l+1). \\
&\quad \left\{ 2 \left\| W_{l,\Phi} - W_{l,\Psi} \right\| + \| \Phi - \Psi \| \left\| W_{l,\Psi} \right\| \nu^{-\varepsilon}(l) \right\} \\
&\leq 2\delta D \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)} \right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \left\| W_{l,\Phi} - W_{l,\Psi} \right\| \\
&\quad + 2\delta D^2 \| \Phi - \Psi \| \left(\frac{\mu(m)}{\mu(n)} \right)^{-\alpha} \sum_{l=n}^{m-1} \left(\frac{\mu(l)}{\mu(l+1)} \right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1) \nu^{-\varepsilon}(l) \nu^{\varepsilon}(n) \\
&\leq 2\delta D \vartheta \left(\frac{\mu(m)}{\mu(n)} \right)^{-\alpha} \left(\frac{\mu(l)}{\mu(n)} \right)^{\alpha} \left\| W_{l,\Phi} - W_{l,\Psi} \right\| + 2\delta D^2 \vartheta \| \Phi - \Psi \| \left(\frac{\mu(m)}{\mu(n)} \right)^{-\alpha}.
\end{aligned}$$

Setting $\Upsilon_m = \left(\frac{\mu(m)}{\mu(n)} \right)^{\alpha} \left\| W_{m,\Phi} - W_{m,\Psi} \right\|$, then

$$\Upsilon_m \leq 2\delta D \vartheta \Upsilon_l + 2\delta D^2 \vartheta \| \Phi - \Psi \|$$

and setting $\Upsilon = \sup\{\Upsilon_m : m \geq n\}$,

$$\Upsilon \leq 2\delta D \vartheta \Upsilon + 2\delta D^2 \Upsilon \| \Phi - \Psi \| .$$

Thus, taking δ sufficiently small so that $2\delta D \vartheta < \frac{1}{2}$ we have

$$\Upsilon \leq 4\delta D^2 \vartheta \| \Phi - \Psi \|$$

and therefore,

$$\left\| W_{m,\Phi} - W_{m,\Psi} \right\| \leq 4\delta D^2 \vartheta \| \Phi - \Psi \| \left(\frac{\mu(m)}{\mu(n)} \right)^{-\alpha} \tag{3.28}$$

Using (3.16) and (3.28) in (3.27) we obtain

$$\begin{aligned}
&\left\| A(\Phi)_{n,\lambda} - A(\Psi)_{n,\lambda} \right\| \\
&\leq 2\delta D \vartheta \cdot 4\delta D^2 \vartheta \| \Phi - \Psi \| \left(\frac{\mu(l)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon}(n) + 2\delta D^2 \vartheta \| \Phi - \Psi \| \left(\frac{\mu(l)}{\mu(n)} \right)^{-\alpha-\varepsilon} \nu^{\varepsilon}(n) \\
&\leq \delta K' \| \Phi - \Psi \| \nu^{\varepsilon}(n)
\end{aligned} \tag{3.29}$$

for $K' = 8\delta D^3 \vartheta^2 + 2D^2 \vartheta > 0$, provided that $\delta K' \nu^{\varepsilon}(n) \leq 1$. This shows that A is a contraction.

Nextly, also given $\Phi, \Psi \in \mathcal{X}, U, V \in \mathcal{F} and\ \lambda \in Y$, set $Z_{l,\Phi,U} = Z_{l,\Phi,U,\lambda}$ and $Z_{l,\Psi,U} = Z_{l,\Psi,U,\lambda}$, we obtain

$$\begin{aligned}
&\left\| B(\Phi, U)_{n,\lambda} - B(\Phi, V)_{n,\lambda} \right\| \\
&\leq \delta D \sum_{l=n}^{\infty} \left(\frac{\mu(l+1)}{\mu(n)} \right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1). \\
&\quad \left\{ 2 \left\| Z_{l,\Phi,U} - Z_{l,\Psi,V} \right\| + \left\| \Phi_{l,\lambda} - \Psi_{l,\lambda} \right\| \left(\left\| Z_{l,\Phi,U} \right\| + \left\| W_{l,\Phi} \right\| \right) \right. \\
&\quad \left. + \left\| U_{l,\lambda} - V_{l,\lambda} \right\| \left\| W_{l,\Phi} \right\| + \left\| W_{l,\Phi} - W_{l,\Psi} \right\| \left(1 + \left\| V_{l,\lambda} \right\| + \left\| \Psi_{l,\lambda} \right\| \right) \right\} .
\end{aligned} \tag{3.30}$$

