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A new integral transform on time scales and its applications

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Abstract

Integral transform methods are widely used to solve the several dynamic equations with initial values or boundary conditions which are represented by integral equations. With this purpose, the Sumudu transform is introduced in this article as a new integral transform on a time scale $\mathbb{T}$ to solve a system of dynamic equations. The Sumudu transform on time scale $\mathbb{T}$ has not been presented before. The results in this article not only can be applied on ordinary differential equations when $\mathbb{T} = \mathbb{R}$, difference equations when $\mathbb{T} = \mathbb{N}_0$, but also, can be applied for q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0}$, where $q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} := q^{\mathbb{Z}} \cup \{0\}$ for $q > 1$ (which has important applications in quantum theory) and on different types of time scales like $\mathbb{T} = h\mathbb{N}_0$, $\mathbb{T} = \mathbb{N}_0^2$ and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers. Finally, we give some applications to illustrate our main results.

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1 Introduction

In the literature there are several integral transforms that are widely used in physics, astronomy as well as in engineering. Watugala [1,2] introduced a new integral transform and named it the Sumudu transform that is defined by the formula

$$F(u) = S[f(t); u] = \frac{1}{u} \int_0^\infty e^{-\frac{t}{u}} f(t)dt, \ u \in (-\tau_1, \ \tau_2), \tag{1.1}$$

and applied it to find the solution of ordinary differential equations in control engineering problems. It appeared like the modification of the well known Laplace transform $L[f(t); u]$, where

$$L[f(t); u] = \int_0^\infty e^{-ut} f(t)dt. \tag{1.2}$$

However in [3,4], some fundamental properties of the Sumudu transform were established. By looking at the properties of this transform one can notice that the Sumudu transform has very special and useful properties and it can help with intricate applications in the sciences and engineering. For example, in [5], the Sumudu transform was extended to distributions (generalized functions) and some of their properties were also studied in [6,7]. Recently Kılıçman et al. applied this transform to solve a system of differential equations, see [8]. Further in [9], a system of fractional linear differential equations were solved analytically by using a new method which was named fractional Sumudu transform. We note that by using the Sumudu transform technique we can reduce the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result. We also note that an interesting fact about the Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except for the factor $n!$. Thus, if $f(t) = \sum_{n=0}^{\infty} a_n t^n$, then $F(u) = \sum_{n=0}^{\infty} n! a_n t^n$; see [10]. Furthermore, Laplace and Sumudu transforms of the Dirac delta function and the Heaviside function satisfy:

$$S_2[H(x,t)] = \pounds_2[\delta(x,t)] = 1,$$

and

$$S_2[\delta(x,\ t)] = \pounds_2[H(x,t)] = \frac{1}{uv}.$$

In this study, authors' purpose is to introduce the Sumudu transform on a time scale and show the applicability of this interesting new transform and its efficiency in solving the linear system of dynamic equations and integral equations. Assume that $\mathbb{T}$ is a time scale such that $\sup \mathbb{T} = \infty$ and fix $t_0 \in \mathbb{T}$. Let $z \in \mathcal{R}$ (the set of regressive functions), then $\ominus z \in \mathcal{R}$ and $e_{\ominus z}(t,\ t_0)$ is well defined. The Laplace transform of the function $f : \mathbb{T} \to \mathbb{R}$ was defined by

$$L\{f\}(z) := \int_{t_0}^{\infty} f(t) e^{\sigma}_{\ominus z}(t, t_0) \Delta t \tag{1.3}$$

for $z \in \Omega\{f\}$, where $\Omega\{f\}$ consists of all complex numbers $z \in \mathcal{R}$ for which the improper integral exists, (see [11-15]).

2 Main results

Definition 2.1 [15] The function $f : \mathbb{T} \to \mathbb{R}$ is said to be of exponential type **I** if there exist constants M, $c > 0$ such that $|f(t)| \le Me^{ct}$. Furthermore, f is said to be of exponential type **II** if there exist constants M, $c > 0$ such that $|f(t)| \le Me_c(t, t_0)$.

Definition 2.2 Assume that $f : \mathbb{T}_0 \to \mathbb{R}$ is a rd-continuous function, then the Sumudu transform of f is

$$S\{f\}(u) := \frac{1}{u} \int_{t_0}^{\infty} f(t) e^{\sigma}_{\ominus \frac{1}{u}}(t, t_0) \Delta t \tag{2.1}$$

for $u \in D\{f\}$, where $D\{f\}$ consists of all complex numbers $u \in \mathcal{R}$ for which the improper integral exists.

Theorem 2.1 (Linearity) Assume that $S\{f\}$ and $S\{g\}$ exist for $u \in D\{f\}$ and $u \in D\{g\}$, respectively, where f and g are rd-continuous functions on $\mathbb{T}$ and α and β are constants. Then

$$S\{\alpha f + \beta g\}(u) = \alpha S\{f\}(u) + \beta S\{g\}(u) \tag{2.2}$$

for $u \in D\{f\} \cap D\{g\}$.

