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Computation of eigenvalues of discontinuous dirac system using Hermite interpolation technique

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Abstract

We use the derivative sampling theorem (Hermite interpolations) to compute eigenvalues of a discontinuous regular Dirac systems with transmission conditions at the point of discontinuity numerically. We closely follow the analysis derived by Levitan and Sargsjan (1975) to establish the needed relations. We use recently derived estimates for the truncation and amplitude errors to compute error bounds. Numerical examples, illustrations and comparisons with the sinc methods are exhibited.

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1 Introduction

Let $\sigma > 0$ and PW_σ^2 be the Paley-Wiener space of all $L^2(\mathbb{R})$-entire functions of exponential type type σ. Assume that $f(t) \in \mathrm{PW}_\sigma^2 \subset \mathrm{PW}_{2\sigma}^2$. Then $f(t)$ can be reconstructed via the sampling series

$$f(t) = \sum_{n=-\infty}^{\infty} \left[f\left(\frac{n\pi}{\sigma}\right) S_n^2(t) + f'\left(\frac{n\pi}{\sigma}\right) \frac{\sin(\sigma t - n\pi)}{\sigma} S_n(t) \right], \tag{1}$$

where $S_n(t)$ is the sequences of sinc functions

$$S_n(t) := \begin{cases} \dfrac{\sin(\sigma t - n\pi)}{(\sigma t - n\pi)}, & t \neq \dfrac{n\pi}{\sigma}, \\ 1, & t = \dfrac{n\pi}{\sigma}. \end{cases} \tag{2}$$

Series (1) converges absolutely and uniformly on $\mathbb{R}$ (cf. [1-4]). Sometimes, series (1) is called the derivative sampling theorem. Our task is to use formula (1) to compute eigenvalues of Dirac systems numerically. This approach is a fully new technique that uses the recently obtained estimates for the truncation and amplitude errors associated with (1) (cf. [5]). Both types of errors normally appear in numerical techniques that use interpolation procedures. In the following we summarize these estimates. The truncation error associated with (1) is defined to be

$$R_N(f)(t) := f(t) - f_N(t), \quad N \in \mathbb{Z}^+, \quad t \in \mathbb{R}, \tag{3}$$

where $f_N(t)$ is the truncated series

$$f_N(t) = \sum_{|n| \le N} \left[f\left(\frac{n\pi}{\sigma}\right) S_n^2(t) + f'\left(\frac{n\pi}{\sigma}\right) \frac{\sin(\sigma t - n\pi)}{\sigma} S_n(t) \right]. \tag{4}$$

It is proved in [5] that if $f(t) \in \mathrm{PW}_\sigma^2$ and $f(t)$ is sufficiently smooth in the sense that there exists $k \in \mathbb{Z}^+$ such that $t^k f(t) \in L^2(\mathbb{R})$, then, for $t \in \mathbb{R}$, $|t| < N\pi/\sigma$, we have

$$\begin{aligned}
|R_N(f)(t)| \le T_{N,k,\sigma}(t) &:= \frac{\xi_{k,\sigma} E_k |\sin \sigma t|^2}{\sqrt{3}(N+1)^k} \left(\frac{1}{(N\pi - \sigma t)^{3/2}} + \frac{1}{(N\pi + \sigma t)^{3/2}} \right) \\
&+ \frac{\xi_{k,\sigma}(\sigma E_k + k E_{k-1})|\sin \sigma t|^2}{\sigma(N+1)^k} \left(\frac{1}{\sqrt{N\pi - \sigma t}} + \frac{1}{\sqrt{N\pi + \sigma t}} \right),
\end{aligned} \tag{5}$$

where the constants E_k and $\xi_{k,\sigma}$ are given by

$$E_k := \sqrt{\int_{-\infty}^{\infty} |t^k f(t)|^2 dt}, \quad \xi_{k,\sigma} := \frac{\sigma^{k+1/2}}{\pi^{k+1}\sqrt{1 - 4^{-k}}}. \tag{6}$$

The amplitude error occurs when approximate samples are used instead of the exact ones, which we can not compute. It is defined to be

$$\begin{aligned}
\mathcal{A}(\varepsilon, f)(t) = \sum_{n=-\infty}^{\infty} \Bigg[&\left\{ f\left(\frac{n\pi}{\sigma}\right) - \tilde{f}\left(\frac{n\pi}{\sigma}\right) \right\} S_n^2(t) \\
&+ \left\{ f'\left(\frac{n\pi}{\sigma}\right) - \tilde{f}'\left(\frac{n\pi}{\sigma}\right) \right\} \frac{\sin(\sigma t - n\pi)}{\sigma} S_n(t) \Bigg], \quad t \in \mathbb{R},
\end{aligned} \tag{7}$$

where $\tilde{f}\left(\frac{n\pi}{\sigma}\right)$ and $\tilde{f} - \left(\frac{n\pi}{\sigma}\right)$ are approximate samples of $f\left(\frac{n\pi}{\sigma}\right)$ and $f'\left(\frac{n\pi}{\sigma}\right)$, respectively. Let us assume that the differences $\varepsilon_n := f\left(\frac{n\pi}{\sigma}\right) - \tilde{f}\left(\frac{n\pi}{\sigma}\right)$, $\varepsilon_n' := f'\left(\frac{n\pi}{\sigma}\right) - \tilde{f}' - \left(\frac{n\pi}{\sigma}\right)$, $n \in \mathbb{Z}$, are bounded by a positive number ε, i.e. $|\varepsilon_n|, |\varepsilon_n'| \le \varepsilon$. If $f(t) \in \mathrm{PW}_\sigma^2$ satisfies the natural decay conditions

$$|\varepsilon_n| \le \left| f\left(\frac{n\pi}{\sigma}\right) \right|, \quad |\varepsilon'_n| \le \left| f'\left(\frac{n\pi}{\sigma}\right) \right|, \tag{8}$$

$$|f(t)| \le \frac{M_f}{|t|^{\ell+1}}, \quad t \in \mathbb{R} - \{0\}, \tag{9}$$

$0 < \lambda \le 1$, then for $0 < \varepsilon \le \min\left\{ \pi/\sigma, \ \sigma/\pi, \ 1/\sqrt{e} \right\}$, we have, [5],

$$\|\mathcal{A}(\varepsilon, f)\|_\infty \le \frac{4e^{1/4}}{\sigma(\ell+1)} \left\{ \sqrt{3}e(1 + \sigma) + ((\pi/\sigma)A + M_f)\rho(\varepsilon) + (\sigma + 2 + \log(2))M_f \right\} \varepsilon \log(1/\varepsilon), \tag{10}$$

where

$$A := \frac{3\sigma}{\pi} \left(|f(0)| + M_f \left(\frac{\sigma}{\pi}\right)^\ell \right), \quad \rho(\varepsilon) := \gamma + 10\log(1/\varepsilon), \tag{11}$$

and $\gamma := \lim_{n \to \infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n \right] \cong 0.577216$ is the Euler-Mascheroni constant.

The classical [6] sampling theorem of Whittaker, Kotel'nikov and Shannon (WKS) for $f \in PW_\sigma^2$ is the series representation

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\sigma}\right) S_n(t), \quad t \in \mathbb{R}, \tag{12}$$

where the convergence is absolute and uniform on $\mathbb{R}$ and it is uniform on compact sets of $\mathbb{C}$ (cf. [6-8]). Series (12), which is of Lagrange interpolation type, has been used to compute eigenvalues of second order eigenvalue problems (see e.g. [9-13]). The use of (12) in numerical analysis is known as the sinc-method established by Stenger (cf. [14-16]). In [11,12], the authors applied (12) and the regularized sinc method to compute eigenvalues of Dirac systems with a derivation of the error estimates as given by [17,18]. The regularized sinc method; a method which is based on (WKS) but applied to regularized functions. Hence avoiding any (multiple) integration and keeping the number of terms in the Cardinal series manageable. It has been demonstrated that the method is capable of delivering higher order estimates of the eigenvalues at a very low cost. The aim of this article is to investigate the possibilities of using Hermite interpolations rather than Lagrange interpolations, to compute the eigenvalues numerically. Notice that, due to Paley-Wiener's theorem [19]$f \in PW_\sigma^2$ if and only if there is $g(\cdot) \in L^2$ $(-\sigma, \sigma)$ such that

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} g(x) e^{ixt} dx. \tag{13}$$

Therefore $f'(t) \in PW_\sigma^2$, i.e, $f'(t)$ also has an expansion of the form (12). However, $f'(t)$ can be also obtained by term-by-term differentiation formula of (12)

$$f'(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\sigma}\right) S_n'(t), \tag{14}$$

see [[6], p. 52] for convergence. Thus the use of Hermite interpolations will not cost any additional computational efforts since the samples $f\left(\frac{n\pi}{\sigma}\right)$ will be used to compute both $f(t)$ and $f'(t)$ according to (12) and (14), respectively. We would like to mention that works in direction of computing eigenvalues with the new method, Hermite interpolation technique, are few (see e.g. [5]). Also articles in computing of eigenvalues with discontinuous are few (see [20-22]). However the computing of eigenvalues by Hermite interpolation technique which has discontinuity conditions, do not exist as for as we know. The next section contains some preliminary results. The method with error estimates are contained in Section three. The last section involves some illustrative examples.

2 The eigenvalue problem

In this section we closely follow the analysis derived by [23] to establish the needed relations (see also [24]). We consider the Dirac system

$$u_2'(x) - r_1(x)u_1(x) = \lambda u_1(x), \quad u_1'(x) + r_2(x)u_2(x) = -\lambda u_2(x), \quad x \in [-1, 0) \cup (0, 1], \tag{15}$$

$$U_1(u) := \sin\alpha u_1(-1) + \cos\alpha u_2(-1) = 0, \tag{16}$$

$$U_2(\boldsymbol{u}) := \sin \beta u_1(1) + \cos \beta u_2(1) = 0, \tag{17}$$

and transmission conditions

$$U_3(\boldsymbol{u}) := u_1(0^-) - \delta u_1(0^+) = 0, \tag{18}$$

$$U_4(\boldsymbol{u}) := u_2(0^-) - \delta u_2(0^+) = 0, \tag{19}$$

where $\lambda \in \mathbb{C}$; the real valued function $r_1(\cdot)$ and $r_2(\cdot)$ are continuous in $[-1, 0)$ and $(0, 1]$, and have finite limits $r_1(0^\pm) := \lim\limits_{x \to 0^\pm} r_1(x), r_2(0^\pm) := \lim\limits_{x \to 0^\pm} r_2(x); \delta \in \mathbb{R}, \alpha, \beta \in [0, \pi)$ and $\delta \neq 0$.