By (3.16), (3.26), and (3.30)

$$
\begin{aligned}
&\left\|Z_{m,\Phi,U} - Z_{m,\Psi,V}\right\| \\
&\leq \delta D \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1). \\
&\left\{2\left\|Z_{l,\Phi,U} - Z_{l,\Psi,V}\right\| + \left\|\Phi_{l,\lambda} - \Psi_{l,\lambda}\right\|\left(\left\|Z_{l,\Phi,U}\right\| + \left\|W_{l,\Phi}\right\|\right)\right. \\
&\left. + \left\|U_{l,\lambda} - V_{l,\lambda}\right\|\left\|W_{l,\Phi}\right\| + \left\|W_{l,\Phi} - W_{l,\Psi}\right\|\left(1 + \left\|V_{l,\lambda}\right\| + \left\|\Psi_{l,\lambda}\right\|\right)\right\} \\
&\leq 2\delta D \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1)\left\|Z_{l,\Phi,U} - Z_{l,\Psi,V}\right\| \\
&+ \left(12\delta^2 D^3\vartheta + 2\delta D^2 + 12\delta^2 D^3\vartheta\right)\|\Phi - \Psi\|. \\
&\sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha}\left(\frac{\mu(l)}{\mu(m)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1)\nu^{\varepsilon}(n) \\
&+ 2\delta D^2\|U - V\| \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha}\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1)\nu^{\varepsilon}(n) \qquad (3.31) \\
&\leq 2\delta D \sum_{l=n}^{m-1} \left(\frac{\mu(m)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1)\left\|Z_{l,\Phi,U} - Z_{l,\Psi,V}\right\| \\
&+ \left(12\delta^2 D^3\vartheta + 2\delta D^2 + 12\delta^2 D^3\vartheta\right)\|\Phi - \Psi\|\,\vartheta\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \\
&+ 2\delta D^2\|U - V\|\,\vartheta\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \\
&\leq 2\delta D\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \sum_{l=n}^{m-1}\left(\frac{\mu(n)}{\mu(l+1)}\right)^{-\alpha} \nu^{\varepsilon-\beta}(l+1)\left\|Z_{l,\Phi,U} - Z_{l,\Psi,V}\right\| \\
&+ \delta K_0\left(\|\Phi - \Psi\| + \|U - V\|\right)\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha}
\end{aligned}
$$

for some positive constant $K_0 = 12\delta D^3\vartheta^2 + 2D^2\vartheta + 12\delta D^3\vartheta^2 > 0$, provided that $\delta \leq 1$. Setting $\Upsilon_m = \left(\frac{\mu(m)}{\mu(n)}\right)^{\alpha}\left\|Z^n_{m,\Phi,U} - Z^n_{m,\Psi,V}\right\|$, we obtain

$$\Upsilon_m \leq 2\delta D\vartheta\Upsilon_l + \delta K_0\left(\|\Phi - \Psi\| + \|U - V\|\right)$$

and setting $\Upsilon = \sup\{\Upsilon_m : m \geq n\}$ we obtain

$$\Upsilon \leq 2\delta D\vartheta\Upsilon + \delta K_0\left(\|\Phi - \Psi\| + \|U - V\|\right)$$

Taking δ sufficiently small so that $2\delta D\vartheta < \frac{1}{2}$ we obtain

$$\Upsilon \leq 2\delta K_0\left(\|\Phi - \Psi\| + \|U - V\|\right)$$

and therefore,

$$\left\|Z_{m,\Phi,U} - Z_{m,\Psi,V}\right\| \leq 2\delta K_0\left(\|\Phi - \Psi\| + \|U - V\|\right)\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha}. \qquad (3.32)$$