Proof. The proof follows directly from Definition 2.2.

We will assume that $\mathbb{T}$ is a time scale with bounded graininess, that is, $0 < \mu_* \leq \mu(t) \leq \mu^*$ for all $t \in \mathbb{T}$. Let $\mathbb{H}$ denotes the Hilger circle, see [15], given by

$$\mathbb{H} = \mathbb{H}_t = \left\{ z \in \mathbb{C} : 0 < \left| z + \frac{1}{\mu(t)} \right| < \frac{1}{\mu(t)} \right\},$$
$$\mathbb{H}_{\min} = \left\{ z \in \mathbb{C} : 0 < \left| z + \frac{1}{\mu^*} \right| < \frac{1}{\mu^*} \right\},$$
$$\mathbb{H}_{\max} = \left\{ z \in \mathbb{C} : 0 < \left| z + \frac{1}{\mu_*} \right| < \frac{1}{\mu_*} \right\}.$$

It is clear that $\mathbb{H}_{\min} \subset \mathbb{H}_t \subset \mathbb{H}_{\max}$. To give an appropriate domain for the transform, which of course is tied to the region of convergence of the integral in (2.1), for any $c > 0$ define the set

$$\begin{aligned} D &= \left\{ z \in \mathbb{C} : \mathrm{Re}_\mu \left(\frac{1}{z} \right) > \mathrm{Re}_\mu (c) \text{ for all } t \in \mathbb{T} \right\} \\ &= \left\{ z \in \mathbb{C} : \frac{1}{z} \in \overline{\mathbb{H}}^c_{\max} \text{ and } \mathrm{Re}_{\mu_*} \left(\frac{1}{z} \right) > \mathrm{Re}_{\mu_*}(c) \text{ for all } t \in \mathbb{T} \right\} \\ &= \left\{ z \in \mathbb{C} : \ominus \frac{1}{z} \in \mathbb{H} \text{ and } \mathrm{Re}_\mu \left(\frac{1}{z} \right) > \mathrm{Re}_\mu (c) \text{ for all } t \in \mathbb{T} \right\}, \end{aligned}$$

where $\overline{\mathbb{H}}^c_{\max}$ denotes the complement of the closure of largest Hilger circle corresponding to $\mu_\cdot$. Note that if $\mu_* = 0$ this set is a right half plane; see [15].

Lemma 2.1 If $\ominus \frac{1}{z} \in \mathbb{H}$ and $\mathrm{Re}_\mu \left(\frac{1}{z} \right) > \mathrm{Re}_\mu (c)$ for all $t \in \mathbb{T}$, then $\frac{1}{\left| 1 + \frac{\mu(t)}{z} \right|} \leq 1$ and $\left(\ominus \frac{1}{z} \oplus c \right) \in \mathbb{H}$.

Proof. Since $\ominus \frac{1}{z} \in \mathbb{H}$ then $\left| \ominus \frac{1}{z} + \frac{1}{\mu(t)} \right| < \frac{1}{\mu(t)}$ which implies $\frac{1}{\left| 1 + \frac{\mu(t)}{z} \right|} \leq 1$. Also, since $\mathrm{Re}_\mu \left(\frac{1}{z} \right) > \mathrm{Re}_\mu (c)$ implies $\left| 1 + \frac{\mu(t)}{z} \right| > |1 + c\mu(t)|$, we see

$$\left| \left(\ominus \frac{1}{z} \oplus c \right) + \frac{1}{\mu(t)} \right| = \left| \frac{1 + c\mu(t)}{\mu(t) \left(1 + \frac{\mu(t)}{z} \right)} \right| < \frac{1}{\mu(t)}.$$

Theorem 2.2 (*Domain of the transform*). The integral $\frac{1}{z} \int_{t_0}^{\infty} f(t) e^\sigma_{\ominus \frac{1}{z}} (t, t_0) \Delta t$ converges absolutely for $z \in D$ if $f(t)$ is of exponential type **II** with exponential constant c.

Proof. If $\ominus \frac{1}{z} \in \mathbb{H}$, then $\frac{1}{\left| 1 + \frac{\mu(t)}{z} \right|} \leq 1$.