Let $\boldsymbol{H}$ be the Hilbert space

$$\boldsymbol{H} := \left\{ \boldsymbol{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \ u_1(x), \ u_2(x) \in L^2(-1, 0) \oplus L^2(0, 1) \right\}. \tag{20}$$

The inner product of $\boldsymbol{H}$ is defined by

$$\langle \boldsymbol{u}(\cdot), \boldsymbol{v}(\cdot) \rangle_H := \int_{-1}^{0} \boldsymbol{u}^\top(x) \bar{\boldsymbol{v}}(x) dx + \delta^2 \int_{0}^{1} \boldsymbol{u}^\top(x) \bar{\boldsymbol{v}}(x) dx, \tag{21}$$

where $\top$ denotes the matrix transpose,

$$\boldsymbol{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \ \boldsymbol{v}(x) = \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} \in \boldsymbol{H}, \quad u_i(\cdot), \ v_i(\cdot) \in L^2(-1, 1), \quad i = 1, 2.$$

Equation (15) can be written as

$$\ell(\boldsymbol{u}) := A\boldsymbol{u}'(x) - P(x)\boldsymbol{u}(x) = \lambda \boldsymbol{u}(x), \tag{22}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P(x) = \begin{pmatrix} r_1(x) & 0 \\ 0 & r_2(x) \end{pmatrix}, \quad \boldsymbol{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}. \tag{23}$$

For functions $u(x)$, which defined on $[-1, 0) \cup (0, 1]$ and has finite limit $u(\pm 0) := \lim\limits_{x \to \pm 0} u(x)$, by $u_{(1)}(x)$ and $u_{(2)}(x)$ we denote the functions

$$u_{(1)}(x) = \begin{cases} u(x), & x \in [-1, 0); \\ u(0^-), & x = 0 \end{cases}, \ u_{(2)}(x) = \begin{cases} u(x), & x \in (0, 1] \\ u(0^+), & x = 0; \end{cases} \tag{24}$$

which are defined on $\Gamma_1 := [-1, 0]$ and $\Gamma_2 := 0[1]$ respectively.

In the following lemma, we will prove that the eigenvalues of the problem (15)-(19) are real.

Lemma 2.1 *The eigenvalues of the problem* (15)-(19) *are real.*

Proof. Assume the contrary that λ_0 is a nonreal eigenvalue of problem (15)-(19). Let $\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$ be a corresponding (non-trivial) eigenfunction. By (15), we have, for $x \in [-1, 0) \cup (0, 1]$,

$$\frac{d}{dx}\{u_1(x)\bar{u}_2(x) - \bar{u}_1(x)u_2(x)\} = (\bar{\lambda}_0 - \lambda_0)\{|u_1(x)|^2 + |u_2(x)|^2\}.$$

Integrating the above equation through [−1, 0) and (0, 1], we obtain

$$(\bar{\lambda}_0 - \lambda_0)\left[\int_{-1}^{0}(|u_1(x)|^2 + |u_2(x)|^2)dx\right] = u_1(0^-)\bar{u}_2(0^-) - \bar{u}_1(0^-)u_2(0^-) \tag{25}$$

$$- [u_1(-1)\bar{u}_2(-1) - \bar{u}_1(-1)u_2(-1)],$$

$$(\bar{\lambda}_0 - \lambda_0)\left[\int_{0}^{1}(|u_1(x)|^2 + |u_2(x)|^2)dx\right] = u_1(1)\bar{u}_2(1) - \bar{u}_1(1)u_2(1) \tag{26}$$

$$- [u_1(0^+)\bar{u}_2(0^+) - \bar{u}_1(0^+)u_2(0^+)].$$

Then from (16), (17) and transmission conditions, we have respectively

$$u_1(-1)\bar{u}_2(-1) - \bar{u}_1(-1)u_2(-1) = 0,$$

$$u_1(1)\bar{u}_2(1) - \bar{u}_1(1)u_2(1) = 0$$

and

$$u_1(0^-)\bar{u}_2(0^-) - \bar{u}_1(0^-)u_2(0^-) = \delta^2[u_1(0^+)\bar{u}_2(0^+) - \bar{u}_1(0^+)u_2(0^+)].$$

Since $\lambda_0 \neq \bar{\lambda}_0$, it follows from the last three equations and (25), (26) that

$$\int_{-1}^{0}(|u_1(x)|^2 + |u_2(x)|^2)dx + \delta^2\int_{0}^{1}(|u_1(x)|^2 + |u_2(x)|^2)dx = 0. \tag{27}$$

Then $u_i(x) = 0$, $i = 1, 2$ and this is contradiction. Consequently, λ_0 must be real.

Lemma 2.2 *Let λ_1 and λ_2 be two different eigenvalues of the problem (15)-(19). Then the corresponding eigenfunctions $u(x, \lambda_1)$ and $v(x, \lambda_2)$ of this problem satisfy the following equality*

$$\int_{-1}^{0}u^{\top}(x, \lambda_1)v(x, \lambda_2)\,dx + \delta^2\int_{0}^{1}u^{\top}(x, \lambda_1)v(x, \lambda_2)\,dx = 0. \tag{28}$$

Proof. By (15) we obtain

$$\frac{d}{dx}\{u_1(x, \lambda_1)v_2(x, \lambda_2) - u_2(x, \lambda_2)v_1(x, \lambda_1)\} = (\lambda_2 - \lambda_1)\{u_1(x, \lambda_1)v_1(x, \lambda_2) + u_2(x, \lambda_1)v_2(x, \lambda_2)\}.$$

Integrating the above equation through [−1, 0) and (0, 1], and taking into account $u(x, \lambda_1)$ and $v(x, \lambda_2)$ satisfy (16)-(19), we obtain (28), where $\lambda_1 \neq \lambda_2$.

Now, we shall construct a special fundamental system of solutions of the Equation (15) for λ not being an eigenvalue. Let us consider the next initial value problem:

$$u_2'(x) - r_1(x)u_1(x) = \lambda u_1(x), \quad u_1'(x) + r_2(x)u_2(x) = -\lambda u_2(x), \quad x \in (-1, 0), \tag{29}$$

$$u_1(-1) = \cos\alpha, \quad u_2(-1) = -\sin\alpha. \tag{30}$$

By virtue of Theorem 1.1 in [23] this problem has a unique solution

$$u_2'(x) - r_1(x)u_1(x) = \lambda u_1(x), \quad u_1'(x) + r_2(x)u_2(x) = -\lambda u_2(x), \quad x \in (0, 1), \text{which is an}$$

entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [-1, 0]$. Similarly, employing the same method as in proof of Theorem 1.1 in [23], we see that the problem

$$u_2'(x) - r_1(x)u_1(x) = \lambda u_1(x), \quad u_1'(x) + r_2(x)u_2(x) = -\lambda u_2(x), \quad x \in (0, 1), \quad (31)$$

$$u_1(1) = \cos \beta, \quad u_2(1) = -\sin \beta. \quad (32)$$

has a unique solution $\boldsymbol{u} = \begin{pmatrix} \chi_{12}(x, \lambda) \\ \chi_{22}(x, \lambda) \end{pmatrix}$ which is an entire function of parameter λ for each fixed $x \in [0.1]$.

Now the functions $\phi_{i2}(x, \lambda)$ and $\chi_{i1}(x, \lambda)$ are defined in terms of $\phi_{i1}(x, \lambda)$ and $\chi_{i2}(x, \lambda)$, $i = 1, 2$, respectively, as follows: The initial-value problem,

$$u_2'(x) - r_1(x)u_1(x) = \lambda u_1(x), \quad u_1'(x) + r_2(x)u_2(x) = -\lambda u_2(x), \quad x \in (0, 1), \quad (33)$$

$$u_1(0) = \frac{1}{\delta}\phi_{11}(0, \lambda), \quad u_2(0) = \frac{1}{\delta}\phi_{21}(0, \lambda), \quad (34)$$

has unique solution $\boldsymbol{u} = \begin{pmatrix} \phi_{12}(x, \lambda) \\ \phi_{22}(x, \lambda) \end{pmatrix}$ for each $\lambda \in \mathbb{C}$.

Similarly, the following problem also has a unique solution $\boldsymbol{u} = \begin{pmatrix} \chi_{11}(x, \lambda) \\ \chi_{21}(x, \lambda) \end{pmatrix}$:

$$u_2'(x) - r_1(x)u_1(x) = \lambda u_1(x), \quad u_1'(x) + r_2(x)u_2(x) = -\lambda u_2(x), \quad x \in (-1, 0), \quad (35)$$

$$u_1(0) = \delta\chi_{12}(0, \lambda), \quad u_2(0) = \delta\chi_{22}(0, \lambda). \quad (36)$$

Let us construct two basic solutions of the equation (15) as

$$\boldsymbol{\phi}(\cdot, \lambda) = \begin{pmatrix} \phi_1(\cdot, \lambda) \\ \phi_2(\cdot, \lambda) \end{pmatrix}, \quad \boldsymbol{\mathcal{X}}(\cdot, \lambda) = \begin{pmatrix} \mathcal{X}_1(\cdot, \lambda) \\ \mathcal{X}_2(\cdot, \lambda) \end{pmatrix},$$

where

$$\phi_1(x, \lambda) = \begin{cases} \phi_{11}(x, \lambda), x \in [-1, 0) \\ \phi_{12}(x, \lambda), x \in (0, 1] \end{cases}, \quad \phi_2(x, \lambda) = \begin{cases} \phi_{21}(x, \lambda), x \in [-1, 0) \\ \phi_{22}(x, \lambda), x \in (0, 1] \end{cases}, \quad (37)$$

$$\chi_1(x, \lambda) = \begin{cases} \chi_{11}(x, \lambda), x \in [-1, 0) \\ \chi_{12}(x, \lambda), x \in (0, 1] \end{cases}, \quad \chi_2(x, \lambda) = \begin{cases} \chi_{21}(x, \lambda), x \in [-1, 0) \\ \chi_{22}(x, \lambda), x \in (0, 1] \end{cases}, \quad (38)$$

By virtue of Equations (34) and (36) these solutions satisfy both transmission conditions (18) and (19). These functions are entire in λ for all $x \in [-1, 0) \cup (0, 1]$.

