Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

An AQCQ-functional equation in paranormed spaces

Choonkil Park[1] and Jung Rye Lee[2*]

* Correspondence: jrlee@daejin.ac.kr
[2]Department of Mathematics, Daejin University, Kyeonggi 487-711, Korea
Full list of author information is available at the end of the article

Abstract

In this article, we prove the Hyers-Ulam stability of an additive-quadratic-cubic-quartic functional equation in paranormed spaces.

Mathematics Subject Classification (2010): Primary 39B82; 39B52; 39B72; 46A99.

Keywords: Hyers-Ulam stability, paranormed space, additive-quadratic-cubic-quartic functional equation.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [1] and Steinhaus [2] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [3-7]). This notion was defined in normed spaces by Kolk [8].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1. [9] Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(1) $P(0) = 0$;
(2) $P(-x) = P(x)$;
(3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality)
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [10] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [12] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of Rassias' theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1990, Rassias [15] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [16] following the same approach as in Rassias [13], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [16], as well as by Rassias and Šemrl [17] that one cannot prove a Rassias-type theorem when $p = 1$ (cf. the books of Czerwik [18], Hyers et al. [19]).

In 1982, Rassias [20] followed the innovative approach of the Rassias' theorem [13] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [21] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [22] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [23] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have extensively been investigated by a number of authors and there are many interesting results concerning this problem (see [24-30]).

Jun and Kim [31] considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.1}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

Lee et al. [32] considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Throughout this article, assume that (X, P) is a Fréchet space and that $(Y, \|\cdot\|)$ is a Banach space.

In this article, we prove the Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + f(2y) + f(-2y) - 4f(y) - 4f(-y) \tag{1.3}$$

in paranormed spaces.

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.3) if and only if the odd mapping $f : X \to Y$ is an additive-cubic mapping, i.e.,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x).$$

It was shown in [[33], Lemma 2.2] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = \frac{1}{6}g(x) - \frac{1}{6}h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.3) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x + 2y) + f(x - 2y) = 4f(x + y) + 4f(x - y) - 6f(x) + 2f(2y) - 8f(y).$$

It was shown in [[34], Lemma 2.1] that $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and that $f(x) = \frac{1}{12}g(x) - \frac{1}{12}h(x)$.

2. Hyers-Ulam stability of the functional equation (1.3): an odd mapping case

For a given mapping f, we define

$$\begin{aligned} Df(x, y) := & f(x + 2y) + f(x - 2y) - 4f(x + y) - 4f(x - y) + 6f(x) \\ & -f(2y) - f(-2y) + 4f(y) + 4f(-y). \end{aligned}$$

In this section, we prove the Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in paranormed spaces: an odd mapping case.

Note that $P(2x) \le 2P(x)$ for all $x \in Y$.

Theorem 2.1. *Let r, θ be positive real numbers with $r > 1$, and let $f : Y \to X$ be an odd mapping such that*

$$P(Df(x, y)) \le \theta(\|x\|^r + \|y\|^r) \tag{2.1}$$

for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(2x) - 8f(x) - A(x)) \le \frac{2^r + 9}{2^r - 2}\theta\|x\|^r \tag{2.2}$$

for all $x \in Y$.

Proof. Letting $x = y$ in (2.1), we get

$$P(f(3y) - 4f(2y) + 5f(y)) \le 2\theta\|y\|^r \tag{2.3}$$

for all $y \in Y$.

Replacing x by $2y$ in (2.1), we get

$$P(f(4y) - 4f(3y) + 6f(2y) - 4f(y)) \le (2^p + 1)\theta\|y\|^r \tag{2.4}$$

for all $y \in Y$.

By (2.3) and (2.4),

$$\begin{aligned} & P(f(4y) - 10f(2y) + 16f(y)) \\ & \quad \le P(4(f(3y) - 4f(2y) + 5f(y))) + P(f(4y) - 4f(3y) + 6f(2y) - 4f(y)) \\ & \quad \le 4P(f(3y) - 4f(2y) + 5f(y)) + P(f(4y) - 4f(3y) + 6f(2y) - 4f(y)) \\ & \quad \le 8\theta\|y\|^r + (2^r + 1)\theta\|y\|^r = (2^r + 9)\theta\|y\|^r \end{aligned} \tag{2.5}$$

for all $y \in Y$. Replacing y by $\frac{x}{2}$ and letting $g(x) := f(2x) - 8f(x)$ in (2.5), we get

$$P\left(g(x) - 2g\left(\frac{x}{2}\right)\right) \le \frac{2^r + 9}{2^r}\theta\|x\|^r$$

for all $x \in Y$. Hence

$$P\left(2^l g\left(\frac{x}{2^l}\right) - 2^m g\left(\frac{x}{2^m}\right)\right) \le \sum_{j=l}^{m-1} \frac{(2^r + 9)2^j}{2^{rj+r}} \theta \|x\|^r \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in Y$. It follows from (2.6) that the sequence $\{2^k g(\frac{x}{2^k})\}$ is Cauchy for all $x \in Y$. Since X is complete, the sequence $\{2^k g(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : Y \to X$ by

$$A(x) := \lim_{k\to\infty} 2^k g\left(\frac{x}{2^k}\right)$$

for all $x \in Y$.

