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Exponential stability of stochastic functional differential equations with Markovian switching and delayed impulses via Razumikhin method

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Abstract

In this article, by using Razumikhin-type technique, we investigate pth moment exponential stability of stochastic functional differential equations with Markovian switching and delayed impulses. Several stability theorems of impulsive hybrid stochastic functional differential equations are derived. It is assumed that the state variables on the impulses can relate to the finite delay. These new results are employed to a class of n-dimensional linear impulsive hybrid stochastic systems with bounded time-varying delay. Moreover, an effective M-matrix method is introduced to study the exponential stability of these hybrid systems. Meanwhile, some examples and simulations are given to show our results.

Keywords: Razumikhin theorem, stochastic functional differential equations, impulse, M-matrix, Markovian chain

1 Introduction

Stochastic differential equation is an emerging field drawing attention from both theoretical and applied disciplines, which has been successfully applied to problems in mechanical, electrical, economics, physics and several fields in engineering. For details, see [1-6] and the references therein. Recently, stability of stochastic differential equations with Markovian switching has received a lot of attention [7-12]. For example, Ji and Chizeck [7] and Mariton [8] studied the stability of a jump linear equation

$$dx(t) = A(r(t))x(t)dt,$$

where $x(t)$ takes values in R^n, $r(t)$ is a Markovian chain taking values in $S = \{1, 2, ..., N\}$. Mao [9] discussed the stability of nonlinear stochastic differential equation with Markovian switching of the form

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))d\omega(t).$$

In [10], Mao studied the stability of stochastic functional differential equation with Markovian switching of the form

$$dx(t) = f(x_t, t, r(t))dt + g(x_t, t, r(t))d\omega(t).$$

Impulsive effects are common phenomena due to instantaneous perturbations at certain moment, such phenomena are described by impulsive differential equation which

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have been used effciently in modelling many practical problems that arise in the fields of engineering, physics, and science as well. So the theory of impulsive differential equations is also attracting much attention in recent years [13-19]. Correspondingly, a lot of stability results of impulsive stochastic functional differential equations have been obtained [20-26]. However, there are few results on the stability of impulsive stochastic differential equation with Markovian switching. In [27], Wu and Sun established some stability criteria of p-moment stability for stochastic differential equations with impulsive jump and Markovian switching.

In this article, we shall extend Razumikhin method [10,12] to investigate the pth moment exponential stability of the following stochastic functional differential equations with Markovian switching and delayed impulse

$$\begin{cases} dx(t) = f(x_t, t, r(t))dt + g(x_t, t, r(t))d\omega(t) & t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k), x_{t_k}, t_k, r(t_k)) & , k = 1, 2, \ldots, \\ x(t) = \xi, \quad t \in [-\tau, 0]. \end{cases}$$

The state variables on the impulses relate to the finite delay, which implies that the impulsive effects are more general than those given in [20,22,23]. Some Theorems on the pth moment exponential stability are derived in the case that the impulsive gain $d_{ik} + \bar{d}_{ik} < 1$ or $d_{ik} + \bar{d}_{ik} \geq 1$. These new results are employed to the n-dimensional impulsive hybrid stochastic systems with bounded time-varying delay. Useful criteria in terms of an M-matrix (see Berman and Plemmons [28]) which can be verified much more easily are established. Meanwhile, examples and simulations are provided to show the impulsive effects play an important role in the stability for hybrid stochastic systems. The rest of this article is organized as follows. In Section 2, stochastic functional differential equations with Markovian switching and delayed impulses together with some definitions of pth moment exponential stability are presented. In Section 3, the Razumikhin-type theorems on pth moment exponential stability for stochastic functional differential equations with Markovian switching and delayed impulses are established. In Section 4, these results will then be applied to the n dimensional hybrid stochastic delay systems and M-matrix method is introduced to verify the stability easily. Finally, examples are given to demonstrate our effective results in Section 5.

2 Preliminaries

Let $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$, R^n denote the n-dimensional Euclidean space with the Euclidean norm $| \cdot |$. If A is a vector or matrix, its transpose is denoted by A^T, and its norm is denoted by $||A|| = \sqrt{\lambda_{\max}(A^T A)}$, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of a matrix. $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_m(t))^T$ is an m-dimensional Brownian motion on a complete probability space $(\Omega, \mathscr{F}, P)$ with a natural filtration $\{\mathscr{F}_t\}_{t \geq 0}$ satisfying the usual conditions, (i.e.$\mathscr{F}_t = \sigma\{\omega(s) : 0 \leq s \leq t\})$.

Let $\tau > 0$ and $PC([-\tau, 0], R^n) = \{\psi : [-\tau, 0] \to R^n | \psi(t^+), \psi(t)$ exist, and $\psi(t^-) = \psi(t)\}$ with the norm $||\psi|| = \sup_{-\tau \leq \theta \leq 0} |\psi(\theta)|,$ where $\psi(t^+)$ and $\psi(t^-)$ denote the right-hand and left-hand limits of function $\psi(t)$ at t.

Denoted by $PC^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$ the family of all bounded, $\mathscr{F}_0$ - measurable, $PC([-\tau, 0]; R^n)$-valued random variables. For $p > 0$, denoted by $PC^p_{\mathscr{F}_t}([-\tau, 0], R^n)$ the family of

all $\mathscr{F}_t$-measurable $PC([-\tau, 0], R^n)$-valued random variables ψ such that $\int_{-\tau}^{0} E|\psi(\theta)|^p d\theta < \infty$. Let $r(t)(t > 0)$ be a right-continuous Markovian chain on the probability space taking values in a finite state space $S = \{1, 2, ..., N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), \text{ if } i \neq j, \\ 1 + \gamma_{ij}\Delta + o(\Delta), \text{ if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j while

$$\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}.$$

We assume that the Markovian chain $r(t)$ is independent of the Brownian motion $\omega(t)$. It is known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of R^+.

Consider the following impulsive hybrid stochastic functional differential equation of the form

$$\begin{cases} dx(t) = f(x_t, t, r(t))dt + g(x_t, t, r(t))d\omega(t), \quad t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k), x_{t_k}, t_k, r(t_k)), \quad k = 1, 2, \ldots, \\ x(t) = \xi, \ t \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $\xi \in PC_{\mathscr{F}_0}^b([-\tau, 0]; R^n), x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and $x(t_k^+) = \lim_{h \to 0+} x(t_k + h), x(t_k) = \lim_{h \to 0-} x(t_k + h), t_k \geq 0$, $x(t_k^+) = \lim_{h \to 0+} x(t_k + h), x(t_k) = \lim_{h \to 0-} x(t_k + h), t_k \geq 0$ are impulsive moments satisfying $t_k < t_{k+1}$ and $\lim_{k \to +\infty} t_k = +\infty$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ represents the jump in the state x at t_k with I_k determining the size of the jump, $f : PC([-\tau, 0]; R^n) \times R^+ \times S \to R^n, g : PC(-\tau, 0];$ $R^n) \times R^+ \times S \to R^{n \times m}, I_k : R^n \times PC([-\tau, 0]; R^n) \times R^+ \times S \to R^n$.

Throughout this article, we assume that f, g and I_k satisfy the necessary conditions for the global existence and uniqueness of solutions for all $t \geq 0$. For any $\xi \in PC_{\mathscr{F}_0}^b([-\tau, 0]; R^n)$, there exists a unique stochastic process satisfying Equation (2.1) denoted by $x(t; \xi)$, which is continuous on the left-hand side and limitable on the right-hand side. Also we assume that $f(0, t, i) \equiv 0, g(0, t, i) \equiv 0$ and $I_k(0, 0, t, i) \equiv 0, k = 1, 2, ...$, which implies that $x(t) \equiv 0$ is an equilibrium solution.