Let $W(\phi, \chi)(\cdot, \lambda)$ denote the Wronskian of $\phi(\cdot, \lambda)$ and $\chi(\cdot, \lambda)$ defined in [[25], p. 194], i.e.,

$$W(\boldsymbol{\phi}, \boldsymbol{\chi})(\cdot, \lambda) := \begin{vmatrix} \phi_1(\cdot, \lambda) & \phi_2(\cdot, \lambda) \\ \chi_1(\cdot, \lambda) & \chi_2(\cdot, \lambda) \end{vmatrix}.$$

It is obvious that the Wronskian

$$\Omega_i(\lambda) := W(\boldsymbol{\phi}, \boldsymbol{\chi})(x, \lambda) = \phi_{1i}(x, \lambda)\chi_{2i}(x, \lambda) - \phi_{2i}(x, \lambda)\chi_{1i}(x, \lambda), \quad x \in \Gamma_i, \quad i = 1, 2 \quad (39)$$

are independent of $x \in \Gamma_i$ and are entire functions. Taking into account (34) and (36), a short calculation gives

$$\Omega_1(\lambda) = \delta^2 \Omega_2(\lambda),$$

for each $\lambda \in \mathbb{C}$.

Corollary 2.3 *The zeros of the functions $\Omega_1(\lambda)$ and $\Omega_2(\lambda)$ coincide.*

Then, we may introduce to the consideration the characteristic function $\Omega(\lambda)$ as

$$\Omega(\lambda) := \Omega_1(\lambda) = \delta^2 \Omega_2(\lambda). \tag{40}$$

In the following lemma, we show that all eigenvalues of the problem (15)-(19) are simple.

Lemma 2.4 *All eigenvalues of problem (15)-(19) are just zeros of the function $\Omega(\lambda)$. Moreover, every zero of $\Omega(\lambda)$ has multiplicity one.*

Proof. Since the functions $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ satisfy the boundary condition (16) and both transmission conditions (18) and (19), to find the eigenvalues of the (15)-(19) we have to insert the functions $\phi_1(x, \lambda)$ and $\phi_2(x, \lambda)$ in the boundary condition (17) and find the roots of this equation.

By (15) we obtain for $\lambda, \mu \in \mathbb{C}, \lambda \neq \mu$,

$$\frac{d}{dx}\{\phi_1(x, \lambda)\phi_2(x, \mu) - \phi_1(x, \mu)\phi_2(x, \lambda)\} = (\mu - \lambda)\{\phi_1(x, \lambda)\phi_1(x, \mu) + \phi_2(x, \lambda)\phi_2(x, \mu)\}.$$

Integrating the above equation through $[-1, 0)$ and $(0, 1]$, and taking into account the initial conditions (30), (34) and (36), we obtain

$$\phi_{12}(1, \lambda)\phi_{22}(1, \mu) - \phi_{12}(1, \mu)\phi_{22}(1, \lambda) =$$
$$(\mu - \lambda)\left(\int_{-1}^{0} (\phi_{11}(x, \lambda)\phi_{11}(x, \mu) + \phi_{21}(x, \lambda)\phi_{21}(x, \mu))\, dx \right. \tag{41}$$
$$\left. + \delta^2 \int_{0}^{1} (\phi_{12}(x, \lambda)\phi_{12}(x, \mu) + \phi_{22}(x, \lambda)\phi_{22}(x, \mu))\, dx \right).$$

Dividing both sides of (41) by $(\lambda - \mu)$ and by letting $\mu \to \lambda$, we arrive to the relation

$$\phi_{22}(1, \lambda)\frac{\partial \phi_{12}(1, \lambda)}{\partial \lambda} - \phi_{12}(1, \lambda)\frac{\partial \phi_{22}(1, \lambda)}{\partial \lambda} = -\left(\int_{-1}^{0} (|\phi_{11}(x, \lambda)|^2 + |\phi_{21}(x, \lambda)|^2)\, dx \right. \tag{42}$$
$$\left. + \delta^2 \int_{0}^{1} (|\phi_{12}(x, \lambda)|^2 + |\phi_{22}(x, \lambda)|^2)\, dx \right).$$

We show that equation

$$\Omega(\lambda) = -\delta^2 (\sin \beta \phi_{12}(1, \lambda) + \cos \beta \phi_{22}(1, \lambda)) = 0 \tag{43}$$

has only simple roots. Assume the converse, i.e., Equation (43) has a double root λ^*, say. Then the following two equations hold

$$\sin \beta \phi_{12}(1, \lambda^*) + \cos \beta \phi_{22}(1, \lambda^*) = 0, \tag{44}$$

$$\sin \beta \frac{\partial \phi_{12}(1, \lambda^*)}{\partial \lambda} + \cos \beta \frac{\partial \phi_{22}(1, \lambda^*)}{\partial \lambda} = 0. \tag{45}$$

The Equations (44) and (45) imply that

$$\phi_{22}(1, \lambda^*)\frac{\partial\phi_{12}(1, \lambda^*)}{\partial\lambda} - \phi_{12}(1, \lambda^*)\frac{\partial\phi_{22}(1, \lambda^*)}{\partial\lambda} \tag{46}$$

Combining (46) and (42), with $\lambda = \lambda^*$, we obtain

$$\int_{-1}^{0}(|\phi_{11}(x, , \lambda^*)|^2 + |\phi_{21}(x, , \lambda^*)|^2)\,dx + \delta^2\int_{0}^{1}(|\phi_{12}(x, , \lambda^*)|^2 + |\phi_{22}(x, , \lambda^*)|^2)\,dx = 0. \tag{47}$$

It follows that $\phi_1(x, \lambda^*) = \phi_2(x, \lambda^*) = 0$, which is impossible. This proves the lemma.

Here $\{\boldsymbol{\phi}(\cdot, \lambda_n)\}_{n=-\infty}^{\infty}$ will be a sequence of eigen-vector-functions of (15)-(19) corresponding to the eigenvalues $\{\lambda_n\}_{n=-\infty}^{\infty}$. Since $\boldsymbol{\chi}(\cdot, \lambda)$ satisfies (17)-(19), then the eigenvalues are also determined via

$$\sin\alpha\chi_{11}(-1, \lambda) + \cos\alpha\chi_{21}(-1, \lambda) = \Omega(\lambda). \tag{48}$$

Therefore $\{\boldsymbol{\chi}(\cdot, \lambda_n)\}_{n=-\infty}^{\infty}$ is another set of eigen-vector-functions which is related by $\{\boldsymbol{\phi}(\cdot, \lambda_n)\}_{n=-\infty}^{\infty}$ with

$$\boldsymbol{\chi}(x, \lambda_n) = c_n\boldsymbol{\phi}(x, \lambda_n), \quad x \in [-1, 0) \cup (0, 1], \quad n \in \mathbb{Z}, \tag{49}$$

Where $c_n \neq 0$ are non-zero constants, since all eigenvalues are simple. Since the eigenvalues are all real, we can take the eigen-vector-functions to be real valued.

Since $\phi(\cdot, \lambda)$ satisfies (16), then the eigenvalues of the problem (15)-(19) are the zeros of the function

$$\Omega(\lambda) = -\delta^2(\sin\beta\phi_{12}(1, \lambda) + \cos\beta\phi_{22}(1, \lambda)). \tag{50}$$

Notice that both $\phi(\cdot, \lambda)$ and $\Omega(\lambda)$ are entire functions of λ. Now we shall transform Equations (15), (30), (34) and (37) into the integral equations (see [25]),

$$\phi_{11}(x, \lambda) = \cos(\lambda(x+1) - \alpha) - \mathcal{S}_{-1,1}\phi_{11}(x, \lambda) - \tilde{\mathcal{S}}_{-1,2}\phi_{21} - (x, \lambda), \tag{51}$$

$$\phi_{21}(x, \lambda) = \sin(\lambda(x+1) - \alpha) + \tilde{\mathcal{S}}_{-1,1}\phi_{11}(x, \lambda) - \mathcal{S}_{-1,2}\phi_{21}(x, \lambda), \tag{52}$$

$$\phi_{12}(x, \lambda) = \frac{1}{\delta}\phi_{11}(0^-, \lambda)\cos(\lambda x) - \frac{1}{\delta}\phi_{21}(0^-, \lambda)\sin(\lambda x) - \mathcal{S}_{0,1}\phi_{12}(x, \lambda) - \tilde{\mathcal{S}}_{0,2}\phi_{22}(x, \lambda), \tag{53}$$

$$\phi_{22}(x, \lambda) = \frac{1}{\delta}\phi_{11}(0^-, \lambda)\sin(\lambda x) + \frac{1}{\delta}\phi_{21}(0^-, \lambda)\cos(\lambda x)^- + \tilde{\mathcal{S}}_{0,1}\phi_{12}(x, \lambda) - \mathcal{S}_{0,2}\phi_{22}(x, \lambda), \tag{54}$$

where $\mathcal{S}_{-1,i}$, $\tilde{\mathcal{S}}_{-1,i}$, $\mathcal{S}_{0,i}$ and $\tilde{\mathcal{S}}_{0,i}$, $i = 1, 2$, are the Volterra integral operators defined by

$$\mathcal{S}_{-1,i}\varphi(x, \lambda) := \int_{-1}^{x}\sin\lambda(x-t)r_i(t)\varphi(t, \lambda)dt, \quad \tilde{\mathcal{S}}_{-1,i}\varphi(x, \lambda) := \int_{-1}^{x}\cos\lambda(x-t)r_i(t)\varphi(t, \lambda)\,dt,$$

$$\mathcal{S}_{0,i}\varphi(x, \lambda) := \int_{0}^{x}\sin\lambda(x-t)r_i(t)\varphi(t, \lambda)dt, \quad \tilde{\mathcal{S}}_{0,i}\varphi(x, \lambda) := \int_{0}^{x}\cos\lambda(x-t)r_i(t)\varphi(t, \lambda)\,dt.$$