Note that

$$\begin{aligned}
\left|\frac{1}{z}\int_{t_0}^{\infty} f(t)e^{\sigma}_{\ominus\frac{1}{z}}(t,t_0)\Delta t\right| &\le M\int_{t_0}^{\infty}\left|\frac{1}{z}e_c(t,t_0)e^{\sigma}_{\ominus\frac{1}{z}}(t,t_0)\right|\Delta t\\
&\le M\int_{t_0}^{\infty}\left|\frac{\frac{1}{z}}{1+\frac{\mu(t)}{z}}\right|\left|e_{\ominus\frac{1}{z}\oplus c}(t,t_0)\right|\Delta t\\
&\le (1+\mu^*)M\int_{t_0}^{\infty}\left|e_{\ominus\frac{1}{z}\oplus c}(t,t_0)\right|\Delta t\\
&= \gamma\int_{t_0}^{\infty}\exp\left(\int_{t_0}^{t}\frac{\log\left|\left(1+\ominus\frac{1}{z}\oplus c\right)\mu(s)\right|}{\mu(s)}\Delta s\right)\Delta t\\
&= \gamma\int_{t_0}^{\infty}\left(\exp\int_{t_0}^{t}\frac{1}{\mu(s)}\log\left|\frac{1+c\mu(s)}{1+\frac{\mu(s)}{z}}\right|\Delta s\right)\Delta t\\
&\le \gamma\int_{t_0}^{\infty}e^{-\alpha t}dt = \frac{(1+\mu^*)M}{\alpha},
\end{aligned}$$

where $\gamma = (1+\mu^{*})M$ and $\alpha = \dfrac{\log\left|\dfrac{1+c\mu_*}{1+\frac{\mu^*}{z}}\right|}{\mu^*}$.

The same estimates used in the proof of the proceeding theorem can be used to show that if $f(t)$ is of exponential type **II** with constant c and $\mathrm{Re}_{\mu}\left(\frac{1}{z}\right) > \mathrm{Re}_{\mu}(c)$, then $\lim_{t\to\infty} e_{\ominus\frac{1}{z}}(t,t_0)f(t) = 0$.

In the following theorem, we state the relationship between Sumudu transform and Laplace transform:

Theorem 2.3 Assume that $f:\mathbb{T}\to\mathbb{R}$ is *rd*-continuous function, then

$$S\{f\}(z) = \frac{1}{z}L\{f\}\left(\frac{1}{z}\right) \tag{2.3}$$

for $z \in D\{f\}$, where $D\{f\}$ consists of all complex numbers $z \in \mathcal{R}$ for which the improper integrals in (1.3) and (2.1) exist.

Proof. By using the definition of the transform we obtain

$$L\{f\}\left(\frac{1}{z}\right) = \int_{t_0}^{\infty} f(t)e^{\sigma}_{\ominus\frac{1}{z}}(t,t_0)\Delta t = zS\{f\}(z). \tag{2.4}$$

Theorem 2.4 If $\mathbb{T} = \mathbb{N}_0$, then

$$S\{f\}(z) = \frac{1}{z+1}Z\{f\}\left(\frac{1}{z}+1\right) \tag{2.5}$$

where $Z\{f\}$ is the Z-transform of f, which is defined by

$$Z\{f\}(u) = \sum_{t=0}^{\infty} \frac{f(t)}{u^t} \tag{2.6}$$

for those complex values of for which this infinite sum converges.

Proof. The proof follows directly from Theorem 2.3 and the relation

$$(u+1)\, L\left\{f\right\}(u) = Z\left\{f\right\}(u+1)\,.$$

Example 2.1 In this example, we find the Sumudu transform of $f(t) \equiv 1$. From Definition 2.2 we have

$$\begin{aligned}
S\{1\}(u) &= \frac{1}{u}\int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{1}{u}}(t,t_0)\Delta t \\
&= \frac{1}{u}\int_{t_0}^{\infty} \left[\mu(t)e^{\Delta}_{\ominus\frac{1}{u}}(t,t_0) + e_{\ominus\frac{1}{u}}(t,t_0)\right]\Delta t \\
&= \frac{1}{u}\int_{t_0}^{\infty} \left[\mu(t)\ominus\frac{1}{u}e_{\ominus\frac{1}{u}}(t,t_0) + e_{\ominus\frac{1}{u}}(t,t_0)\right]\Delta t \\
&= \frac{1}{u}\int_{t_0}^{\infty} \left[1+\mu(t)\ominus\frac{1}{u}\right]e_{\ominus\frac{1}{u}}(t,t_0)\Delta t \int_{t_0}^{\infty} \\
&= \frac{1}{u}\int_{t_0}^{\infty} \left[1+\frac{\mu(t)(-\frac{1}{u})}{1+(\frac{1}{u})\mu(t)}\right]e_{\ominus\frac{1}{u}}(t,t_0)\Delta t \\
&= -\int_{t_0}^{\infty} \left[\frac{(-\frac{1}{u})}{1+(\frac{1}{u})\mu(t)}\right]e_{\ominus\frac{1}{u}}(t,t_0)\Delta t \\
&= \int_{t_0}^{\infty} \left(\ominus\frac{1}{u}\right)e_{\ominus\frac{1}{u}}(t,t_0)\Delta t = -e_{\ominus\frac{1}{u}}(t,t_0)\Big|_{t_0}^{\infty} = 1.
\end{aligned}$$