Let $\mathscr{C}_1^2(R^n \times [-\tau, \infty) \times S; R^+)$ be the family of all nonnegative functions $V(x, t, i)$ on $R^n \times [-\tau, \infty) \times S$ which are continuous on $R^n \times (t_{k-1}, t_k] \times S, V_t, V_x, V_{xx}$ are continuous on $R^n \times (t_{k-1}, t_k] \times S$. For each $V \in \mathscr{C}_1^2(R^n \times [-\tau, \infty) \times S; R^+)$, we define an operator $\mathscr{L}V : PC_{\mathscr{F}_t}^b([-\tau, 0]; R^n) \times (t_{k-1}, t_k] \times S \to R$ associated with Equation (2.1) as follows:

$$\mathscr{L}V(\phi, t, i) = V_t(\phi(0), t, i) + V_x(\phi(0), t, i)f(\phi, t, i)$$

$$+ \frac{1}{2}\text{trace}[g^T(\phi, t, i)V_{xx}(\phi(0), t, i)g(\phi, t, i)] + \sum_{j=1}^{N} \gamma_{ij}V(\phi(0), t, j),$$

where

$$V_t(x, t, i) = \frac{\partial V(x,t,i)}{\partial t}, \quad V_x(x, t, i) = \left(\frac{\partial V(x,t,i)}{\partial x_1}, \cdots, \frac{\partial V(x,t,i)}{\partial x_n} \right), \quad V_{xx}(x, t, i) = \left(\frac{\partial^2 V(x,t,i)}{\partial x_i x_j} \right)_{n \times n}.$$

Definition 2.1. The zero solution of Equation (2.1) are said to be pth moment exponentially stable if there exists $\eta > 0$ such that for any initial values $\xi \in PC_{\mathcal{F}_0}^b([-\tau, 0]; R^n)$ and $t \geq 0$

$$\limsup_{t \to \infty} \frac{1}{t} \log(E|x(t; \xi)|^p) \leq -\eta.$$

Remark 2.1. When $p = 2$, it is often called to be exponentially stable in mean square.

3 Stability analysis

In the following, we shall establish some criteria on pth moment exponential stability for Equation (2.1).

Theorem 3.1. Assume that $V \in \mathscr{C}_1^2(R^n \times [-\tau, \infty) \times S; R^+)$ and constants $p > 0, c_1 > 0, c_2 > 0, d_{ik} \geq 0, \bar{d}_{ik} \geq 0, d_{ik}^2 + \bar{d}_{ik}^2 \neq 0, \delta > 0, \lambda > 0, \eta_i \geq 0, i \in S, k = 1, 2, \ldots$ such that

(i) $c_1|x|^p \leq V(x, t, i) \leq c_2|x|^p$ for all $(t, x) \in R^n \times [-\tau, \infty) \times S$;

(ii) for all $t \in (t_{k-1}, t_k]$ and $i \in S, E\mathscr{L}V(\phi, t, i) \leq \eta_i EV(\phi(0), t, i)$ whenever

$$E\left[\min_{1 \leq i \leq N} V(\phi(\theta), t + \theta, i) \right] < q e^{\lambda \tau} E\left[\max_{1 \leq i \leq N} EV(\phi(0), t, i) \right];$$

(iii) for all $i \in S$,

$EV(\phi(0) + I_k(\phi(0), \phi(\theta), t_k, i), t_k^+, i) \leq d_{ik} EV(\phi(0), t_k, i) + \bar{d}_{ik} \sup_{-\tau \leq \theta < 0} EV(\phi(\theta), t_k, i)$;

(iv) $\sup_{1 \leq k < +\infty} \{t_k - t_{k-1}\} \leq \delta$;

(v) for any $i \in S, \lambda + \eta_i \leq \frac{\ln q}{\delta}$,

then the zero solution of Equation (2.1) is pth moment exponentially stable with pth moment exponent λ, where

$$\phi = \{\phi(\theta) | -\tau \leq \theta \leq 0\} \in PC_{\mathcal{F}_t}^p([-\tau, 0]; R^n), q = \frac{1}{\max_{i \in S, 1 \leq k < +\infty} \{d_{ik} + \bar{d}_{ik} e^{\lambda \tau}\}} > 1.$$

Proof. For any $\xi \in PC_{\mathcal{F}_0}^b([-\tau, 0]; R^n)$, we denote the solution $x(t) = x(t; \xi)$ of (2.1) and extend $r(t) = r(0) = r_0$ for all $t \in [-\tau, 0]$. Let ε be small enough such that $t + \varepsilon \in (t_{k-1}, t_k)$. By generalized Itô formula, we have

$$EV(x(t + \varepsilon), t + \varepsilon, r(t + \varepsilon)) = EV(x(t), t, r(t)) + \int_t^{t+\varepsilon} E\mathscr{L}V(x(s), s, r(s))ds. \qquad (3.1)$$

Let $\varepsilon \to 0$, it follows that for $t \in (t_{k-1}, t_k]$

$$D^+EV(x(t), t, r(t)) = E\mathscr{L}V(x(t), t, r(t)). \qquad (3.2)$$

Let $W(t) = e^{\lambda t} EV(x(t), t, i)$, we have for $t \in (t_{k-1}, t_k]$

$$D^+W(t) = \lambda e^{\lambda t} EV(x(t), t, i) + e^{\lambda t} D^+ E\mathscr{L}V(x(t), t, i). \qquad (3.3)$$

From (*iii*), we have

$$W(t_k^+) = e^{\lambda t_k} EV(x(t_k^+), t_k^+, i) \le d_{ik} W(t_k) + \bar{d}_{ik} e^{\lambda \tau} \sup_{-\tau \le \theta \le 0} W(t_k + \theta). \tag{3.4}$$

Taking $M > 0$ such that

$$\sup_{-\tau \le \theta \le 0} W(\theta) < \frac{M}{q}, \tag{3.5}$$

we can claim that for $t \ge -\tau$

$$W(t) < M. \tag{3.6}$$

It is easy to see that $W(t) < M$ for $t \in [-\tau, 0]$. Now, we shall prove that

$$W(t) < M, t \in (0, t_1]. \tag{3.7}$$

Otherwise, there exists a $t^* \in (0, t_1]$ such that

$$W(t^*) = M, W(t) < M, -\tau < t < t^*. \tag{3.8}$$

In view of the continuity of $W(t)$ in $[0, t_1]$, there exists a $t^{**} \in (0, t^*)$ such that

$$W(t^{**}) = \frac{M}{q}, W(t) > \frac{M}{q}, \ t \in (t^{**}, t^*]. \tag{3.9}$$

For $t \in [t^{**}, t^*]$, $\theta \in [-\tau, 0]$, we have

$$qW(t) > W(t + \theta). \tag{3.10}$$

Then we obtain

$$qe^{\lambda \tau} \left[\max_{1 \le i \le N} EV(x(t), t, i) \right] \ge qe^{-\lambda(t-\tau)} W(t) > e^{-\lambda(t-\tau)} W(t + \theta)$$
$$\ge \min_{1 \le i \le N} EV(x(t + \theta), t + \theta, i). \tag{3.11}$$

Together with (3.3) and (ii), for $t \in [t^{**}, t^*]$, we have

$$D^+ W(t) \le \lambda e^{\lambda t} EV(x(t), t, i) + \eta_i e^{\lambda t} EV(x(t), t, i)$$
$$= (\lambda + \eta_i) e^{\lambda t} EV(x(t), t, i) = (\lambda + \eta_i) W(t). \tag{3.12}$$

Thus

$$M = W(t^*) \le W(t^{**}) e^{(\lambda + \eta_i)(t^* - t^{**})} = \frac{M}{q} e^{(\lambda + \eta_i)(t^* - t^{**})} < \frac{M}{q} e^{(\lambda + \eta_i) t_1} \le M. \tag{3.13}$$

This is a contradiction. Hence (3.7) holds. From (iii), we obtain

$$W(t_1^+) \le d_{i1} W(t_1) + \bar{d}_{i1} e^{\lambda \tau} \sup_{-\tau \le \theta \le 0} W(t_1 + \theta) < \frac{M}{q} < M. \tag{3.14}$$