For convenience, we define the constants

$$c_1 := \int_{-1}^{0} [|r_1(t)| + |r_2(t)|] \, dt, \quad c_2 := c_1 \exp(c_1),$$

$$c_3 := \int_{0}^{1} [|r_1(t)| + |r_2(t)|] \, dt, \quad c_4 := c_2 + \frac{2}{|\delta|}(1 + c_2).$$

(55)

Define $h_{-1,i}(\cdot, \lambda)$ and $h_{0,i}(\cdot, \lambda)$, $i = 1, 2$, to be

$$\left.\begin{array}{l} h_{-1,1}(x, \lambda) := \mathcal{S}_{-1,1}\phi_{11}(x, \lambda) + \tilde{\mathcal{S}}_{-1,2}\phi_{21}(x, \lambda), \\ h_{-1,2}(x, \lambda) := \tilde{\mathcal{S}}_{-1,1}\phi_{11}(x, \lambda) + \mathcal{S}_{-1,2}\phi_{21}(x, \lambda), \end{array}\right\}$$

(56)

$$\left.\begin{array}{l} h_{0,1}(x, \lambda) := \mathcal{S}_{0,1}\phi_{12}(x, \lambda) + \tilde{\mathcal{S}}_{0,2}\phi_{22}(x, \lambda), \\ h_{0,2}(x, \lambda) := \tilde{\mathcal{S}}_{0,1}\phi_{12}(x, \lambda) - \mathcal{S}_{0,2}\phi_{22}(x, \lambda). \end{array}\right\}$$

(57)

Lemma 2.5 *The functions $h_{-1,1}(x, \lambda)$ and $h_{-1,2}(x, \lambda)$ are entire in λ for any fixed $x \in [-1, 0)$ and satisfy the growth condition*

$$|h_{-1,1}(x, \lambda)|, \, |h_{-1,2}(x, \lambda)| \leq 2c_2 e^{|\Im\lambda|(x+1)}, \quad \lambda \in \mathbb{C}.$$

(58)

Proof. Since $h_{-1,1}(x, \lambda) = \mathcal{S}_{-1,1}\phi_{11}(x, \lambda) + \tilde{\mathcal{S}}_{-1,2}\phi_{21}(x, \lambda)$, then from (51) and (52) we obtain $h_{-1,1}(x, \lambda) = \mathcal{S}_{-1,1}\cos(\lambda(x+1)-\alpha) + \tilde{\mathcal{S}}_{-1,2}\sin(\lambda(x+1)-\alpha) - \mathcal{S}_{-1,1}h_{-1,1}-(x, \lambda) + \tilde{\mathcal{S}}_{-1,2}h_{-1,2}(x, \lambda)$ Using the inequalities $|\sin z| \leq e^{|\Im z|}$ and $|\cos z| \leq e^{|\Im z|}$ for $z \in \mathbb{C}$, leads for $\lambda \in \mathbb{C}$ to

$$\begin{aligned} |h_{-1,1}(x, \lambda)| \leq & \, |\mathcal{S}_{-1,1}\cos(\lambda(x+1)-\alpha)| + |\tilde{\mathcal{S}}_{-1,2}\sin(\lambda(x-+1)-\alpha)| + |\mathcal{S}_{-1,1}h_{-1,1}(x, \lambda)| \\ & + |\tilde{\mathcal{S}}_{-1,2}h_{-1,2}(x, \lambda)| \\ \leq & \, e^{|\Im\lambda|(x+1)}\int_{-1}^{x}[|r_1(t)||h_{-1,1}(t, \lambda)| + |r_2(t)||h_{-1,2}(t, \lambda)|]\,e^{-|\Im\lambda|(t+1)}\,dt \\ & + 2e^{|\Im\lambda|(x+1)}\int_{-1}^{x}[|r_1(t)| + |r_2(t)|]\,dt \\ \leq & \, 2c_1 e^{|\Im\lambda|(x+1)} + e^{|\Im\lambda|(x+1)}\int_{-1}^{x}[|r_1(t)||h_{-1,1}(t, \lambda)| + |r_2(t)||h_{-1,2}(t, \lambda)|]\,e^{-|\Im\lambda|(t+1)}\,dt. \end{aligned}$$

The above inequality can be reduced to

$$e^{-|\Im\lambda|(x+1)}|h_{-1,1}(x, \lambda)| \leq 2c_1 + \int_{-1}^{x}[|r_1(t)||h_{-1,1}(t, \lambda)| + |r_2(t)||h_{-1,2}(t, \lambda)|]\,e^{-|\Im\lambda|(t+1)}\,dt. \quad (59)$$

Similarly, we can prove that

$$e^{-|\Im\lambda|(x+1)}|h_{-1,2}(x, \lambda)| \leq 2c_1 + \int_{-1}^{x}[|r_1(t)||h_{-1,1}(t, \lambda)| + |r_2(t)||h_{-1,2}(t, \lambda)|]\,e^{-|\Im\lambda|(t+1)}\,dt. \quad (60)$$

Then from (58) and (59) and and Lemma 3.1 of [[25], pp. 204], we obtain (58).

In a similar manner, we will prove the following lemma for $h_{0,1}(\cdot, \lambda)$ and $h_{0,2}(\cdot, \lambda)$.

Lemma 2.6 *The functions $h_{0,1}(x, \lambda)$ and $h_{0,2}(x, \lambda)$ are entire in λ for any fixed $x \in (0, 1]$ and satisfy the growth condition*

$$|h_{0,1}(x, \lambda)|, |h_{0,2}(x, \lambda)| \le 2c_3 c_4 e^{|\Im \lambda|(x+1)}, \quad \lambda \in \mathbb{C}. \tag{61}$$

Proof. Since $h_{0,1}(x, \lambda) = \mathcal{S}_{0,1}\phi_{11}(x, \lambda) + \tilde{\mathcal{S}}_{0,2}\phi_{21}(x, \lambda)$, then from (53) and (54) we obtain

$$h_{0,1}(x, \lambda) = \frac{1}{\delta}\phi_{11}(0^-, \lambda)\mathcal{S}_{0,1}\cos(\lambda x) - \frac{1}{\delta}\phi_{21}(0^-, \lambda)\tilde{\mathcal{S}}_{0,1}\sin(\lambda x) - \mathcal{S}_{0,1}h_{-1,2}(x, \lambda)$$
$$+ \frac{1}{\delta}\phi_{11}(0^-, \lambda)\tilde{\mathcal{S}}_{0,2}\sin(\lambda_X) + \frac{1}{\delta}\phi_{21}(0^-, \lambda)\tilde{\mathcal{S}}_{0,2}\cos(\lambda_X) + \tilde{\mathcal{S}}_{0,2}h_{-1,2}(x, \lambda).$$

Then from (51) and (52) and Lemma 2.5, we get

$$h_{0,1}(x, \lambda) \le \frac{1}{|\delta|}|\phi_{11}(0^-, \lambda)||\mathcal{S}_{0,1}\cos(\lambda x)| + \frac{1}{|\delta|}|\phi_{21}(0^-, \lambda)||\mathcal{S}_{0,1}\sin(\lambda x)|$$
$$+ |\mathcal{S}_{0,1}h_{-1,2}(x, \lambda)| + \frac{1}{|\delta|}|\phi_{11}(0^-, \lambda)||\tilde{\mathcal{S}}_{0,2}\sin(\lambda x)|$$
$$+ \frac{1}{|\delta|}|\phi_{21}(0^-, \lambda)||\tilde{\mathcal{S}}_{0,2}\cos(\lambda x)| - +|\tilde{\mathcal{S}}_{0,2}h_{-1,2}(x, \lambda)|$$
$$\le 2\left(c_2 + \frac{2}{|\delta|}(1 + c_2)\right)c_3 e^{|\Im \lambda|(x+1)} = 2c_3 c_4 e^{|\Im \lambda|(x+1)}.$$

Similarly, we can prove that

$$h_{0,2}(x, \lambda) \le 2c_3 c_4 e^{|\Im \lambda|(x+1)}.$$

3 The numerical scheme

In this section we derive the method of computing eigenvalues of problem (15)-(19) numerically. The basic idea of the scheme is to split $\Omega(\lambda)$ into two parts a known part $\mathcal{K}(\lambda)$ and an unknown one $\mathcal{U}(\lambda)$. Then we prove that $\mathcal{U}(\lambda)$ has an expansion of the form (1). We then approximate $\mathcal{U}(\lambda)$ in two stages. First by truncating the sampling expansion (4) and then by approximating the samples, using standard methods of solving ordinary differential equations. This produces both a truncation error and an amplitude error. We apply forms (4) and (7) to derive an estimate of the error of the technique. We first split $\Omega(\lambda)$ into two parts:

$$\Omega(\lambda) := \mathcal{K}(\lambda) + \mathcal{U}(\lambda), \tag{62}$$

where $\mathcal{U}(\lambda)$ is the unknown part involving integral operators

$$\mathcal{U}(\lambda) := \delta^2 \sin\beta h_{0,1}(1, \lambda) - \delta^2 \cos\beta h_{0,2}(1, \lambda) - \delta \sin(\lambda - \beta)h_{-1,1}(0^-, \lambda) - \delta \cos(\lambda + \beta)h_{-1,2}(0^-, \lambda), \tag{63}$$

and $\mathcal{K}(\lambda)$ is the known part

$$\mathcal{K}(\lambda) := -\delta \sin(2\lambda - \alpha + \beta). \tag{64}$$

Then, from Lemmas 2.5 and 2.6, we have the following result.

Lemma 3.1 *The function $\mathcal{U}(\lambda)$ is entire in λ and the following estimate holds*

$$|\mathcal{U}(\lambda)| \le M e^{2|\Im \lambda|}, \tag{65}$$

where

$$M := 4|\delta|(c_2 + |\delta|c_3 c_4). \tag{66}$$

Proof. From (63), we have

$$
\begin{aligned}
|\mathcal{U}(\lambda)| \leq\ & |\delta|^2 |\sin\beta| |h_{0,1}(1,\ \lambda)| + |\delta|^2 |\cos\beta| |h_{0,2}(1,\ \lambda)| \\
& + |\delta| |\sin(\lambda-\beta)| |h_{-1,1}(0^-,\ \lambda)| + |\delta| |\cos(\lambda+\beta)| |h_{-1,2}(0^-,\ \lambda)|.
\end{aligned}
$$

Using the inequalities $|\sin\lambda| \leq e^{|\Im\lambda|}$ and $|\cos\lambda| \leq e^{|\Im\lambda|}$ for $\lambda \in \mathbb{C}$, and Lemmas 2.5 and 2.6 imply (65).