Using (3.32) in (3.30) we obtain

$$\begin{aligned}
&\left\|B(\Phi,U)_{n,\lambda}-B(\Phi,V)_{n,\lambda}\right\| \\
&\le \delta D\sum_{l=n}^{\infty}\left(\frac{\mu(l+1)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon-\beta}(l+1)\cdot\Bigg[4\delta K_0\left(\|\Phi-\Psi\|+\|U-V\|\right)\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \\
&\quad+\|\Phi-\Psi\|\left(12\delta D^2\vartheta\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha}+2D\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon}(n)\right) \\
&\quad+2D\,\|U-V\|\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon}(n)+12\delta D^2\vartheta\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha}\|\Phi-\Psi\|\Bigg] \\
&\le 4\delta^2DK_0\left(\|\Phi-\Psi\|+\|U-V\|\right)\sum_{l=n}^{\infty}\left(\frac{\mu(l+1)\mu(m)}{\mu^2(n)}\right)^{-\alpha}\nu^{\varepsilon-\beta}(l+1) \\
&\quad+\delta D\sum_{l=n}^{\infty}\left(\frac{\mu(l+1)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon-\beta}(l+1)\cdot\Big[\left(12\delta^2D^3\vartheta+2\delta D^2+12\delta^2D^3\vartheta\right) \\
&\quad\|\Phi-\Psi\|\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon}(n)+2\delta D^2\,\|U-V\|\left(\frac{\mu(l)}{\mu(n)}\right)^{-\alpha}\nu^{\varepsilon}(n)\Big] \\
&\le \delta L\left(\|\Phi-\Psi\|+\|U-V\|\right)
\end{aligned}$$

for some positive constant $L = 4\delta DK_0\vartheta + \delta DK_0 > 0$, provided that $\delta L \le 1$. It follows from (3.29) and the above inequality that for δ sufficiently small the operator S is a contraction.

Now we proceed with the proof of Theorem 3.1. By Lemma 3.5 and its proof, there exists a unique pair $(\overline{\Phi},\overline{U}) \in \mathcal{X}\times\mathcal{F}$ such that $S(\overline{\Phi},\overline{U}) = (\overline{\Phi},\overline{U})$ and $\overline{\Phi}$ is the unique sequence in $\mathcal{X}$ such that $A(\overline{\Phi})_{n,\lambda} = \overline{\Phi}_{n,\lambda}$ for each $n \in J$, $\lambda \in Y$. Namely, $\overline{\Phi}$ is the unique solution of Equation (3.12) as well as Equation (3.11). Together with (3.9) this implies that if $\xi \in E_n$, then

$$m \mapsto \left(W^n_{m,\lambda}\xi, \overline{\Phi}_{m,\lambda}\left(W^n_{m,\lambda}\xi\right)\right)$$

is a solution of (3.7) and (3.8). This means (3.3) holds.

Let Φ be another sequence for (3.3). If $\xi \in E_n$, then

$$(\xi, \Phi_{n,\lambda}\xi) \in E^{\lambda}_n \quad\text{and}\quad \mathcal{A}_\lambda(m,n)(\xi,\Phi_{n,\lambda}\xi) \in E^{\lambda}_m$$

Thus, if (x_m, y_m) is the solution of Equation (1.1) with $x_n = \xi$ and $y_n = \Phi_{n,\lambda}\xi$, then $y_m = \Phi_{m,\lambda}x_m$ for $m \ge n$. This means (3.7) and (3.8) hold. Furthermore, the sequence $x_m = W^n_{m,\lambda}\xi$ satisfies (3.9) and (3.11) holds. So $\Phi = \overline{\Phi}$.