Next, we shall show that

$$W(t) < M, \quad t \in (t_1, t_2]. \tag{3.15}$$

If it does not hold, there exists a $t_1^* \in (t_1, t_2]$ such that

$$W(t_1^*) = M, W(t) < M, \quad t \in [-\tau, t_1^*]. \tag{3.16}$$

In view of the continuity of $W(t)$ in $(t_1, t_2]$, there exists a $t_1^{**} \in (t_1, t_1^*)$ such that

$$W(t_1^{**}) = \frac{M}{q}, W(t) > \frac{M}{q}, \quad t \in (t_1^{**}, t_1^*]. \tag{3.17}$$

For $t \in [t_1^{**}, t_1^*], \theta \in [-\tau, 0]_{,,}$ we have $qW(t) > W(t + \theta)$. It follows from (3.3) and (ii) that for $t \in [t_1^{**}, t_1^*]$

$$D^+W(t) \le (\lambda + \eta_i)W(t). \tag{3.18}$$

Then, we have

$$M = W(t_1^*) \le W(t_1^{**})e^{(\lambda+\eta_i)(t_1^*-t_1^{**})} = \frac{M}{q}e^{(\lambda+\eta_i)(t_1^*-t_1^{**})} < \frac{M}{q}e^{(\lambda+\eta_i)(t_2-t_1)} \le M, \tag{3.19}$$

which is a contradiction. Thus, (3.15) holds. By induction, we can prove that for $k = 1, 2, \ldots$

$$W(t) < M, \quad t \in (t_{k-1}, t_k]. \tag{3.20}$$

Therefore, we have for $t \ge -\tau$

$$W(t) < M. \tag{3.21}$$

From (i) and the above inequality, we have

$$c_1 E|x(t)|^p \le EV(t) \le Me^{-\lambda t}, \tag{3.22}$$

which implies that

$$E|x(t)|^p \le \frac{M}{c_1}e^{-\lambda t}. \tag{3.23}$$

The proof of Theorem 3.1 is complete.

Remark 3.1. In Theorem 3.1, the zero solution of hybrid stochastic functional differential equations without impulses is allowed to be unstable. In this case, the delayed impulses are key in stabilizing the hybrid stochastic equations. It requires the nearest impulse time interval must be sufficiently small and the maximal impulsive gain $\max\limits_{i\in S, 1\le k<+\infty}\{d_{ik} + \bar{d}_{ik}\} < 1$.

Theorem 3.2. Assume that $V \in \mathscr{C}_1^2(R^n \times [-\tau, \infty) \times S; R^+)$ and constants $p > 0, c_1 > 0, c_2 > 0, d_{ik} \ge 0, \bar{d}_{ik} \ge 0, d_{ik}^2 + \bar{d}_{ik}^2 \ne 0, \gamma_i > 0, \mu > 0, \lambda > 0, i \in S, k = 1, 2, \ldots$ such that

(i) $c_1|x|^p \le V(t, x) \le c_2|x|^p$ for all $(t, x) \in [-\tau, \infty) \times R^n$;

(ii) for all $t \in (t_{k-1}, t_k]$ and $i \in S, \varepsilon > 0$ $E\mathscr{L}V(\phi, t, i) \le -\gamma_i EV(\phi(0), t, i)$ whenever $E[\min\limits_{1\le i\le N} V(\phi(\theta), t + \theta, i)] \le qe^{\lambda\tau}E[\max\limits_{1\le i\le N} EV(\phi(0), t, i)]$;

(iii) for all $i \in S$, $EV(\phi(0) + I_k(\phi(0), \phi(\theta), t_k, i), t_k^+, i) \le d_{ik}EV(\phi(0), t_k, i) + \bar{d}_{ik} \sup\limits_{-\tau\le\theta\le0} EV(\phi(\theta), t_k, i)$;

(iv) $\inf\limits_{1\le k<+\infty}\{t_k - t_{k-1}\} \ge \mu$;

(v) for any $i \in S, \gamma_i - \lambda > \frac{\ln q}{\mu}$,

then the zero solution of Equation (2.1) is pth moment exponentially stable, where

$$\phi = \{\phi(\theta)| - \tau \le \theta \le 0\} \in PC_{\mathscr{F}_t}^p([-\tau, 0]; R^n), q = \frac{1}{\max\limits_{i\in S, 1\le k<+\infty}\{d_{ik} + \bar{d}_{ik}e^{\lambda\tau}\}} > 1.$$

Proof. Since (v) holds, we can choose sufficiently small $\varepsilon > 0$ such that $\gamma_i - \lambda \geq \dfrac{\ln(q + \varepsilon)}{\mu}$ for all $i \in S$. Let $W(t) = e^{\lambda t} EV(x(t), t, i)$, we have for $t \in (t_{k-1}, t_k]$

$$D^+W(t) = \lambda e^{\lambda t}EV(x(t), t, i) + e^{\lambda t}D^+E\mathscr{L}V(x(t), t, i). \tag{3.24}$$

From (iii), we have

$$W(t_k^+) = e^{\lambda t_k}EV(x(t_k^+),\ t_k^+,\ i) \leq d_{ik}W(t_k) + \bar{d}_{ik}e^{\lambda \tau} \sup_{-\tau \leq \theta \leq 0} W(t_k + \theta). \tag{3.25}$$

Taking $M > 0$ such that

$$\sup_{-\tau \leq \theta \leq 0} W(\theta) < \frac{M}{q+\varepsilon}, \tag{3.26}$$

we shall show that for $t \geq -\tau$

$$W(t) < M. \tag{3.27}$$

It is easy to see that $W(t) < M$ for $t \in [-\tau, 0]$. Now, we shall prove that

$$W(t) < M, t \in (0,\ t_1]. \tag{3.28}$$

If it does not hold, there exists a $t^* \in (0, t_1]$ such that

$$W(t^*) = M, W(t) < M, -\tau < t < t^*. \tag{3.29}$$

In view of the continuity of $W(t)$ in $[0, t_1]$, there exists a $t^{**} \in [0, t^*)$ such that

$$W(t^{**}) = \frac{M}{q+\varepsilon}, W(t) > \frac{M}{q+\varepsilon},\quad t \in (t^{**},\ t^*]. \tag{3.30}$$

For $t \in [t_\varepsilon^{**},\ t^*], \theta \in [-\tau,\ 0]$

$$(q + \varepsilon)W(t) > W(t + \theta). \tag{3.31}$$

Then

$$(q + \varepsilon)e^{\lambda \tau} \left[\max_{1 \leq i \leq N} EV(x(t),\ t,\ i) \right] \geq \min_{1 \leq i \leq N} EV(x(t + \theta),\ t + \theta,\ i). \tag{3.32}$$

Together with (3.24) and (ii), for $t \in [t^{**},\ t^*]$, we have

$$D^+W(t) \leq \lambda e^{\lambda t}EV(x(t), t, i) + e^{\lambda t}E\mathscr{L}V(x(t), t, i) \leq (\lambda - \gamma_i)W(t) \leq 0. \tag{3.33}$$

Thus

$$M = W(t^*) \leq W(t^{**}) = \frac{M}{q+\varepsilon} < M. \tag{3.34}$$

This is a contradiction. Next, we shall show that

$$W(t_1) \leq \frac{M}{q+\varepsilon}. \tag{3.35}$$

If it does not hold, we have

$$W(t_1) > \frac{M}{q+\varepsilon}. \tag{3.36}$$

Since the continuity of $W((t)$ in $[0, t_1]$, there exists a $\bar{t} \in [0, t_1)$ such that

$$W(\bar{t}_\varepsilon) = \frac{M}{q+\varepsilon}, W(t) > \frac{M}{q+\varepsilon}, \ t \in (\bar{t}, t_1]. \tag{3.37}$$

For $t \in [\bar{t}, t_1], \theta \in [-\tau, 0]$, we have

$$(q + \varepsilon)W(t) > W(t + \theta). \tag{3.38}$$

By (3.24) and (ii), for $t \in [\bar{t}, t_1], \theta \in [-\tau, 0]$, we have

$$D^+W(t) \le (\lambda - \gamma_i)W(t) \le 0. \tag{3.39}$$

Thus

$$W(t_1) \le W(\bar{t}) = \frac{M}{q+\varepsilon}, \tag{3.40}$$

which is a contradiction. It follows from (3.25), (3.35) and (3.28) that

$$W(t_1^+) \le d_{i1} W(t_1) + \bar{d}_{i1} e^{\lambda \tau} \sup_{-\tau \le \theta \le 0} W(t_1 + \theta) \le \frac{(d_{i1} + \bar{d}_{i1} e^{\lambda \tau})M}{q+\varepsilon} < M. \tag{3.41}$$