Let $\theta \in (0,\ 1)$ and $m \in \mathbb{Z}^+$, $m > 1$ be fixed. Let $\mathcal{G}_{\theta,m}(\lambda)$ be the function

$$
\mathcal{G}_{\theta,m}(\lambda) := \left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^m \mathcal{U}(\lambda), \quad \lambda \in \mathbb{C}. \tag{67}
$$

Lemma 3.2 $\mathcal{G}_{\theta,m}(\lambda)$ *is an entire function of* λ *which satisfies the estimate*

$$
|\mathcal{G}_{\theta,m}(\lambda)| \leq \frac{c_0^m M}{(1+\theta|\lambda|)^m} e^{|\Im\lambda|(2+m\theta)}, \quad \lambda \in \mathbb{C}. \tag{68}
$$

Moreover, $\lambda^{m-1}\mathcal{G}_{\theta,m}(\lambda) \in L^2(\mathbb{R})$ *and*

$$
E_{m-1}(\mathcal{G}_{\theta,m}) = \sqrt{\int_{-\infty}^{\infty} |\lambda^{m-1}\mathcal{G}_{\theta m}(\lambda)|^2 d\lambda} \leq \sqrt{2}c_0^m M \xi_0, \tag{69}
$$

where

$$
\xi_0 := \sqrt{\frac{1}{\theta^{2m-1}(2m-1)}}.
$$

Proof. Since $\mathcal{U}(\lambda)$ is entire, then also $\mathcal{G}_{\theta,m}(\lambda)$ is entire in λ. Combining the estimates $\left|\dfrac{\sin z}{z}\right| \leq \dfrac{c_0}{1+|z|} e^{|\Im z|}$, where $c_0 \simeq 1.72$, cf. [26], and (65), we obtain

$$
\begin{aligned}
|\lambda^{m-1}\mathcal{G}_{\theta,m}(\lambda)| &\leq \left(\frac{c_0}{1+\theta|\lambda|}\right)^m e^{|\Im z|m\theta} \cdot M|\lambda|^{m-1} e^{2|\Im\lambda|} \\
&= \frac{c_0^m M |\lambda|^{m-1}}{(1+\theta|\lambda|)^m} e^{|\Im\lambda|(2+m\theta)}, \quad \lambda \in \mathbb{C}.
\end{aligned} \tag{70}
$$

Therefore if $\lambda \in \mathbb{R}$ we have

$$
|\lambda^{m-1}\mathcal{G}_{\theta,m}(\lambda)| \leq \frac{c_0^m M |\lambda|^{m-1}}{(1+\theta|\lambda|)^m}, \quad \lambda \in \mathbb{R}, \tag{71}
$$

i.e. $\lambda^{m-1}\mathcal{G}_{\theta,m}(\lambda) \in L^2(\mathbb{R})$. Moreover

$$
\int_{-\infty}^{\infty} |\lambda^{m-1}\mathcal{G}_{\theta,m}(\lambda)|^2 d\lambda \leq c_0^{2m} M^2 \int_{-\infty}^{\infty} \frac{|\lambda|^{2m-2}}{(1+\theta|\lambda|)^{2m}} d\lambda = 2c_0^{2m} M^2 \xi_0^2. \tag{72}
$$

A direct and important result of Lemma 51 is that $\mathcal{G}_{\theta,m}(\lambda)$ belongs to the Paley-Wienerz space PW_σ^2 with $\sigma = 1+m\theta$. Since $\mathcal{G}_{\theta,m}(\lambda) \in \mathrm{PW}_\sigma^2 \subset \mathrm{PW}_{2\sigma}^2$, then we can reconstruct the functions $\mathcal{G}_{\theta,m}(\lambda)$ via the following sampling formula

$$
\mathcal{G}_{\theta,m}(\lambda) = \sum_{n=-\infty}^{\infty} \left[\mathcal{G}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) S_n^2(\lambda) + \mathcal{G}'_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \frac{\sin(\sigma\lambda - n\pi)}{\sigma} S_n(\lambda) \right]. \tag{73}
$$

Let $N \in \mathbb{Z}^+$, $N > m$ and approximate $\mathcal{G}_{\theta,m}(\lambda)$ by its truncated series $\mathcal{G}_{\theta,m,N}(\lambda)$, where

$$\mathcal{G}_{\theta,m,N}(\lambda) := \sum_{n=-N}^{N} \left[\mathcal{G}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) S_n^2(\lambda) + \mathcal{G}'_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \frac{\sin(\sigma\lambda - n\pi)}{\sigma} S_n(\lambda) \right]. \tag{74}$$

Since all eigenvalues are real, then from now on we restrict ourselves to $\lambda \in \mathbb{R}$. Since $\lambda^{m-1} \mathcal{G}_{\theta,m}(\lambda) \in L^2(\mathbb{R})$, the truncation error, cf. (5), is given for

$$|\mathcal{G}_{\theta,m}(\lambda) - \mathcal{G}_{\theta,m,N}(\lambda)| \leq T_{N,m-1,\sigma}(\lambda), \tag{75}$$

where

$$\begin{aligned}
T_{N,m-1,\sigma}(\lambda) := {} & \frac{\xi_{m-1,\sigma} E_{m-1} |\sin\sigma\lambda|^2}{\sqrt{3}(N+1)^{m-1}} \left(\frac{1}{(N\pi - \sigma\lambda)^{3/2}} + \frac{1}{(N\pi + \sigma\lambda)^{3/2}} \right) \\
& + \frac{\xi_{m-1,\sigma}(\sigma E_{m-1} + (m-1)E_{m-2})|\sin\sigma\lambda|^2}{\sigma(N+1)^{m-1}} \left(\frac{1}{\sqrt{N\pi - \sigma\lambda}} + \frac{1}{\sqrt{N\pi + \sigma\lambda}} \right).
\end{aligned} \tag{76}$$

The samples $\left\{ \mathcal{G}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \right\}_{n=-N}^{N}$ and $\left\{ \mathcal{G}'_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \right\}_{n=-N}^{N}$, in general, are not known explicitly. So we approximate them by solving numerically $2N + 1$ initial value problems at the nodes $\left\{ \frac{n\pi}{\sigma} \right\}_{n=-N}^{N}$. Let $\left\{ \tilde{\mathcal{G}}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \right\}_{n=-N}^{N}$ and $\left\{ \tilde{\mathcal{G}}'_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \right\}_{n=-N}^{N}$ be the approximations of the samples of $\left\{ \mathcal{G}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \right\}_{n=-N}^{N}$ and $\left\{ \mathcal{G}'_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \right\}_{n=-N}^{N}$, respectively. Now we define $\tilde{\mathcal{G}}_{\theta,m,N}(\lambda)$, which approximates $\mathcal{G}_{\theta,m,N}(\lambda)$

$$\tilde{\mathcal{G}}_{\theta,m,N}(\lambda) := \sum_{n=-N}^{N} \left[\tilde{\mathcal{G}}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) S_n^2(\lambda) + \tilde{\mathcal{G}}'_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \frac{\sin(\sigma\lambda - n\pi)}{\sigma} S_n(\lambda) \right], \quad N > m. \tag{77}$$

Using standard methods for solving initial problems, we may assume that for $|n| < N$,

$$\left| \mathcal{G}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) - \tilde{\mathcal{G}}_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \right| < \varepsilon, \quad \left| \mathcal{G}'_{\theta,m}\left(\frac{n\pi}{\sigma}\right) - \tilde{\mathcal{G}}'_{\theta,m}\left(\frac{n\pi}{\sigma}\right) \right| < \varepsilon, \tag{78}$$

for a sufficiently small ε. From Lemma 3.2 we can see that $\mathcal{G}_{\theta,m}(\lambda)$ satisfies the condition (9) when $m > 1$ and therefore whenever $0 < \varepsilon \leq \min\{\pi/\sigma, \sigma/\pi, 1/\sqrt{e}\}$ we have

$$|\mathcal{G}_{\theta,m,N}(\lambda) - \tilde{\mathcal{G}}_{\theta,m,N}(\lambda)| \leq \mathcal{A}(\varepsilon), \quad \lambda \in \mathbb{R}, \tag{79}$$

where there is a positive constant $M_{\mathcal{G}_{\theta,m}}$ for which, cf. (10),

$$\mathcal{A}(\varepsilon) := \frac{2e^{1/4}}{\sigma} \left\{ \sqrt{3}e(1 + \sigma) + \left(\frac{\pi}{\sigma}A + M_{\mathcal{G}_{\theta,m}} \right) \rho(\varepsilon) + (\sigma + 2 + \log(2))M_{\mathcal{G}_{\theta,m}} \right\} \varepsilon \log(1/\varepsilon). \tag{80}$$

Here

$$A := \frac{3\sigma}{\pi} \left(|\mathcal{G}_{\theta,m}(0)| + \frac{\sigma}{\pi} M_{\mathcal{G}_{\theta,m}} \right), \quad \rho(\varepsilon) := \gamma + 10\log(1/\varepsilon), \quad \ell = 1.$$

In the following we use the technique of [27] to determine enclosure intervals for the eigenvalues. Let λ^* be an eigenvalue, that is

$$\Omega(\lambda^*) = \mathcal{K}(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*} \right)^{-m} \mathcal{G}_{\theta,m}(\lambda^*) = 0.$$

Then it follows that

$$\mathcal{K}(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\tilde{\mathcal{G}}_{\theta,m,N}(\lambda^*) = \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\tilde{\mathcal{G}}_{\theta,m,N}(\lambda^*) - \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\mathcal{G}_{\theta,m}(\lambda^*)$$

$$= \left[\left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\tilde{\mathcal{G}}_{\theta,m,N}(\lambda^*) - \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\mathcal{G}_{\theta,m,N}(\lambda^*)\right]$$

$$+ \left[\left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\mathcal{G}_{\theta,m,N}(\lambda^*) - \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\mathcal{G}_{\theta,m}(\lambda^*)\right]$$

and so

$$\left|\mathcal{K}(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\tilde{\mathcal{G}}_{\theta,m,N}(\lambda^*)\right| \leq \left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m}(T_{N,m-1,\sigma}(\lambda^*) + \mathcal{A}(\varepsilon)).$$