Let $(x_n, y_n) \in E^{\lambda}_n$, then for each $m \ge n$ we have

$$(x_m, y_m) = \mathcal{A}_\lambda(m,n)(x_n, y_n)$$

and

$$x_m = W_{m,\lambda}x_n \quad\text{and}\quad y_m = \Phi_{m,\lambda}x_m.$$

Therefore

$$(x_m, y_m) = (\mathrm{Id}_{E_m} + \Phi_{m,\lambda})W_{m,\lambda}x_n.$$

By (3.16)

$$\left\|(x_m, y_m)\right\| \leq 4D\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \nu^{\varepsilon}(n)\, \|x_n\| .$$

On the other hand,

$$\left\|(x_n, y_n)\right\| = \left\|x_n + y_n\right\| \geq \left\|x_n - y_n\right\| = \|x_n\| - \left\|\Phi_{n,\lambda} x_n\right\| \geq (1-\kappa)\, \|x_n\| .$$

Thus

$$\left\|(x_m, y_m)\right\| \leq \frac{4D}{1-\kappa}\left(\frac{\mu(m)}{\mu(n)}\right)^{-\alpha} \nu^{\varepsilon}(n)\, \left\|x_n, y_n\right\| ,$$

which implies that (3.4) holds with $D' = \dfrac{4D}{1-\kappa} > 0$.

For the C^1 regularity of the maps $\lambda \mapsto \Phi_{n,\lambda}$ we consider the pair $(\Phi^1, U^1) = (0, 0) \in \mathcal{X} \times \mathcal{F}$. Clearly,

$$U^1_{n,\lambda} = \frac{d}{d\lambda}\Phi^1_{n,\lambda}$$

for each $n \in J$ and $\lambda \in Y$. We define a sequence $(\Phi^m, U^m) \in \mathcal{X} \times \mathcal{F}$ by

$$(\Phi^{m+1}, U^{m+1}) = S(\Phi^m, U^m) = (A(\Phi^m), B(\Phi^m, U^m))$$

For a given $m \in J$, if $\lambda \mapsto \Phi^m_{n,\lambda}$ is of class C^1 for each $n \in J$, and $U^m_{n,\lambda} = \dfrac{d}{d\lambda}\Phi^m_{m,\lambda}$ for every $n \in J$ and $\lambda \in Y$, then the linear operators $W_{m,\lambda}$ and $Z_{m,\lambda}$ satisfy $Z_{m,\lambda} = \dfrac{d}{d\lambda}W_{m,\lambda}$ for $m \geq n$ and $\lambda \in Y$. Therefore we can apply Leibniz's rule to conclude that $\lambda \mapsto \Phi^{m+1}_{n,\lambda}$ is of class C^1 for every $n \in J$, with

$$\begin{aligned} U^{m+1}_{n,\lambda} &= B(\Phi^m, U^m)_{n,\lambda} \\ &= -\sum_{l=n}^{\infty} \frac{\partial}{\partial\lambda}\left[Q_n\mathcal{A}(l+1, n)^{-1}B_l(\lambda)(W_{l,\lambda} + \Phi^m_{l,\lambda}W_{l,\lambda})\right] \\ &= \frac{d}{d\lambda}A(\Phi^m)_{n,\lambda} \\ &= \frac{d}{d\lambda}\Phi^{m+1}_{n,\lambda} \end{aligned} \tag{3.33}$$

for each $n \in J$ and $\lambda \in Y$.

Moveover, if $(\overline{\Phi}, \overline{U})$ is the unique fixed point for the contraction map S. then the sequence $\left(\Phi^m_{n,\lambda}\right)_{m\in\mathbb{N}}$ converge uniformly to $\overline{\Phi}_{n,\lambda}$ and the sequence $\left(U^m_{n,\lambda}\right)_{m\in\mathbb{N}}$ converge uniformly to $\overline{U}_{n,\lambda}$ for each $n \in J$ and $\lambda \in Y$.

We know that if a sequence f_m of C^1 functions converges uniformly, and its derivatives f'_m also converges uniformly, then the limit of f_m is of class C^1, and its derivative is the limit of f'_m. Therefore, by (3.33) each function $\lambda \mapsto \overline{\Phi}_{n,\lambda}$ is of class C^1, and

$$\frac{d}{d\lambda}\overline{\Phi}_{n,\lambda} = \overline{U}_{n,\lambda}$$

for each $n \in J$ and $\lambda \in Y$. This completes the proof of Theorem 3.1.

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The author declares that they have no competing interests.

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