Furthermore, we can prove that

$$W(t) < M, \quad t \in (t_1, t_2]. \tag{3.42}$$

Indeed, there exists a $\bar{t}_1 \in (t_1, t_2]$ such that

$$W(\bar{t}_1) = M, W(t) < M, \ t \in [-\tau, \bar{t}_1). \tag{3.43}$$

If $W(t) > \frac{M}{q+\varepsilon}$ for $t \in (t_1, \bar{t}_1)$. Then $(q + \varepsilon)W(t) > W(t + \theta)$ for $t \in (t_1, \bar{t}_1), \theta \in [-\tau, 0]$. Thus by (3.24) and (ii), for $t \in (t_1, \bar{t}_1)$, we have $D^+ W(t) \le (\lambda - \gamma)W(t) \le 0$. It follows that

$$W(\bar{t}_1) \le W(t_1^+) < M. \tag{3.44}$$

This is a contradiction. If there exists a $\bar{\bar{t}}_1 \in (t_1, \bar{t}_1)$ such that

$$W(\bar{\bar{t}}_1) \le \frac{M}{q+\varepsilon}. \tag{3.45}$$

In view of the continuity of $W(t)$ in $(t_1, t_2]$, there exists a $\bar{\bar{t}}_1' \in [\bar{\bar{t}}_1, \bar{t}_1)$ such that

$$W(\bar{\bar{t}}_1') = \frac{M}{q+\varepsilon}, W(t) > \frac{M}{q+\varepsilon}, t \in (\bar{\bar{t}}_1', \bar{t}_1]. \tag{3.46}$$

Then for $t \in [\bar{\bar{t}}_1', \bar{t}_1], \ \theta \in [-\tau, 0], \ (q + \varepsilon) \ W(t) > W(t + \theta)$. Thus for $W(\bar{t}_1) \le W(\bar{\bar{t}}_1) = \frac{M}{q+\varepsilon} < M, D^+W(t) \le (\lambda - \gamma)W(t) \le 0$. It follows that

$$W(\bar{t}_1) \le W(\bar{\bar{t}}_1') = \frac{M}{q+\varepsilon} < M, \tag{3.47}$$

which leads to a contradiction. Moreover, we can conclude that $W(t_2) \le \frac{M}{q+\varepsilon}$. If this does not hold, we have $W(t_2) > \frac{M}{q+\varepsilon}$. To prove the conclusion, two cases are to be considered.

Case (*i*). For $t \in (t_1, t_2], W(t) > \frac{M}{q+\varepsilon}$. From (3.26), (3.28) and (3.43), we have for $t \in (t_1, t_2], \theta \in [-\tau, 0]$

$$(q + \varepsilon)W(t) > W(t + \theta). \tag{3.48}$$

Then by (3.24) and (ii), for $t \in (t_1, t_2]$

$$D^+W(t) \leq (\lambda - \gamma_i) W(t). \tag{3.49}$$

Thus

$$W(t_2) \leq W\left(t_1^+\right) e^{(\lambda - \gamma_i)(t_2 - t_1)} < M e^{(\lambda - \gamma_i)(t_2 - t_1)} \leq \frac{M}{q+\varepsilon}, \tag{3.50}$$

which leads to a contradiction.

Case (ii). There exists a $\tilde{t}_{1\varepsilon} \in (t_1, t_2]$ such that $W\left(\tilde{t}_1\right) \leq \frac{M}{q+\varepsilon}$. Since $W(t_2) > \frac{M}{q+\varepsilon}$ and in view of the continuity of $W(t)$ in $(t_1, t_2]$, there exists a $\tilde{t}_1^* \in \left[\tilde{t}_1, t_2\right]$ such that

$$W\left(\tilde{t}_1^*\right) = \frac{M}{q+\varepsilon}, W(t) > \frac{M}{q+\varepsilon}, \quad t \in (\tilde{t}_1^*, t_2]. \tag{3.51}$$

For $t \in (\tilde{t}_1^*, t_2]$, $\theta \in [-\tau, 0]$, we have

$$\left(q + \varepsilon\right) W(t) > W(t + \theta). \tag{3.52}$$

It follows from (3.24) and (ii) that for $t \in (\tilde{t}_1^*, t_2]$

$$D^+W(t) \leq (\lambda - \gamma_i) W(t) \leq 0. \tag{3.53}$$

Thus

$$W(t_2) \leq W\left(\tilde{t}_1^*\right) = \frac{M}{q}. \tag{3.54}$$

This is a contradiction. By induction, we can prove that for $k = 1, 2, \ldots$

$$W(t) < M, \quad t \in (t_{k-1}, t_k]. \tag{3.55}$$

Therefore, we have for $t \geq -\tau$

$$W(t) < M. \tag{3.56}$$

The proof of Theorem 3.1 is complete.

Remark 3.2. In Theorem 3.2, it is permitted that the maximal impulsive gain $\max\limits_{i \in S, 1 \leq k < +\infty} \left\{d_{ik} + \bar{d}_{ik}\right\} \geq 1$. This means that the hybrid stochastic equation not only can achieve exponential stability but also is exponential stability with delayed impulses. In this case, it requires that the minimal impulse time interval must be sufficiently large such that the hybrid stochastic differential equations with delayed impulses can make keep its stability property.

4 Some consequences

In the following, we shall apply the above new results to a class of linear impulsive hybrid stochastic systems by using Lyapunov function and M-matrix method.

Consider the following n-dimensional impulsive hybrid stochastic delay differential equation

$$\begin{cases} dx(t) = [A(r(t))x(t) + B(r(t))x(t - \tau(t))]\,dt \\ \quad + [C(r(t))x(t) + D(r(t))x(t - \tau(t))]\,d\omega(t), \quad t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k), x(t_k - \tau(t_k)), t_k, r(t_k)), \quad k = 1, 2, \dots, \\ x(t) = \xi, \quad t \in [-\tau, 0], \end{cases} \quad (4.1)$$

where $0 \leq \tau(t) \leq \tau$ is continuous, τ is a positive constant. For convenience, we denote $A(r(t)) = A_i$, $B(r(t)) = B_i$, $C(r(t)) = C_i$, $D(r(t)) = D_i$, where $A_i = (a_{uv}(i))_{n \times n}$, $B_i = (b_{uv}(i))_{n \times n}$, $C_i = (c_{uv}(i))_{n \times n}$, $D_i = (d_{uv}(i))_{n \times n}$.

Theorem 4.1. Assume that there exist symmetric positive definite matrices Q_i and constants $d_{ik} \geq 0$, $\bar{d}_{ik} \geq 0$, $d_{ik}^2 + \bar{d}_{ik}^2 \neq 0$, $\delta > 0$, $\lambda > 0$, $\rho_i > 0$, $\sigma_i > 0$, $\bar{\eta}_i \geq 0$, η_i, $i \in S$, $k = 1, 2, \dots$ such that

(i) for all $i \in S$

$$\Gamma_i = Q_i A_i + A_i^T Q_i + \rho_i Q_i^2 + C_i^T Q_i C_i + \sigma_i C_i^T C_i + \sum_{j=1}^{N} \gamma_{ij} Q_j - \eta_i Q \leq 0, \quad (4.2)$$

and

$$\bar{\Gamma}_i = \rho_i^{-1} B_i^T B_i + \sigma_i^{-1} D_i^T Q_i^2 D_i + D_i^T Q_i D_i - \bar{\eta}_i Q_i \leq 0, \quad (4.3)$$

where Γ_i, $\bar{\Gamma}_i \leq 0$, $0 \leq i \leq N$ mean that matrices Γ_i, $\bar{\Gamma}_i$ are negative semi-definite;