Therefore, we get

$$S\{1\}(u) = 1. \tag{2.7}$$

Example 2.2 In particular, $S\{e_\alpha(t, t_0)\}(u)$ is given by

$$S\{e_\alpha(t,t_0)\}(u) = \frac{1}{1-\alpha u}. \tag{2.8}$$

For,

$$S\{e_\alpha(t,t_0)\}(u) = \frac{1}{u}L\{e_\alpha(t,t_0)\}\left(\frac{1}{u}\right) = \frac{1}{u}\left(\frac{1}{\frac{1}{u}-\alpha}\right) = \frac{1}{1-\alpha u}.$$

From the above example, we have

$$S\{\cosh_\alpha(t,t_0)\}(u) = \frac{1}{1-\alpha^2u^2},$$

$$S\{\sinh_\alpha(t,t_0)\}(u)=\frac{\alpha u}{1-\alpha^2u^2},$$

$$S\{\cos_\alpha(t,t_0)\}(u)=\frac{1}{1-\alpha^2u^2},$$

and

$$S\{\sin_\alpha(t,t_0)\}(u)=\frac{\alpha u}{1-\alpha^2u^2}.$$

Example 2.3 In the following, we make use of (2.5) and (2.6) to find $S\{\alpha^t\}(u)$ where $t\in\mathbb{T}=\mathbb{N}_0$. In fact,

$$\begin{aligned}S\{\alpha^t\}(u)&=\frac{1}{u+1}\sum_{t=t_0}^{\infty}\left(\frac{\alpha}{1+\frac{1}{u}}\right)^t\\&=\frac{1}{1+(1-\alpha)u},\ |\alpha|\le\left|1+\frac{1}{u}\right|.\end{aligned}$$

The following results are derived using integration by parts.

Theorem 2.5 Assume that $f:\mathbb{T}\to\mathbb{C}$ is of exponential type **II** such that f^Δ is a *rd*-continuous function and $\lim_{t\to\infty}e_{\ominus\frac{1}{z}}(t,t_0)f(t)=0$, then

$$S\{f^\Delta\}(u)=\frac{S\{f\}(u)-f(t_0)}{u}. \tag{2.9}$$

Proof. Integration by parts yields

$$\begin{aligned}S\{f^\Delta\}(u)&=\frac{1}{u}\int_{t_0}^{\infty}f^\Delta(t)e^\sigma_{\ominus\frac{1}{u}}(t,t_0)\Delta t\\&=\frac{1}{u}\left[f(t)e_{\ominus\frac{1}{u}}(t,t_0)\right]_{t_0}^{\infty}+\frac{1}{u^2}\int_{t_0}^{\infty}f(t)e^\sigma_{\ominus\frac{1}{u}}(t,t_0)\Delta t\\&=\frac{S\{f\}(u)-f(t_0)}{u}.\end{aligned}$$

Remark 2.1 Similarly,

$$S\{f^{\Delta\Delta}\}(u)=\frac{1}{u^2}S\{f\}(u)-\frac{1}{u^2}f(t_0)-\frac{1}{u}f^\Delta(t_0).$$

More generally, we have

$$S\{f^{\Delta^n}\}(u)=\frac{1}{u^n}S\{f\}(u)-\sum_{k=0}^{n-1}\frac{1}{u^{n-k}}f^{\Delta^k}(t_0).$$

This is because from Theorem 2.5, we have

$$\begin{aligned}S\{f^{\Delta^{n+1}}\}(u)=S\{(f^{\Delta^n})^\Delta\}(u)&=\frac{S\{f^{\Delta^n}\}(u)-f^{\Delta^n}(t_0)}{u}\\&=\frac{1}{u}\left[\left(\frac{1}{u^n}S\{f\}(u)-\sum_{k=0}^{n-1}\frac{1}{u^{n-k}}f^{\Delta^k}(t_0)\right)-f^{\Delta^n}(t_0)\right]\\&=\frac{1}{u^{n+1}}S\{f\}(u)-\sum_{k=0}^{n}\frac{1}{u^{n-k+1}}f^{\Delta^k}(t_0).\end{aligned}$$

Theorem 2.6 Assume $h : \mathbb{T} \to \mathbb{C}$ is a *rd*-continuous function. If

$$H(t) := \int_{t_0}^{t} h(\tau)\Delta\tau \tag{2.10}$$

for $t \in \mathbb{T}$, then

$$S\{H\}(u) = u\, S\{h\}(u) \tag{2.11}$$

for those regressive $u \in \mathbb{C} - \{0\}$ satisfying

$$\lim_{t\to\infty}\{H(t)e_{\ominus\frac{1}{u}}(t, t_0)\} = 0. \tag{2.12}$$

Proof. Integrating by parts we have

$$\begin{aligned}
S\{H\}(u) &= \frac{1}{u}\int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{1}{u}}(t, t_0)H(t)\Delta t \\
&= \int_{t_0}^{\infty}\left(\ominus\frac{1}{u}\right)e^{\sigma}_{\ominus\frac{1}{u}}(t, t_0)H(t)\Delta t \\
&= \left[e_{\ominus\frac{1}{u}}(t, t_0)H(t)\right]_{t_0}^{t\to\infty} + \int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{1}{u}}(t, t_0)H^{\Delta}(t)\Delta t \\
&= -H(t_0) + \int_{t_0}^{\infty} e^{\sigma}_{\ominus\frac{1}{u}}(t, t_0)h(t)\Delta t \\
&= u\, S\,\{h\}\,(u)\,.
\end{aligned}$$