Since $\mathcal{K}(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\tilde{\mathcal{G}}_{\theta,m,N}(\lambda^*)$ is given and, $\left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m}(T_{N,m-1,\sigma}(\lambda^*) + \mathcal{A}(\varepsilon))$ has computable upper bound, we can define an enclosure for λ^*, by solving the following system of inequalities

$$-\left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m}(T_{N,m-1,\sigma}(\lambda^*) + \mathcal{A}(\varepsilon)) \leq \mathcal{K}(\lambda^*)$$

$$+ \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}\tilde{\mathcal{G}}_{\theta,m,N}(\lambda^*) \leq \left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m}(T_{N,m-1,\sigma}(\lambda^*) + \mathcal{A}(\varepsilon)). \tag{81}$$

Its solution is an interval containing λ^*, and over which the graph $\mathcal{K}(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m} \times \tilde{\mathcal{G}}_{\theta,m,N}(\lambda^*)$ is squeezed between the graphs

$$-\left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m}(T_{N,m-1,\sigma}(\lambda^*) + \mathcal{A}(\varepsilon)) \tag{82}$$

and

$$\left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m}(T_{N,m-1,\sigma}(\lambda^*) + \mathcal{A}(\varepsilon)). \tag{83}$$

Using the fact that

$$\tilde{\mathcal{G}}_{\theta,m,N}(\lambda) \to \mathcal{G}_{\theta,m}(\lambda)$$

uniformly over any compact set, and since λ^* is a simple root, we obtain for large N and sufficiently small ε

$$\frac{\partial}{\partial\lambda}\left(\mathcal{K}(\lambda) + \left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m}\tilde{\mathcal{G}}_{\theta,m,N}(\lambda)\right) \neq 0$$

in a neighborhood of λ^*. Hence the graph of $\mathcal{K}(\lambda) + \left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m}\tilde{\mathcal{G}}_{\theta,m,N}(\lambda)$ intersects the graphs $-\left|\frac{\sin\theta\lambda}{\theta\lambda}\right|^{-m}(T_{N,m-1,\sigma}(\lambda) + \mathcal{A}(\varepsilon))$ and $\left|\frac{\sin\theta\lambda}{\theta\lambda}\right|^{-m}(T_{N,m-1,\sigma}(\lambda) + \mathcal{A}(\varepsilon))$ at two points with abscissa $a_-(\lambda^*, N, \varepsilon) \leq a_+(\lambda^*, N, \varepsilon)$ the interval

$$I_{\varepsilon,N} := [a_-(\lambda^*, N, \varepsilon), a_+(\lambda^*, N, \varepsilon)]$$

and in particular $\lambda^* \in I_{\varepsilon,N}$. Summarizing the above discussion, we arrive at the following lemma which is similar to that of [27] for Sturm-Liouville problems.

Lemma 3.3 *For any eigenvalue λ^*, we can find $N_0 \in \mathbb{Z}^+$ and sufficiently small ε such that $\lambda^* \in I_{\varepsilon,N}$ for $N > N_0$. Moreover*

$$[a_-(\lambda^*, N, \varepsilon), a_+(\lambda^*, N, \varepsilon)] \rightarrow \{\lambda^*\} \text{ as } N \rightarrow \infty \text{ and } \varepsilon \rightarrow 0. \tag{84}$$

Proof. Since all eigenvalues of (15)-(19) are simple, then for large N and sufficiently small ε we have $\frac{\partial}{\partial \lambda} \left(\mathcal{K}(\lambda) + \left(\frac{\sin \theta \lambda}{\theta \lambda} \right)^{-m} \tilde{\mathcal{G}}_{\theta,m,N}(\lambda) \right) > 0$, in a neighborhood of λ^*.
Choose N_0 such that

$$\mathcal{K}(\lambda) + \left(\frac{\sin \theta \lambda}{\theta \lambda} \right)^{-m} \tilde{\mathcal{G}}_{\theta,m,N_0}(\lambda) = \pm \left| \frac{\sin \theta \lambda}{\theta \lambda} \right|^{-m} (T_{N_0,m-1,\sigma}(\lambda) + \mathcal{A}(\varepsilon))$$

has two distinct solutions which we denote by $a_-(\lambda^*,N_0,\varepsilon) \leq a_+(\lambda^*,N_0,\varepsilon)$. The decay of $T_{N,m-1,\sigma}(\lambda) \rightarrow 0$ as $N \rightarrow \infty$ and $\mathcal{A}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ will ensure the existence of the solutions $a_-(\lambda^*,N,\varepsilon)$ and $a_+(\lambda^*,N,\varepsilon)$ as $N \rightarrow \infty$ and $\varepsilon \rightarrow 0$. For the second point we recall that $\tilde{\mathcal{G}}_{\theta,m,N}(\lambda) \rightarrow \mathcal{G}_{\theta,m}(\lambda)$ as $N \rightarrow \infty$ and as $\varepsilon \rightarrow 0$. Hence by taking the limit we obtain

$$\mathcal{K}(a_+(\lambda^*, \infty, 0)) + \left(\frac{\sin \theta \lambda^*}{\theta \lambda^*} \right)^{-m} \mathcal{G}_{\theta,m}(a_+(\lambda^*, \infty, 0)) = 0,$$

$$\mathcal{K}(a_-(\lambda^*, \infty, 0)) + \left(\frac{\sin \theta \lambda^*}{\theta \lambda^*} \right)^{-m} \mathcal{G}_{\theta,m}(a_-(\lambda^*, \infty, 0)) = 0,$$

that is $\Omega(a_+) = \Omega(a_-) = 0$. This leads us to conclude that $a_+ = a_- = \lambda^*$, since λ^* is a simple root.

Let $\tilde{\Omega}_N(\lambda) := \mathcal{K}(\lambda) + \left(\frac{\sin \theta \lambda}{\theta \lambda} \right)^{-m} \tilde{\mathcal{G}}_{\theta,m,N}(\lambda)$. Then (75) and (79) imply

$$\left| \Omega(\lambda) - \tilde{\Omega}_N(\lambda) \right| \leq \left| \frac{\sin \theta \lambda}{\theta \lambda} \right|^{-m} (T_{N,m-1,\sigma}(\lambda) + \mathcal{A}(\varepsilon)), \quad |\lambda| < \frac{N\pi}{\sigma} \tag{85}$$

and θ is chosen sufficiently small for which $|\theta \lambda| < \pi$. Let λ^* be an eigenvalue and λ_N be its approximation. Thus $\Omega(\lambda^*) = 0$ and $\tilde{\Omega}_N(\lambda_N) = 0$. From (85) we have $|\tilde{\Omega}_N(\lambda^*)| \leq \left| \frac{\sin \theta \lambda^*}{\theta \lambda^*} \right|^{-m} (T_{N,m-1,\sigma}(\lambda^*) + \mathcal{A}(\varepsilon))$. Now we estimate the error $|\lambda^* - \lambda_N|$, for the eigenvalue λ^*.

Lemma 3.4 *Let λ^* be an eigenvalue of (15)-(19). For sufficient large N we have the following estimate*

$$|\lambda^* - \lambda_N| < \left| \frac{\sin \theta \lambda_N}{\theta \lambda_N} \right|^{-m} \frac{T_{N,m-1,\sigma'}(\lambda_N) + \mathcal{A}(\varepsilon)}{\inf_{\zeta \in I_{\varepsilon,N}} |\Omega'(\zeta)|}. \tag{86}$$

Proof. Since $\Omega(\lambda_N) - \tilde{\Omega}_N(\lambda_N) = \Omega(\lambda_N) - \Omega(\lambda^*)$, then from (85) and after replacing λ by λ_N we obtain

$$|\Omega(\lambda_N) - \Omega(\lambda^*)| \le \left| \frac{\sin \theta \lambda_N}{\theta \lambda_N} \right|^{-m} (T_{N,m-1,\sigma}(\lambda_N) + \mathcal{A}(\varepsilon)). \tag{87}$$

Using the mean value theorem yields that for some $\zeta \in J_{\varepsilon,N} := [\min(\lambda^*, \lambda_N), \max(\lambda^*, \lambda_N)]$,

$$|(\lambda^* - \lambda_N)\Omega'(\zeta)| \le \left| \frac{\sin \theta \lambda_N}{\theta \lambda_N} \right|^{-m} (T_{N,m-1,\sigma}(\lambda_N) + \mathcal{A}(\varepsilon)), \quad \zeta \in J_{\varepsilon,N} \subset I_{\varepsilon,N}. \tag{88}$$

Since the eigenvalues are simple, then for sufficiently large $N \inf_{\zeta \in I_{\varepsilon,N}} |\Omega'(\zeta)| > 0$ and we get (86).