(ii) for all $i \in S$

$$E\left[\left(x(t_k + I_k(x(t_k), x(t_k - \tau(t_k)), t_k, i)\right)^T Q_i (x(t_k + I_k(x(t_k), x(t_k - \tau(t_k)), t_k, i))\right]$$
$$\leq d_{ik} E\left[x^T(t_k) Q_i x(t_k)\right] + \bar{d}_{ik} E\left[x^T(t_k - \tau(t_k)) Q_i x(t_k - \tau(t_k))\right]; \quad (4.4)$$

(iii) $\displaystyle\sup_{1 \leq k < +\infty}\{t_k - t_{k-1}\} \leq \delta$;

(iv) for all $i \in S$, $\eta_i + \lambda + \dfrac{\bar{\eta}_i q \alpha_2^2 e^{\lambda \tau}}{\alpha_1^2} \leq \dfrac{\ln q}{\delta}$,

then the zero solution of Equation (4.1) is exponentially stable in the mean square, where $\alpha_1 = \displaystyle\min_{1 \leq i \leq N} \lambda_{\min}(Q_i)$, $\alpha_2 = \displaystyle\max_{1 \leq i \leq N} \lambda_{\max}(Q_i)$, $q = \dfrac{1}{\displaystyle\max_{i \in S, 1 \leq k < +\infty}\left\{d_{ik} + \bar{d}_{ik} e^{\lambda \tau}\right\}} > 1$.

Proof. We define $V \in \mathscr{C}^{2,1}\left(R^n \times [-\tau, \infty) \times S; R^+\right)$ by $V(x, t, i) = x^T(t) Q_i x(t)$. Clearly

$$\alpha_1 |x|^2 \leq V(x, t, i) \leq \alpha_2 |x|^2. \quad (4.5)$$

Then for $t \in (t_{k-1}, t_k]$

$$\begin{aligned}
\mathscr{L}V(t, \phi) &= 2\phi^T(0) Q_i [A_i \phi(0) + B_i \phi(-\tau(t))] + [C_i \phi(0) \\
&\quad + D_i \phi(-\tau(t))]^T \times Q_i [C_i \phi(0) + D_i \phi(-\tau(t))] + \sum_{j=1}^{N} \gamma_{ij} \phi^T(0) Q_j \phi(0) \\
&= \phi^T(0)\left(Q_i A_i + A_i^T Q_i\right)\phi(0) + 2\phi^T(0) Q_i B_i \phi(-\tau(t)) \\
&\quad + \phi^T(0) C_i^T Q_i C_i \phi(0) + \phi^T(0) C_i^T Q_i D_i \phi(-\tau(t)) + \phi^T(-\tau(t)) D_i^T Q_i C_i \phi(0) \\
&\quad + \phi^T(-\tau(t)) D_i^T Q_i D_i \phi(-\tau(t)) + \sum_{j=1}^{N} \gamma_{ij} \phi^T(0) Q_j \phi(0).
\end{aligned} \quad (4.6)$$

In view of for any vectors $x, y \in R^n$, scalar $\varepsilon > 0$, the following inequality holds

$$2x^T y \leq \varepsilon x^T x + \varepsilon^{-1} y^T y, \tag{4.7}$$

then it follows that

$$2\phi^T(0) Q_i B_i \phi(-\tau(t)) \leq \rho_i \phi^T(0) Q_i^2 \phi(0) + \rho_i^{-1} \phi(-\tau(t)) B_i^T B_i \phi(-\tau(t)) \tag{4.8}$$

and

$$\phi^T(0) C_i^T Q_i D_i \phi(-\tau(t)) + \phi^T(-\tau(t)) D_i^T Q_i C_i \phi(0)$$
$$\leq \sigma_i \phi^T(0) C_i^T C_i \phi(0) + \sigma_i^{-1} \phi^T(-\tau(t)) D_i^T Q_i^2 D_i \phi(-\tau(t)). \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.6) we can derive that

$$\mathscr{L} V(\phi, t, i) \leq \phi^T(0) \Big(Q_i A_i + A_i^T Q_i + \rho_i Q^2 + C_i^T Q_i C_i + \sigma_i C_i^T C_i$$
$$+ \sum_{j=1}^{N} \gamma_{ij} Q_j - \eta_i Q_i \Big) \phi(0) + \phi^T(-\tau(t)) \Big(\rho_i^{-1} B_i^T B_i + \sigma_i^{-1} D_i^T Q_i^2 D_i$$
$$+ D_i^T Q_i D_i - \bar{\eta}_i Q_i \Big) \phi(-\tau(t)) + \eta_i \phi^T(0) Q_i \phi(0) + \bar{\eta}_i \phi^T(-\tau(t)) Q_i \phi(-\tau(t)) \tag{4.10}$$
$$\leq \eta_i \phi^T(0) Q_i \phi(0) + \bar{\eta}_i \phi^T(-\tau(t)) Q_i \phi(-\tau(t)).$$

Then

$$E \mathscr{L} V(\phi, t, i) \Big] \leq \eta_i E V(\phi(0), t, i) + \bar{\eta}_i \alpha_2 E |\phi(-\tau(t))|^2. \tag{4.11}$$

Next, if for $\forall \theta \in [-\tau, 0]$, $E\left[\min_{1 \leq i \leq N} V(\phi(\theta), t, i) \right] < q e^{\lambda \tau} E\left[\max_{1 \leq i \leq N} V(\phi(\theta), t, i) \right]$,

we have

$$E|\phi(\theta)|^2 < \frac{q \alpha_2 e^{\lambda \tau}}{\alpha_1} E|\phi(0)|^2, \forall \theta \in [-\tau, 0]. \tag{4.12}$$

Thus

$$E\left[\mathscr{L} V(\phi, t, i) \right] \leq \eta_i E V(\phi(0), t, i) + \frac{\bar{\eta}_i q \alpha_2^2 e^{\lambda \tau}}{\alpha_1} E|\phi(0)|^2 \leq \left(\eta_i + \frac{\bar{\eta}_i q \alpha_2^2 e^{\lambda \tau}}{\alpha_1^2} \right) E V(\phi(0), t, i). \tag{4.13}$$

For $t = t_k$, it follows from (*ii*) that

$$E V\big(\phi(0) + I_k(\phi, \phi(-\tau(t_k))), t_k, i), t_k^+, i\big)$$
$$\leq d_{ik} E V(\phi(0), t_k, i) + d_{ik} E V(\phi(-\tau(t_k)), t_k - \tau(t_k), i) \tag{4.14}$$

Consequently, the conclusions follow from Theorem 3.1. This completes the proof.

Theorem 4.2. Assume that exist symmetric positive definite matrices Q_i and constants $d_{ik} \geq 0, \bar{d}_{ik} \geq 0, d_{ik}^2 + \bar{d}_{ik}^2 \neq 0, \varepsilon_i > 0, \kappa_i > 0, \zeta_i > 0, \bar{\zeta}_i \geq 0, \mu > 0, \lambda > 0, i \in S, k = 1, 2, \ldots$ such that (4.4) holds and the following conditions hold:

(i) for all $i \in S$

$$\Theta_i = Q_i A_i + A_i^T Q_i + \varepsilon_i Q_i^2 + C_i^T Q_i C_i + \kappa_i C_i^T C_i + \sum_{j=1}^{N} \gamma_{ij} Q_j + \zeta_i Q_i \leq 0 \tag{4.15}$$

and

$$\bar{\Theta}_i = \varepsilon_i^{-1} B_i^T B_i + \kappa_i^{-1} D_i^T Q_i^2 D_i + D_i^T Q_i D_i - \bar{\zeta}_i Q_i \leq 0; \tag{4.16}$$

(ii) $\inf\limits_{1 \le k < +\infty} \{t_k - t_{k-1}\} \ge \mu$;

(iii) for all $i \in S$, $\zeta_i - \frac{\bar{\zeta}_i q \alpha_2^2 e^{\lambda \tau}}{\alpha_1^2} - \lambda > \frac{\ln q}{\mu}$,

then the zero solution of Equation (4.1) is exponentially stable in the mean square,
where $q = \max\limits_{i \in S, 1 \le k < +\infty} \left\{ d_{ik} + \bar{d}_{ik} e^{\lambda \tau} \right\} \ge 1$.