Remark 2.2 One can get the result in (2.11) from (2.3) and the relation

$$L\{H(t)\}(u) = \frac{1}{u}L\{h(t)\}(u).$$

Remark 2.3 From (2.11) we have

$$S\,\{t\}\,(u) = u,$$

$$S\{t^2\} = u^2 + S\{\sigma\,(t)\}\,(u)\,.$$

Theorem 2.7 Assume $f : \mathbb{T} \to \mathbb{C}$ is a *rd*-continuous function and $L\{f(t)\}(u) = F_L(u)$. Then

$$L\{e^{\sigma}_{\ominus\alpha}(t, t_0)f(t)\}(u) = F_L(u \oplus \alpha). \tag{2.13}$$

Proof. Since

$$e^{\sigma}_{\ominus\alpha}(t, t_0)e^{\sigma}_{\ominus u}(t, t_0) = e^{\sigma}_{\ominus(u\oplus\alpha)}(t, t_0)$$

and $L\{f(t)\}(u) = F_L(u)$, then

$$\begin{aligned} L\{e^{\sigma}_{\ominus\alpha}(t,t_0)f(t)\}(u) &= \int_{t_0}^{\infty} e^{\sigma}_{\ominus u}(t,t_0)(e^{\sigma}_{\ominus\alpha}(t,t_0)f(t))\Delta t \\ &= \int_{t_0}^{\infty} e^{\sigma}_{\ominus(u\oplus\alpha)}(t,t_0)f(t)\Delta t \\ &= F_L(u\oplus\alpha). \end{aligned}$$

From (2.3) and (2.13) we have the following result.

Theorem 2.8 Assume $f : \mathbb{T} \to \mathbb{C}$ is a *rd*-continuous function and $S\{f(t)\}(u) = F_S(u)$. Then

$$S\{e^{\sigma}_{\ominus\alpha}(t,t_0)f(t)\}(u) = \frac{1}{u}F_L\left(\frac{1}{u}\oplus\alpha\right) = \frac{1}{u}\frac{1}{(\frac{1}{u}\oplus\alpha)}F_S\left(\frac{1}{(\frac{1}{u}\oplus\alpha)}\right). \tag{2.14}$$

In [15], the convolution of two functions f, g is defined by

$$(f*g)(t) = \int_{t_0}^{t} \hat{f}(t,\ \sigma(s))g(s)\Delta s \quad \text{for } t \in \mathbb{T}_0,$$

where $\hat{f}$ is the shift of f and

$$L\left\{f*g\right\}(u) = L\{f\}(u)\,L\{g\}(u)\,.$$

Remark 2.4 When $\mathbb{T} = \mathbb{R}$, then

$$(f*g)(t) = \int_{t_0}^{t} f(t-s)g(s)ds$$

which coincides with the classical definition.

In the following, we present the relation between the Sumudu transform of the convolution of two functions on a time scale $\mathbb{T}$ and the product of Sumudu transform of f and g.

Theorem 2.9 Assume that f, g are regulated functions on $\mathbb{T}$, then

$$S\left\{f\ *\ g\right\}(u)\ =\ u\,S\left\{f\right\}(u)\ \ S\left\{g\right\}(u)\,.$$

Proof. Since

$$\begin{aligned} S\{f*g\}(u) &= \frac{1}{u}L\{f\ *\ g\}\left(\frac{1}{u}\right) = \frac{1}{u}L\{f\}\left(\frac{1}{u}\right)L\{g\}\left(\frac{1}{u}\right) \\ &= u\left(\frac{1}{u}L\{f\}\left(\frac{1}{u}\right)\right)\left(\frac{1}{u}L\{g\}\left(\frac{1}{u}\right)\right) \\ &= u\,S\left\{f\right\}(u)\ \ S\left\{g\right\}(u)\,. \end{aligned}$$

3 Applications

I. To find the solution of a homogeneous dynamic equation with constant coefficients in the form

$$\sum_{k=0}^{n} a_k y^{\Delta^k} = 0.$$

Applying Sumudu transform we get

$$a_0 S\{y\}(u) + \sum_{k=1}^{n} a_k \left[\frac{1}{u^k} S\{y\}(u) - \frac{1}{u^k} y(0) - \frac{1}{u^{k-1}} y^{\Delta}(0) - \cdots - \frac{1}{u} y^{\Delta^{k-1}}(0)\right] = 0,$$

or,

$$\left(\sum_{k=0}^{n} \frac{a_k}{u^k}\right) S\{y\}(u) = \sum_{k=1}^{n} a_k \sum_{i=0}^{k-1} \frac{y^{\Delta^i}(0)}{u^{k-i}} = \psi\ \varphi(0), \tag{3.1}$$