4 Numerical examples

This section includes two detailed worked examples illustrating the above technique. By E_S and E_H we mean the absolute errors associated with the results of the classical sinc method and our new method (Hermite interpolations) respectively. All examples are computed in [22] with the classical sinc method. We indicate in these examples the effect of the amplitude error in the method by determining enclosure intervals for different values of ε. We also indicate the effect of the parameters m and θ by several choices. Every example is accompanied with six figures illustrating $\Omega(\lambda)$, $\tilde{\Omega}(\lambda)$ and the enclosure curves dominating the zeros. Recall that $a_{\pm}(\lambda)$ are defined by

$$a \pm (\lambda) = \Omega_N(\lambda) \pm \left| \frac{\sin \theta \lambda}{\theta \lambda} \right|^{-m} (T_{N,m-1,\sigma}(\lambda) + \mathcal{A}(\varepsilon)), \quad |\lambda| < \frac{N\pi}{\sigma}. \tag{89}$$

Recall also that the enclosure interval $I_{\varepsilon,N} := [a_-, a_+]$ is determined by solving

$$a_{\pm}(\lambda) = 0, |\lambda| < \frac{N\pi}{\sigma}. \tag{90}$$

Example 1 *Consider the system*

$$u_2'(x) - r(x)u_1(x) = \lambda u_1(x), \quad u_1'(x) + r(x)u_2(x) = -\lambda u_2(x), \quad x \in [-1, 0) \cup (0, 1], \tag{91}$$

$$u_1(-1) + u_2(-1) = 0, \quad u_1(1) + u_2(1) = 0, \tag{92}$$

$$u_1(0^-) - 2u_1(0^+) = 0, \quad u_2(0^-) - 2u_2(0^+) = 0. \tag{93}$$

Here

$$r_1(x) = r_2(x) = r(x) = \begin{cases} x^2, & x \in [-1, 0) \\ x, & (0, 1], \end{cases} \tag{94}$$

$\alpha = \beta = \dfrac{\pi}{4}$ *and* $\delta = 2$. *Direct calculations give*

$$\mathcal{K}(\lambda) = -2 \sin[2\lambda] \tag{95}$$

and

$$\Omega(\lambda) = -2 \sin\left[\frac{5}{6} + 2\lambda \right], \tag{96}$$

therefore the eigenvalues are $\lambda_k = \dfrac{6k\pi - 5}{12}, k \in \mathbb{Z}$. *The following four tables indicate the application of our technique to this problem and the effect of ε, θ and m (Tables 1, 2, 3and 4).*

In the following, the Figures 1and 2illustrate the comparison between $\Omega(\lambda)$ and $\tilde{\Omega}_N(\lambda)$ for different values of m and θ.

Figures 3 and 4, for N = 20, m = 6 and θ = 1/7, illustrate the enclosure intervals for $\varepsilon = 10^{-8}$ and $\varepsilon = 10^{-12}$ respectively.

Also, Figures 5 and 6*illustrate the enclosure intervals for $\varepsilon = 10^{-8}$ and $\varepsilon = 10^{-12}$ respectively, but for m = 10, θ =1/5.*

Table 1 N = 20, m = 6, and θ = 1/7

λ_k Sinc $\lambda_{k,N}$	Exact λ_k	Hermite $\lambda_{k,N}$	E_S	E_H
λ_{-2} −3.5582594249986099	−3.5582593202564599	−3.5582593202488253	1.047×10^{-7}	7.634×10^{-12}
λ_{-1} −1.9874629152275372	−1.9874629934615633	−1.9874629934659096	7.823×10^{-8}	4.346×10^{-12}
λ_0 −0.4166665411688605	−0.4166666666666667	−0.41666666665625623	1.255×10^{-7}	1.041×10^{-11}
λ_1 1.1541296373132959	1.1541296601282299	1.1541296601279385	2.281×10^{-8}	2.913×10^{-13}
λ_2 2.7249258102486835	2.7249259869231266	2.7249259869365776	1.767×10^{-7}	1.345×10^{-11}

Table 2 N = 20, m = 10 and θ = 1/5

λ_k Sinc $\lambda_{k,N}$	Exact λ_k	Hermite $\lambda_{k,N}$	E_S	E_H
λ_{-2} −3.558259320291224	−3.5582593202564599	−3.5582593202564596	3.476×10^{-11}	4.441×10^{-16}
λ_{-1} −1.9874629934481942	−1.9874629934615633	−1.9874629934615631	1.3369×10^{-11}	2.220×10^{-16}
λ_0 −0.416666666672866	−0.4166666666666667	−0.41666666666666635	6.199×10^{-12}	3.331×10^{-16}
λ_1 1.1541296601306603	1.1541296601282299	1.1541296601282297	2.431×10^{-12}	2.220×10^{-16}
λ_2 2.724925986925671	2.7249259869231266	2.7249259869231270	2.544×10^{-12}	4.440×10^{-16}

Table 3 The approximation $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval [a_-, a_+] for different values of ε, where N = 20, m = 6, θ = 1/7

| λ_k Exact λ_k	[a_-, a_+], $\varepsilon = 10^{-8}$	[a_-, a_+], $\varepsilon = 10^{-12}$	$\lambda_{k,N}$
λ_{-2} −3.5582593202564599	[−3.753326, −3.387861]	[−3.565657, −3.551497]	−3.5582593202488253
λ_{-1} −1.9874629934615633	[−2.134039, −1.835158]	[−1.991487, −1.983156]	−1.9874629934659096
λ_0 −0.4166666666666667	[−0.562608, −0.277479]	[−0.427095, −0.406713]	−0.41666666665625623
λ_1 1.1541296601282299	[1.017775, 1.295214]	[1.153811, 1.154454]	1.1541296601279385
λ_2 2.7249259869231266	[2.561929, 2.895812]	[2.711312, 2.738729]	2.7249259869365776

$E_5(\mathcal{G}_{\theta,m}) = 1.7659 \times 10^6, E_4(\mathcal{G}_{\theta,m}) = 278897, \ell = 1, M_{\mathcal{G}_{\theta,m}} = 4563.57$

Table 4 For N = 20, m = 10 and θ = 1/5, $\lambda_{k,N}$ and λ_k are all inside the interval [a_-, a_+] for different values of ε

λ_k Exact λ_k	[a_-, a_+], $\varepsilon = 10^{-8}$	[a_-, a_+], $\varepsilon = 10^{-12}$	$\lambda_{k,N}$
λ_{-2} −3.5582593202564599	[−3.739980, −3.404715]	[−3.560364, −3.556152]	−3.5582593202564596
λ_{-1} −1.9874629934615633	[−2.080163, −1.899068]	[−1.988611, −1.986313]	−1.9874629934615631
λ_0 −0.4166666666666667	[−0.487002, −0.346801]	[0.417553, −0.415779]	−0.41666666666666635
λ_1 1.1541296601282299	[1.079274, 1.230862]	[1.153170, 1.155090]	1.1541296601282297
λ_2 2.7249259869231266	[2.614866, 2.845166]	[2.723467, 2.726388]	2.7249259869231270

$E_9(\mathcal{G}_{\theta,m}) = 1.15498 \times 10^9, E_8(\mathcal{G}_{\theta,m}) = 2.44207 \times 10^8, \ell = 1, M_{\mathcal{G}_{\theta,m}} = 20378$
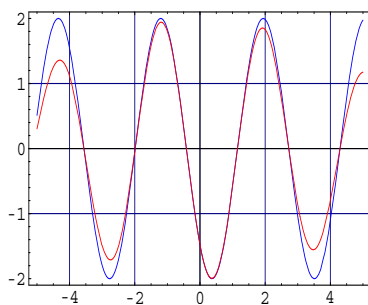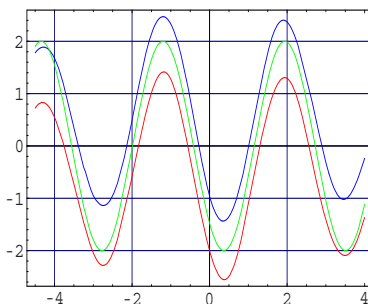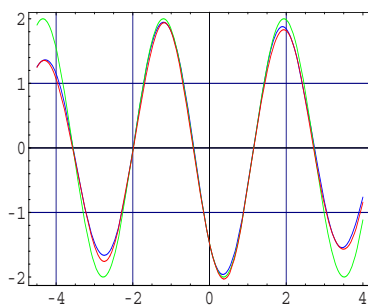
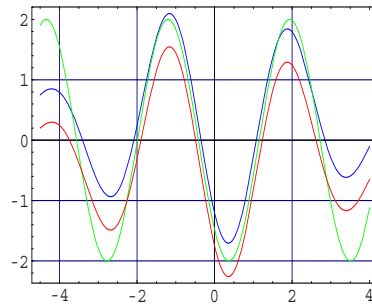**Figure 1** $\Omega(\lambda)$, $\tilde{\Omega}_N(\lambda)$ with $N = 20$, $m = 6$ and $\theta = 1/7$.
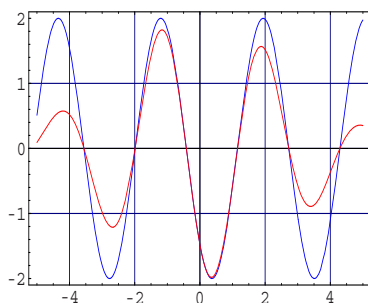
Figure 2 $\Omega(\lambda)$, $\tilde{\Omega}_N(\lambda)$ with $N = 20$, $m = 10$ and $\theta = 1/5$.
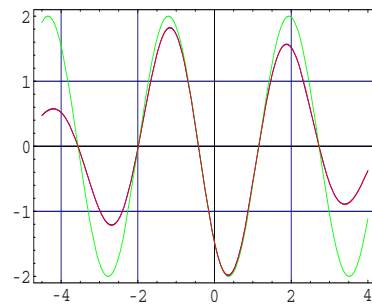

Figure 3 a_+, $\Omega(\lambda)$, a_- with $N = 20$, $m = 6$, $\theta = 1/7$ and $\varepsilon = 10^{-8}$.

Figure 4 a_+, $\Omega(\lambda)$, a_- with $N = 20$, $m = 6$, $\theta = 1/7$ and $\varepsilon = 10^{-12}$.

Figure 5 a_+, $\Omega(\lambda)$, a_- with $N = 20$, $m = 10$, $\theta = 1/5$ and $\varepsilon = 10^{-8}$.

Figure 6 a_+, $\Omega(\lambda)$, a_- with $N = 20$, $m = 10$, $\theta = 1/5$ and $\varepsilon = 10^{-12}$.