Proof. By (iii), we choose sufficiently small $\varepsilon > 0$ such that
$\zeta_i - \frac{\bar{\zeta}_i (q + \varepsilon) \alpha_2^2 e^{\lambda \tau}}{\alpha_1^2} - \lambda > \frac{\ln q}{\mu}$. We define

$V(x, t, i) = x^T(t) Q_i x(t)$. Similar to the proof of Theorem 4.1, we get for $t \in (t_{k-1}, t_k]$

$$E \mathscr{L} V(\phi, t, i) \le \zeta_i EV(\phi(0), t, i) + \bar{\zeta}_i \alpha_2 E |\phi(-\tau(t))|^2. \tag{4.17}$$

For $\forall \theta \in [-\tau, 0]$, $E \left[\min\limits_{1 \le i \le N} V(\phi(\theta), t, i) \right] \le (q + \varepsilon) e^{\lambda \tau} E \left[\max\limits_{1 \le i \le N} V(\phi(0), t, i) \right]$, we
have

$$E |\phi(\theta)|^2 \le \frac{(q + \varepsilon) \alpha_2 e^{\lambda \tau}}{\alpha_1} E |\phi(0)|^2. \tag{4.18}$$

Thus

$$E \left[\mathscr{L} V(\phi, t, i) \right] \le \left(-\zeta_i + \frac{\bar{\zeta}_i (q+\varepsilon) \alpha_2^2 e^{\lambda \tau}}{\alpha_1^2} \right) EV(\phi(0), t, i). \tag{4.19}$$

Therefore, the conclusions follow from Theorem 3.2.

In the following, we shall establish tractable exponential stability conditions. To this
end, we take

$Q_i = r_i I$, $\rho_i = \varepsilon_i = \frac{\|B_i\|}{r_i}$, $\sigma_i = \kappa_i = \frac{r_i \|D_i\|}{\|C_i\|}$, $\eta_i = \frac{1}{r_i} \left[\lambda_{\max} \left(r_i A_i + r_i A_i^T + \sum_{j=1}^N \gamma_{ij} r_j I \right) + r_i \|B_i\| + r_i \|C_i\|^2 + r_i \|C_i\| \|D_i\| \right]$,

$\zeta_i = -\frac{1}{r_i} \left[\lambda_{\max} \left(r_i A_i + r_i A_i^T + \sum_{j=1}^N \gamma_{ij} r_j I \right) + r_i \|B_i\| + r_i \|C_i\|^2 + r_i \|C_i\| \|D_i\| \right]$, $\bar{\eta}_i = \bar{\zeta}_i = \|B_i\| + \|C_i\| \|D_i\| + \|D_i\|^2$.

Corollary 4.1. Assume that there exist constants
$d_{ik} \ge 0$, $\bar{d}_{ik} \ge 0$, $d_{ik} + \bar{d}_{ik} \ne 0$, $\delta > 0$, $\lambda > 0$, $r_i > 0$, $i \in S$, $k = 1, 2, \ldots$ such that
(i) for all $i \in S$

$$\begin{aligned} E \left[(x(t_k) + I_k(x(t_k), x(t_k - \tau(t_k)), t_k, i))^T (x(t_k) + I_k(x(t_k), x(t_k - \tau(t_k)), t_k, i)) \right] \\ \le d_{ik} E \left[x^T(t_k) x(t_k) \right] + \bar{d}_{ik} E \left[x^T(t_k - \tau(t_k)) x(t_k - \tau(t_k)) \right]; \end{aligned} \tag{4.20}$$

(ii) $\sup\limits_{1 \le k < +\infty} \{t_k - t_{k-1}\} \le \delta$;

(iii) for all $i \in S$

$$\begin{aligned} \frac{1}{r_i} &\left[\lambda_{\max} \left(r_i A_i + r_i A_i^T + \sum_{j=1}^N \gamma_{ij} r_j I \right) + r_i \|B_i\| + r_i \|C_i\|^2 + r_i \|C_i\| \|D_i\| \right] \\ &+ \lambda + \frac{q \alpha_2^2 e^{\lambda \tau}}{\alpha_1^2} \left[\|B_i\| + \|C_i\| \|D_i\| + \|D_i\|^2 \right] \le \frac{\ln q}{\delta}, \end{aligned} \tag{4.21}$$

then the zero solution of Equation (4.1) is exponentially stable in the mean square,
where $\alpha_1 = \min\limits_{1 \le i \le N} r_i$, $\alpha_2 = \max\limits_{1 \le i \le N} r_i$, $q = \frac{1}{\max\limits_{i \in S, 1 \le k < +\infty} \left\{ d_{ik} + \bar{d}_{ik} e^{\lambda \tau} \right\}} > 1$.

Corollary 4.2. Assume that there exist constants
$d_{ik} \ge 0$, $\bar{d}_{ik} \ge 0$, $d_{ik} + \bar{d}_{ik} \ne 0$, $\mu > 0$, $\lambda > 0$, $r_i > 0$, $i \in S$, $k = 1, 2, \ldots$ such that
(4.20) holds and the following conditions are satisfied:

(i) $\displaystyle\inf_{1 \leq k < +\infty} \{t_k - t_{k-1}\} \geq \mu$;

(ii) for all $i \in S$

$$\frac{1}{r_i}\left[\lambda_{\max}\left(r_i A_i + r_i A_i^T + \sum_{j=1}^{N} \gamma_{ij} r_j I\right) + r_i \|B_i\| + r_i\|C_i\|^2 + r_i\|C_i\|\|D_i\|\right] \tag{4.22}$$
$$+\lambda + \frac{q\alpha_2^2 e^{\lambda\tau}}{\alpha_1^2}\left[\|B_i\| + \|C_i\|\|D_i\| + \|D_i\|^2\right] + \frac{\ln q}{\mu} < 0,$$

then the zero solution of Equation (4.1) is exponentially stable in the mean square,
where $q = \displaystyle\max_{i \in S, 1 \leq k < +\infty}\left\{d_{ik} + \bar{d}_{ik}e^{\lambda\tau}\right\} \geq 1$.

Next, we apply Corollaries 4.1 and 4.2 to establish some very useful criteria in terms
of M matrix which can be verified much more easily. If A is a vector or matrix, by A
$\gg 0$ we mean all elements of A are positive. If A_1 and A_2 are vectors or matrices with
same dimensions we write $A_1 \gg A_2$ if and only if $A_1 - A_2 \gg 0$. Moreover, we also
adopt here the traditional natation by letting

$$Z^{N \times N} = \left\{A = \left(a_{ij}\right)_{N \times N} | a_{ij} \leq 0, i \neq j\right\}. \tag{4.23}$$

Definition 4.1. (see [12,28]) A square matrix $A = (a_{ij})_{N \times N}$ is called a nonsingular M
matrix if A can be expressed in the form $A = sI - B$ with $s > \rho(B)$ while all the elements
of B are nonnegative, where I is the identity matrix and $\rho(B)$ the spectral radius of B.