where

$$\psi = \left(\tfrac{1}{u}\,\tfrac{1}{u^2} \cdot \cdot \tfrac{1}{u^n}\right) \begin{pmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ a_2 & a_3 & \cdot & \cdot & a_n & 0 \\ a_3 & a_4 & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_n & 0 & \cdot & \cdot & \cdot \\ a_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

$$\varphi(0) = \left(y(0) y^{\Delta}(0) \ldots y^{\Delta^{n-1}}(0)\right)^T$$

Example 3.1 Consider the following dynamic equation

$$y^{\Delta\Delta}(t) - y^{\Delta}(t) - 6y(t) = 0, \quad y(0) = 1, y^{\Delta}(0) = -7, \tag{3.2}$$

then $a_0 = -6$, $a_1 = -1$, $a_2 = 1$. Consequently from (3.1) we have

$$\left(-6 - \frac{1}{u} + \frac{1}{u^2}\right) S\{y\}(u) = \left(\tfrac{1}{u}\,\tfrac{1}{u^2}\right) \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -7 \end{pmatrix};$$

hence,

$$S\{y\}(u) = -\frac{1}{1-3u} + \frac{2}{1+2u}.$$

Therefore,

$$y(t) = -e_3(t, 0) + 2e_{-2}(t, 0). \tag{3.3}$$

Remarks

(1) When $\mathbb{T} = \mathbb{R}$ the Equation (3.2) becomes

$$y''(t) - y'(t) - 6y(t) = 0, \quad y(0) = 1,\ y'(0) = -7$$

and then from (3.3) its solution is

$$y(t) = -e^{3t} + 2e^{-2t}.$$

(2) When $\mathbb{T} = \mathbb{Z}$ the Equation (3.2) becomes as a difference equation

$$\Delta\Delta y(t) - \Delta y(t) - 6y(t) = 0,\ y(0) = 1, \Delta y(0) = -7$$

and from (3.3) it has a solution

$$y(t) = -(4)^t + 2(-1)^t, \quad t \in \mathbb{Z}.$$

(3) When $\mathbb{T} = q^{\mathbb{N}_0} \cup \{0\}$ the Equation (3.2) becomes

$$y^{\Delta_q\Delta_q}(t) - y^{\Delta_q}(t) - 6y(t) = 0, \quad y(0) = 1,\ y^{\Delta_q}(0) = -7$$

where $y^{\Delta_q}(t) = \dfrac{y(qt) - y(t)}{(q-1)t}$, $t \geq 1$ Then from (3.3), we have

$$y(1) = -6,\ y(t) = -\prod_{s\in[1,t)} [1 + 3(q-1)s] + 2\prod_{s\in[1,t)} [1 - 2(q-1)s],\ t > 1.$$

II. To find the solution of a system of dynamic equations with constant coefficients in the form

$$P(D)\,Y = F,$$

where,

$$P(D) = \begin{bmatrix} P_{11}(D)\ldots P_{1n}(D) \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \cdot \\ P_{n1}(D)\ldots P_{nn}(D) \end{bmatrix},\ Y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix},\ F = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix}$$

$$P_{ij}(D) = \sum_{k=0}^{n_i} a_{ik}D^k,\ D^k y = y^{\Delta^k}.$$

Analog to (3.1), we have

$$S\{P(D)Y\}(u) = (S\{P_{ij}(D)y_j\}(u)) = S\{F\}(u).$$

Since

$$\begin{aligned} S\{P_{ij}(D)y_j\}(u) &= a_{i0}S\{y_j\}(u) + \\ &\sum_{k=1}^{n_i} a_{ik}\left[\frac{1}{u^k}S\{y_j\}(u) - \frac{1}{u^k}y_j(0) - \frac{1}{u^{k-1}}y_j^{\Delta}(0) - \cdots - \frac{1}{u}y_j^{\Delta^{k-1}}(0)\right] \\ &= \left(\sum_{k=0}^{n_i}\frac{a_{ik}}{u^k}\right)S\{y_j\}(u) - \sum_{k=1}^{n_i} a_{ik}\sum_{\ell=0}^{k-1}\frac{y_j^{\Delta^\ell}(0)}{u^{k-\ell}} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^{n_i} a_{ik}\sum_{\ell=0}^{k-1}\frac{y_j^{\Delta^\ell}(0)}{u^{k-\ell}} &= \begin{pmatrix} \frac{1}{u} \\ \frac{1}{u^2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{u^{n_i}} \end{pmatrix}^T \begin{pmatrix} a_{i1} & a_{i2} & \cdot & \cdot & \cdot & a_{in_i} \\ a_{i2} & a_{i3} & \cdot & \cdot & a_{in_i} & 0 \\ a_{i3} & a_{i4} & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_{in_i} & 0 & \cdot & \cdot & \cdot \\ a_{in_i} & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} y_j(0) \\ y_j^{\Delta}(0) \\ \cdot \\ \cdot \\ \cdot \\ y_j^{\Delta^{n_i-1}}(0) \end{pmatrix} \\ &= \Psi_{ij}\Phi_j(0), \end{aligned}$$