Example 2 *In this example we consider the system*

$$u_2'(x) - r(x)u_1(x) = \lambda u_1(x), \quad u_1'(x) + r(x)u_2(x) = -\lambda u_2(x), \quad x \in [-1, 0) \cup (0, 1], \quad (97)$$

$$\sqrt{3}u_1(-1) + u_2(-1) = 0, \quad u_1(1) + \sqrt{3}u_2(1) = 0, \quad (98)$$

$$u_1(0^-) - 3u_1(0^+) = 0, \quad u_2(0^-) - 3u_2(0^+) = 0, \quad (99)$$

where

$$r_1(x) = r_2(x) = r(x) = \begin{cases} x, & x \in [-1, 0) \\ x^2 + 1, & (0, 1], \end{cases} \quad (100)$$

$\alpha = \dfrac{\pi}{3}, \beta = \dfrac{\pi}{6}$ *and* $\delta = 3$. *Direct calculations give*

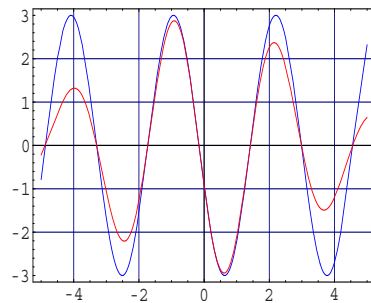$$\mathcal{K}(\lambda) = 3 \sin\left[\frac{\pi}{6} - 2\lambda\right] \quad (101)$$

and

$$\Omega(\lambda) = \frac{3}{2}\left(\cos\left[\frac{5}{6} + 2\lambda\right] - \sqrt{3}\sin\left[\frac{5}{6} + 2\lambda\right]\right), \quad (102)$$
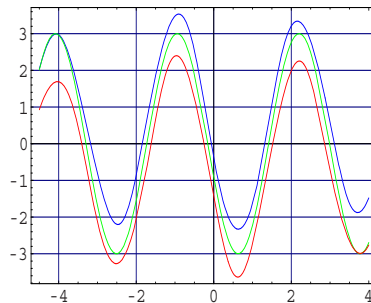
therefore the eigenvalues are $\lambda_k = \dfrac{(6k + 1)\pi - 5}{12}, k \in \mathbb{Z}$. *As in the previous example, Figures 7, 8, 9, 10, 11 and 12 illustrate the results of Tables 5, 6, 7 and 8. Also, Figures 5 and 6 illustrate the enclosure intervals for* $\varepsilon = 10^{-8}$ *and* $\varepsilon = 10^{-12}$ *respectively, but for* $m = 9$, $\theta = 2/11$.
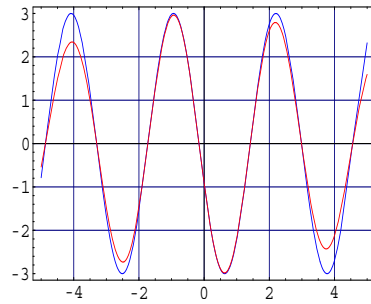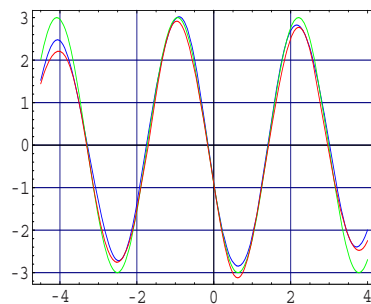
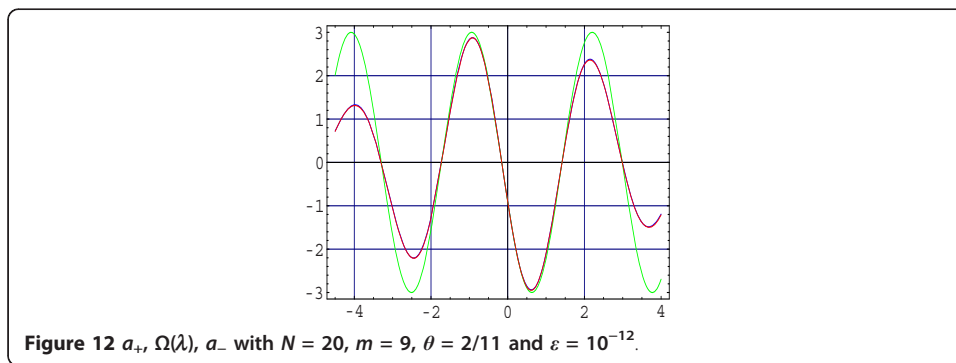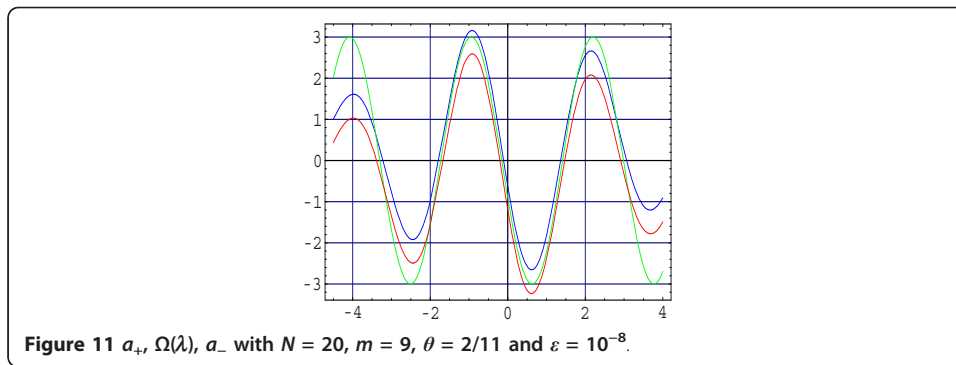**Figure 7** $\Omega(\lambda)$, $\tilde{\Omega}_N(\lambda)$ with $N = 20$, $m = 5$ and $\theta = 2/15$.

Figure 8 $\Omega(\lambda)$, $\tilde{\Omega}_N(\lambda)$ with $N = 20$, $m = 9$ and $\theta = 2/11$.

Figure 9 a_+, $\Omega(\lambda)$, a_- with $N = 20$, $m = 5$, $\theta = 2/15$ and $\varepsilon = 10^{-8}$.

Figure 10 a_+, $\Omega(\lambda)$, a_- with $N = 20$, $m = 5$, $\theta = 2/15$ and $\varepsilon = 10^{-12}$.

Figure 11 a_+, $\Omega(\lambda)$, a_- with $N = 20$, $m = 9$, $\theta = 2/11$ and $\varepsilon = 10^{-8}$.

Figure 12 a_+, $\Omega(\lambda)$, a_- with $N = 20$, $m = 9$, $\theta = 2/11$ and $\varepsilon = 10^{-12}$.

Table 5 $N = 20$, $m = 5$, $\theta = 2/15$

λ_k Sinc $\lambda_{k,N}$	Exact λ_k	Hermite $\lambda_{k,N}$	E_S	E_H
λ_{-2} −3.2964601812375145	−3.2964599324573105	−3.2964599325093795	2.488×10^{-7}	5.207×10^{-11}
λ_{-1} −1.725663818527078	−1.725663605662414	−1.7256636055362218	2.129×10^{-7}	1.262×10^{-10}
λ_0 −0.15486730070281368	−0.15486727886751722	−0.15486727888681906	2.183×10^{-8}	1.930×10^{-11}
λ_1 1.4159289890168527	1.4159290479273794	1.4159290479703164	5.891×10^{-8}	4.294×10^{-11}
λ_2 2.9867250912171013	2.986725374722276	2.9867253745880475	2.835×10^{-7}	1.342×10^{-10}

Table 6 $N = 20$, $m = 9$, $\theta = 2/11$

λ_k Sinc $\lambda_{k,N}$	Exact λ_k	Hermite $\lambda_{k,N}$	E_S	E_H
λ_{-2} −3.2964599325179624	−3.2964599324573105	−3.2964599324573127	6.065×10^{-11}	2.220×10^{-15}
λ_{-1} −1.7256636056618686	−1.725663605662414	−1.725663605662413	5.453×10^{-13}	8.881×10^{-16}
λ_0 −0.1548672788373793	−0.15486727886751722	−0.1548672788675186	3.013×10^{-11}	1.388×10^{-15}
λ_1 1.415929047873631	1.4159290479273794	1.4159290479273827	5.374×10^{-11}	3.331×10^{-15}
λ_2 2.986725374801227	2.986725374722276	2.9867253747222686	7.895×10^{-11}	7.549×10^{-15}

Table 7 For $N = 20$, $m = 5$ and $\theta = 2/15$, the approximation $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$ for different values of ε

λ_k Exact λ_k	$[a_-, a_+]$, $\varepsilon = 10^{-5}$	$[a_-, a_+]$, $\varepsilon = 10^{-10}$	$\lambda_{k,N}$
λ_{-2} −3.2964599324573105	[−3.402745, −3.177441]	[−3.305475, −3.286114]	−3.2964599325093795
λ_{-1} −1.725663605662414	[−1.840530, −1.615062]	[−1.749858, −1.702197]	−1.7256636055362218
λ_0 −0.15486727886751722	[−0.250393, −0.067671]	[−0.158786, −0.151287]	−0.15486727888681906
λ_1 1.4159290479273794	[1.323736, 1.520703]	[1.408023, 1.424863]	1.4159290479703164
λ_2 2.986725374722276	[2.863300, 3.109311]	[2.960260, 3.012514]	2.9867253745880475

$E_4(\mathcal{G}_{\theta,m}) = 319577$, $E_3(\mathcal{G}_{\theta,m}) = 48315.5$, $\ell = 1$, $M_{\mathcal{G}_{\theta,m}} = 4400.77$

Table 8 With $N = 20$, $m = 9$ and $\theta = 2/11$, $\lambda_{k,N}$ and λ_k are all inside the interval $[a_-, a_+]$ for different values of ε

λ_k Exact λ_k	$[a_-, a_+]$, $\varepsilon = 10^{-5}$	$[a_-, a_+]$, $\varepsilon = 10^{-10}$	$\lambda_{k,N}$
λ_{-2} −3.2964599324573105	[−3.381001, −3.215245]	[−3.297352, −3.295553]	−3.2964599324573127
λ_{-1} −1.725663605662414	[−1.781207, −1.671159]	[−1.725760, −1.725566]	−1.725663605662413
λ_0 −0.15486727886751722	[−0.202971, −0.107279]	[−0.155364, −0.154374]	−0.1548672788675186
λ_1 1.4159290479273794	[1.362486, 1.469718]	[1.414521, 1.417324]	1.4159290479273827
λ_2 2.986725374722276	[2.912282, 3.065032]	[2.984366, 2.989098]	2.9867253747222686

$$E_8(\mathcal{G}_{\theta,m}) = 4.61204 \times 10^8, E_7(\mathcal{G}_{\theta,m}) = 8.92709 \times 10^7, \ell = 1, M_{\mathcal{G}_{\theta,m}} = 20713.1$$

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Authors' contributions
The authors have equal contributions to each part of this article. All the authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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