Remark 4.1. If A is a nonsingular M-matrix, then A has nonpositive off-diagonal and
positive diagonal entries, that is

$$a_{ii} > 0, \quad \text{while} \quad a_{ij} \leq 0, \quad i \neq j.$$

Corollary 4.3. There exist constants
$d_{ik} \geq 0$, $\bar{d}_{ik} \geq 0$, $d_{ik} + \bar{d}_{ik} \neq 0$, $i \in S$, $k = 1, 2, \ldots$ such that (4.20) holds. If there
exists $\lambda > 0$ such that $\Xi - \Gamma$ is a nonsingular and for any $i \in S$

$$r_i\left[\lambda + \frac{q\alpha_2^2 e^{\lambda\tau}}{\alpha_1^2}\left(\|B_i\| + \|C_i\|\|D_i\| + \|D_i\|^2\right)\right] \leq 1, \tag{4.24}$$

then the zero solution of Equation (4.1) is exponentially stable in the mean square,
where Ξ is the diagonal matrix

$$\Xi = \text{diag}\left[-\lambda_{\max}\left(A_1 + A_1^T\right) - \|B_1\| - \|C_1\|^2 - \|C_1\|\|D_1\|\right.$$
$$\left. + \frac{\ln q}{\delta}, \ldots, -\lambda_{\max}\left(A_N + A_N^T\right) - \|B_N\| - \|C_N\|^2 - \|C_N\|\|D_N\| + \frac{\ln q}{\delta}\right],$$
$$(r_1, \ldots, r_N) = (\Xi - \Gamma)^{-1}\mathbf{1} \gg 0, \mathbf{1} = (1, \ldots, 1)^T, \alpha_1 = \min_{1 \leq i \leq N} r_i, \alpha_2 = \max_{1 \leq i \leq N} r_i, q = \frac{1}{\max\limits_{i \in S, 1 \leq k < +\infty}\left\{d_{ik} + \bar{d}_{ik}e^{\lambda\tau}\right\}} > 1.$$

Proof. We conclude that all the elements of $(\Xi - \Gamma)^{-1}$ are nonnegative. So we have
(4.24) holds, namely all r_i are positive. Note

$$(-\Xi + \Gamma)\mathbf{r} = -\mathbf{1}, \tag{4.25}$$

that is

$$\lambda_{\max}\left(A_i + A_i^T\right)r_i + \sum_{j=1}^{N}\gamma_{ij}r_j + \left(\|B_i\| + \|C_i\|^2 + \|C_i\|\|D_i\| - \frac{\ln q}{\delta}\right)r_i = -1. \tag{4.26}$$

Then we have for $i \in S$

$$
\begin{aligned}
\frac{1}{r_i} & \left[\lambda_{\max} \left(r_i A_i + r_i A_{i^T} + \sum_{j=1}^{N} \gamma_{ij} r_j I \right) + r_i \|B_i\| + r_i \|C_i\|^2 \right. \\
& \left. + r_i \|C_i\| \|D_i\| - \frac{r_i \ln q}{\delta} \right] + \frac{q \alpha_{22} e^{\lambda \tau}}{\alpha_{12}} \left[\|B_i\| + \|C_i\| \|D_i\| + \|D_i\|^2 \right] \\
& = \frac{1}{r_i} \left[\lambda_{\max} (r_i A_i + r_i A_{i^T}) + \sum_{j=1}^{N} \gamma_{ij} r_j + r_i \|B_i\| + r_i \|C_i\|^2 \right. \\
& \left. + r_i \|C_i\| \|D_i\| - \frac{r_i \ln q}{\delta} \right] + \lambda + \frac{q \alpha_{22} e^{\lambda \tau}}{\alpha_{12}} \left(\|B_i\| + \|C_i\| \|D_i\| + \|D_i\|^2 \right) \\
& = -\frac{1}{r_i} + \lambda + \frac{q \alpha_{22} e^{\lambda \tau}}{\alpha_{12}} \left(\|B_i\| + \|C_i\| \|D_i\| + \|D_i\|^2 \right) \leq 0.
\end{aligned}
\tag{4.27}
$$

The required conclusion follows from Corollary 4.1.

Corollary 4.4. There exist constants $d_{ik} \geq 0$, $\bar{d}_{ik} \geq 0$, $d_{ik} + \bar{d}_{ik} \neq 0$, $i \in S$, $k = 1, 2, \ldots$ such that (4.17) holds. If there exists $\lambda > 0$ such that $\bar{\Xi} - \Gamma$ is a nonsingular and for any $i \in S$

$$
r_i \left[\lambda + \frac{q \alpha_2^2 e^{\lambda \tau}}{\alpha_1^2} \left(\|B_i\| + \|C_i\| \|D_i\| + \|D_i\|^2 \right) \right] < 1,
\tag{4.28}
$$

then the zero solution of Equation (4.1) is exponentially stable in the mean square, where $\overline{\Xi}$ is the diagonal matrix denoted by

$$
\begin{aligned}
\bar{\Xi} = \text{diag} & \left[-\lambda_{\max} \left(A_1 + A_1^T \right) - \|B_1\| - \|C_1\|^2 - \|C_1\| \|D_1\| \right. \\
& \left. -\frac{\ln q}{\mu}, \ldots, -\lambda_{\max} \left(A_N + A_N^T \right) - \|B_N\| - \|C_N\|^2 - \|C_N\| \|D_N\| - \frac{\ln q}{\mu} \right],
\end{aligned}
$$

$(r_1, \ldots, r_N) = (\bar{\Xi} - \Gamma)^{-1} \mathbf{1} \gg 0$, $\alpha_1 = \min\limits_{1 \leq i \leq N} r_i$, $\alpha_2 = \max\limits_{i \in S, 1 \leq i \leq N} r_i$, $q = \max\limits_{1 \leq k < +\infty} \left\{ d_{ik} + \bar{d}_{ik} e^{\lambda \tau} \right\} \geq 1$.

5 Examples and numerical simulations

In this section, two examples are provided to illustrate our results.

Example 5.1. Let $\omega(t)$ be a scalar Brownian motion. Let $r(t)$, $t \geq 0$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$
\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}.
$$

Consider the following scalar hybrid impulsive stochastic delay system

$$
\begin{cases}
dx(t) = \left[A_{r(t)} x(t) + B_{r(t)} x(t-1) \right] dt + \left[C_{r(t)} x(t) + D_{r(t)} x(t-1) \right] d\omega(t), & t \geq 0, \quad t \neq t_k, \\
\Delta x(t_k) = -0.6 x(t_k) + 0.4 x(t_k - 1), & k = 1, 2, \ldots,
\end{cases}
\tag{5.1}
$$

where $t_k = 0.003k$, $\tau(t) = 1$, $A_1 = -1$, $A_2 = -2$, $B_1 = 1$, $B_2 = 2$, $C_1 = 1$, $C_2 = 2$, $D_1 = 2$, $D_2 = 3$, $d_{1k} = d_{2k} = 0.32$, $\bar{d}_{1k} = \bar{d}_{2k} = 0.32$. Taking $r_1 = r_2 = 1$, we see that there exists $\lambda > 0$ such that

$$
0.32 + 0.32 e^{\lambda} < 1,
$$

$$
\begin{aligned}
\frac{1}{r_1} & \left[\lambda_{\max} \left(r_1 A_1 + r_1 A_1^T + \sum_{j=1}^{2} \gamma_{1j} r_j I \right) + r_1 \|B_1\| \right. \\
& \left. + r_1 \|C_1\|^2 + r_1 \|C_1\| \|D_1\| \right] + \lambda + \frac{q \alpha_2^2 e^{\lambda \tau}}{\alpha_1^2} \left[\|B_1\| + \|C_1\| \|D_1\| + \|D_1\|^2 \right] \\
& = 2 + \lambda + \frac{7 e^{\lambda}}{0.32 + 0.32 e^{\lambda}} \leq -\frac{\ln(0.32 + 0.32 e^{\lambda})}{0.003}
\end{aligned}
$$

and

$$\frac{1}{r_2}\left[\lambda_{\max}\left(r_2A_2 + r_2A_2^T + \sum_{j=1}^{2}\gamma_{2j}r_jI\right) + r_2\|B_2\|\right.$$

$$\left. + r_2\|C_2\|^2 + r_2\|C_2\|\|D_2\|\right] + \lambda + \frac{q\alpha_2^2 e^{\lambda\tau}}{\alpha_1^2}\left[\|B_2\| + \|C_2\|\|D_2\| + \|D_2\|^2\right]$$

$$= 8 + \lambda + \frac{17e^\lambda}{0.32 + 0.32e^\lambda} \leq -\frac{\ln(0.32 + 032e^\lambda)}{0003}.$$

By Corollary 4.1, the zero solution of Equation (5.1) is exponentially stable in the mean square. Figure 1 depicts $x(t)$ of Equation (5.1) is exponentially stable in the mean square. Figure 2 depicts $x(t)$ of Equation (5.1) without delayed impulses.