$$\Phi_j(0) = \left(y_j(0)y_j^{\Delta}(0)\ldots y_j^{\Delta^{n-1}}(0)\right)^T,$$

then,

$$\left[\left(\sum_{k=0}^{n_i} \frac{a_{ik}}{u^k}\right) S\{y_j\}(u)\right] = \left[\sum_{k=1}^{n_i} a_{ik} \sum_{\ell=0}^{k-1} \frac{y_j^{\Delta^\ell}(0)}{u^{k-\ell}}\right] + S\{F\}(u)$$

i.e.,

$$\left(P\left(\frac{1}{u}\right)\right)_{n\times n} (S\{Y\}(u))_{n\times 1} = \Psi_{n\times m}(\Phi(0))_{m\times 1} + (S\{F\}(u))_{n\times 1} \tag{3.4}$$

where, Ψ is $n \times m$ matrix, $m = \sum_{j=1}^{n} d_j, d_j = \max_{1\le i\le n}\{\deg P_{ij}(D)\}$. Consequently,

$$S\{Y\} = \left[P\left(\frac{1}{u}\right)\right]^{-1} \left[\Psi\Phi(0) + S\{F\}(u)\right]. \tag{3.5}$$

Example 3.2 To solve the following system of two dynamic equations

$$\begin{aligned} &-x^{\Delta\Delta} + 3y^{\Delta} + 2x = 2e_3(t,\ 0),\ x(0) = 1,\ x^{\Delta}(0) = 5,\\ &y^{\Delta\Delta} - 4x^{\Delta} + 3y = -9\cos_2(t,\ 0),\ y(0) = 2,\ y^{\Delta}(0) = 3, \end{aligned}$$

we have,

$$\begin{aligned} P\left(\frac{1}{u}\right) &= \begin{bmatrix} 2 - \frac{1}{u^2} & \frac{3}{u} \\ -\frac{4}{u} & 3 + \frac{1}{u^2} \end{bmatrix}, S\{Y\} = S\left\{\begin{bmatrix} x \\ y \end{bmatrix}\right\},\\ \Phi(0) &= \begin{bmatrix} x(0) \\ x^{\Delta}(0) \\ y(0) \\ y^{\Delta}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \\ 3 \end{bmatrix},\\ \Psi_{11} &= \begin{bmatrix} \frac{1}{u} & \frac{1}{u^2} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{u^2} & \frac{-1}{u} \end{bmatrix} = -\Psi_{22}\\ \Psi_{12} &= \begin{bmatrix} \frac{1}{u} & \frac{1}{u^2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\\ \Psi_{21} &= \begin{bmatrix} \frac{1}{u} & \frac{1}{u^2} \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}\\ \Psi &= \begin{bmatrix} \frac{-1}{u^2} & \frac{-1}{u} & \frac{3}{u} & 0 \\ \frac{-4}{u} & 0 & \frac{1}{u^2} & \frac{1}{u} \end{bmatrix}\\ S\{F\} = S&\left\{\begin{bmatrix} 2e_3(t,0) \\ -9\cos_2(t,0) \end{bmatrix}\right\} = \begin{bmatrix} \frac{2}{1-3u} \\ \frac{-9}{1+4u^2} \end{bmatrix}. \end{aligned}$$

Since

$$S\{Y\} = P^{-1}(u)[\Psi\Phi(0) + S\{F\}(u)].$$

Then

$$\begin{aligned} S\{Y\} &= \frac{1}{(1-3u)(1+4u^2)} \begin{bmatrix} 1 + 2u - 2u^2 \\ 2 - 3u + 4u^2 \end{bmatrix}\\ &= \begin{bmatrix} \frac{1}{1-3u} + \frac{2u}{1+4u^2} \\ \frac{1}{1-3u} + \frac{1}{1+4u^2} \end{bmatrix}. \end{aligned}$$

Consequently,

$$\begin{bmatrix} x \\ y \end{bmatrix} = S^{-1}\left\{\begin{bmatrix} \frac{1}{1-3u} + \frac{2u}{1+4u^2} \\ \frac{1}{1-3u} + \frac{1}{1+4u^2} \end{bmatrix}\right\} = \begin{bmatrix} e_3(t,0) + \sin_2(t,0) \\ e_3(t,0) + \cos_2(t,0) \end{bmatrix}.$$

i.e., $x(t) = e_3\,(t, 0) + \sin_2(t, 0)$ and $y(t) = e_3(t, 0) + \cos_2\,(t, 0)$.