Remark 5.1. From Figures 1 and 2, although hybrid stochastic delay system without impulses may be exponentially unstable in the mean square, adding delayed impulses may lead to exponentially stable in the mean square, which implies that impulses may change the stable behavior of an system.

Example 5.2. Let $r(t)$, $t \geq 0$ be a right-continuous Markov chain taking values in $S = \{1, 2, 3\}$ with the generator

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

Consider the following the 3-D hybrid impulsive stochastic delay system

$$\begin{cases} dx(t) = \left[A_{r(t)}x(t) + B_{r(t)}x(t - 0.5)\right]dt + \left[C_{r(t)}x(t) + D_{r(t)}x(t - 0.5)\right]d\omega(t), & t \geq 0, \ t \neq t_k, \\ \Delta x(t_k) = -0.6x(t_k) + 0.4x(t_k - 0.5), & k = 1, 2, \ldots, \end{cases} \quad (5.2)$$

where

$$A_1 = \begin{pmatrix} -1 & -1 \\ -0.5 & 0.4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.3 & 0.4 \\ 0.2 & 0.1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.6 & 0.8 \\ -1 & 0.6 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 & 0.6 \\ 0.7 & -0.2 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.8 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0.6 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.7 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$t_k = 0.02k$, $\tau(t) = 0.5$, $\|B_1\| = 1.1817$, $\|B_2\| = 1.3650$, $\|C_1\| = 0.8$, $\|C_2\| = 1$, $\|D_1\| = \|D_2\| = 1$. It is easy to see that $d_{1k} = d_{2k} = 0.20$, $\bar{d}_{1k} = \bar{d}_{2k} = \bar{d}_{3k} = 0.20$. By
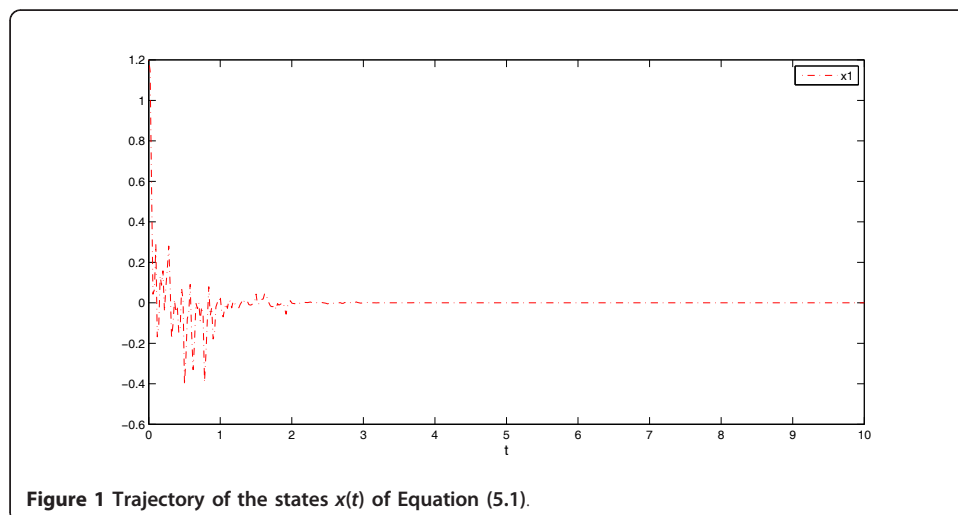
Figure 1 Trajectory of the states $x(t)$ of Equation (5.1).

Figure 2 Trajectory of the states $x(t)$ of Equation (5.1) without delay impulses.
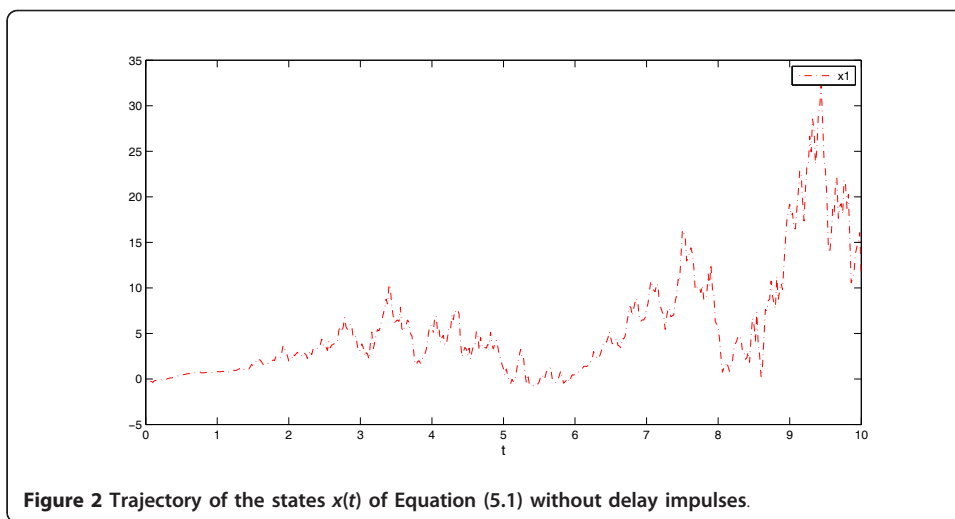
computation, we have

$$\lambda_{\max}\left(A_i + A_i^T\right) = \begin{cases} 1.4518, i = 1, \\ 0.5211, i = 2. \end{cases}$$
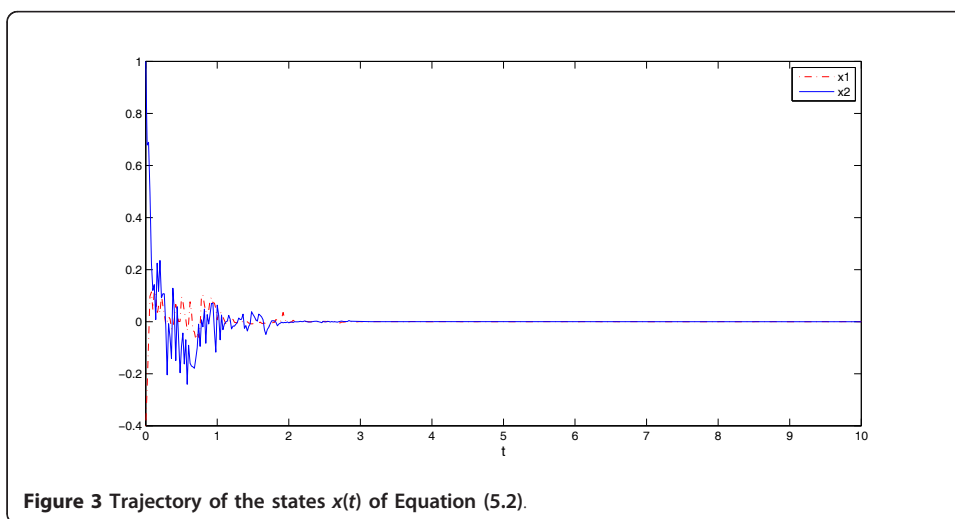
Taking $\lambda = 0.5$, we have

$$\varXi = \text{diag}\left[35.1065, 35.2939\right].$$

Hence

$$\varXi - \varGamma = \begin{pmatrix} 16.0354 & -2 \\ -1 & 36.2939 \end{pmatrix}.$$

$\varXi - \varGamma$ is a nonsingular M-matrix. Then

$$(r_1, r_2, r_3)^T = (\varXi - \varGamma)^{-1}\mathbf{1} = (0.0285, 0.0284)^T.$$

Figure 3 Trajectory of the states $x(t)$ of Equation (5.2).

Compute

$$r_i \left[0.5 + \frac{\alpha_2^2 e^{\frac{1}{4}}}{\alpha_1^2} \left(\|B_i\| + \|C_i\| \, \|D_i\| + \|D_i\|^2 \right) \right] \leq 1, \quad i = 1, 2.$$

By Corollary 4.3, the zero solution of Equation (5.2) is exponentially stable in the mean square. Figure 3 depicts $x_1(t)$, $x_2(t)$ of Equation (5.2) is exponentially stable in the mean square.

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Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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