II. To find the solution dynamic integral equation we provide the following example.

Example 3.3 Consider the integral equation

$$h(t) = e_2(t,\ 0) + 4\int_0^t h(\tau)\Delta\tau.$$

Applying the Sumudu transform we get

$$S\{h\}(u) = \frac{1}{1-2u} + 4u\ S\{h\}\,(u).$$

Then we get

$$S\{h\}(u) = \frac{1}{(1-2u)(1-4u)} = \frac{-1}{1-2u} + \frac{2}{1-4u}$$

and consequently from (2.8), we have

$$h(t) = -e_2(t,\ 0) + 2e_4(t,\ 0).$$

Example 3.4 For solving the following equation

$$x(t) = \cos_2(t,\ 0) + 3(x(t) * \sin_3(t,\ 0)).$$

Applying Sumudu transform we get

$$S\{x\} = \frac{1+9u^2}{1+4u^2} = \frac{1}{4}\left[9 - \frac{5}{1+4u^2}\right].$$

Then

$$x(t) = \frac{1}{4}S^{-1}\left\{\left[9 - \frac{5}{1+4u^2}\right]\right\} = \frac{1}{4}\,(9 - 5\cos_2\,(t,\ 0))\,,$$

III. Assuming that $\mathbb{T}$ has constant graininess $\mu(t) \equiv \mu$, we can find $S(e^{\sigma}_{\ominus\alpha}(t,\ 0))$, $S(e_\beta(t,\ 0)e^{\sigma}_{\ominus\alpha}(t,\ 0))$, $S(e_\beta(t,\ 0)e^{\sigma}_{\ominus\alpha}(t,\ 0))$, $S(e^{\sigma}_{\ominus\alpha}(t,\ 0)\cos_\beta(t,\ 0))$ and $S(e^{\sigma}_{\ominus\alpha}(t,\ 0)\sin_\beta(t,\ 0))$.

From (2.14), we have

$$S(e^{\sigma}_{\ominus\alpha}(t,\ 0)) = \frac{1}{u}\frac{1}{\left(\frac{1}{u}\oplus\alpha\right)}, \tag{3.6}$$

$$S(te^{\sigma}_{\ominus\alpha}(t,\ 0))(u) = \frac{1}{u}\frac{1}{\left(\frac{1}{u}\oplus\alpha\right)^2}, \tag{3.7}$$

$$S(e_\beta(t,\ 0)e^{\sigma}_{\ominus\alpha}(t,\ 0))(u) = \frac{1}{u}\frac{1}{\left(\frac{1}{u}\oplus\alpha\right)-\beta}, \tag{3.8}$$

$$S(e^{\sigma}_{\ominus\alpha}(t,\ 0)\cos_{\beta}(t,\ 0)) = \frac{1}{u}\frac{\left(\frac{1}{u}\oplus\alpha\right)}{\left(\frac{1}{u}\oplus\alpha\right)^2+\beta^2}, \tag{3.9}$$

and

$$S\left(e^{\sigma}_{\ominus\alpha}(t,\ 0)\sin_{\beta}(t,\ 0)\right) = \frac{1}{u}\frac{\beta}{\left(\frac{1}{u}\oplus\alpha\right)^2+\beta^2}.$$

When $\mathbb{T} = \mathbb{R}$ (as a special case) the results in (3.6), (3.7), (3.8), and (3.9) become

$$S\{e^{-\alpha t}\}(u) = \frac{1}{(1+\alpha u)},$$

$$S\{te^{-t}\}(u) = \frac{u}{(1+\alpha u)^2},$$

$$S\{e^{\beta t}e^{-\alpha t}\}(u) = \frac{1}{1+(\alpha-\beta)u},$$

$$S\{e^{-\alpha t}\cos\beta t\}(u) = \frac{(1+\alpha u)}{(1+\alpha u)^2+\beta^2u^2}$$

which coincide with previous results in [3]. Thus it seems that the present results are more general.

4 Conclusion

In this article, the Sumudu transform is introduced as a new integral transform on a time scale $\mathbb{T}$ in order to solve system of dynamic equations. Further, Sumudu transform on time scale $\mathbb{T}$ is not presented before. The results in this article not only can be applied to ordinary differential equations when $\mathbb{T} = \mathbb{R}$, difference equations when $\mathbb{T} = \mathbb{N}_0$, but also, can be applied for q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0}$, where $q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} := q^{\mathbb{Z}} \cup \{0\}$ for q>1 which has several important applications in quantum theory and on different types of time scales like $\mathbb{T} = h\mathbb{N}_0$, $\mathbb{T} = \mathbb{N}_0^2$, and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers. Regarding the comparison between Sumudu and Laplace transform, see for example, [4-7,16]. For example when $\mathbb{T} = \mathbb{R}$, Maxwell's equations were solved for transient electromagnetic waves propagating in lossy conducting media, see [16] where the Sumudu transform of Maxwell's differential equations yields a solution directly in the time domain, which neutralizes the need to perform the inverse Sumudu transform.

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Competing interests

The authors declare that they have no competing interests.

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