By (2.1),

$$P(DA(x,y)) = \lim_{k\to\infty} P\left(2^k Dg\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right) \le \frac{2^k\theta}{2^{rk}}(2^r + 8)(\|x\|^r + \|y\|^r) = 0$$

for all $x, y \in Y$. So $DA(x, y) = 0$. Since $g : Y \to X$ is odd, $A : Y \to X$ is odd. So the mapping $A : Y \to X$ is additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.2). So there exists an additive mapping $A : Y \to X$ satisfying (2.2).

Now, let $T : Y \to X$ be another additive mapping satisfying (2.2). Then we have

$$\begin{aligned} P(A(x) - T(x)) &= P\left(2^p A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right) \\ &\le P\left(2^q\left(A\left(\frac{x}{2^q}\right) - g\left(\frac{x}{2^q}\right)\right)\right) + P\left(2^q\left(T\left(\frac{x}{2^q}\right) - g\left(\frac{x}{2^q}\right)\right)\right) \\ &\le \frac{2(2^r+9)2^q}{(2^r-2)2^{rq}} \theta \|x\|^r, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in Y$. So we can conclude that $A(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of A. Thus the mapping $A : Y \to X$ is a unique additive mapping satisfying (2.2).

Theorem 2.2. *Let r be a positive real number with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\|Df(x,y)\| \le P(x)^r + P(y)^r \tag{2.7}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(2x) - 8f(x) - A(x)\| \le \frac{9+2^r}{2-2^r} P(x)^r \tag{2.8}$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.7), we get

$$\|f(3y) - 4f(2y) + 5f(y)\| \le 2P(y)^r \tag{2.9}$$

for all $y \in X$.

Replacing x by $2y$ in (2.7), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \le (2^p + 1)P(y)^r \tag{2.10}$$

for all $y \in X$.

By (2.9) and (2.10),

$$\begin{aligned}&\|f(4y)-10f(2y)+16f(y)\|\\&\quad\le\|4(f(3y)-4f(2y)+5f(y))\|+\|f(4y)-4f(3y)+6f(2y)-4f(y)\|\\&\quad\le4\|f(3y)-4f(2y)+5f(y)\|+\|f(4y)-4f(3y)+6f(2y)-4f(y)\|\\&\quad\le8P(y)^r+(2^r+1)P(y)^r=(2^r+9)P(y)^r\end{aligned}\tag{2.11}$$

for all $y \in X$. Replacing y by x and letting $g(x) := f(2x) - 8f(x)$ in (2.11), we get

$$\left\|g(x)-\frac{1}{2}g(2x)\right\|\le\frac{2^r+9}{2}P(x)^r$$

for all $x \in X$. Hence

$$\left\|\frac{1}{2^l}g(2^lx)-\frac{1}{2^m}g(2^mx)\right\|\le\sum_{j=l}^{m-1}\frac{(2^r+9)2^rj}{2^{j+1}}P(x)^r\tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.12) that the sequence $\{\frac{1}{2^k}g(2^kx)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^k}g(2^kx)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x):=\lim_{k\to\infty}\frac{1}{2^k}g(2^kx)$$

for all $x \in X$.

By (2.7),

$$\|DA(x,y)\|=\lim_{k\to\infty}\left\|\frac{1}{2^k}Dg(2^kx,2^ky)\right\|\le\frac{2^{rk}}{2^k}(2^r+8)(P(x)^r+P(y)^r)=0$$

for all $x, y \in X$. So $DA(x, y) = 0$. Since $g : X \to Y$ is odd, $A : X \to Y$ is odd. So the mapping $A : X \to Y$ is additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.8). So there exists an additive mapping $A : X \to Y$ satisfying (2.8).

Now, let $T : X \to Y$ be another additive mapping satisfying (2.8). Then we have

$$\begin{aligned}\|A(x)-T(x)\|&=\left\|\frac{1}{2^q}A(2^qx)-\frac{1}{2^q}T(2^qx)\right\|\\&\le\left\|\frac{1}{2^q}(A(2^qx)-g(2^qx))\right\|+\left\|\frac{1}{2^q}(T(2^qx)-g(2^qx))\right\|\\&\le\frac{2(9+2^r)2^{rq}}{(2-2^r)2^q}P(x)^r,\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (2.8).

Theorem 2.3. *Let r, θ be positive real numbers with $r > 3$, and let $f : Y \to X$ be an odd mapping satisfying* (2.1). *Then there exists a unique cubic mapping $C : Y \to X$ such that*

$$P(f(2x)-2f(x)-C(x))\le\frac{2^r+9}{2^r-8}\theta\|x\|^r$$

for all $x \in Y$.

Proof. Replacing y by $\frac{x}{2}$ and letting $g(x) := f(2x) - 2f(x)$ in (2.5), we get

$$P\left(g(x) - 8g\left(\frac{x}{2}\right)\right) \leq \frac{2^r + 9}{2^r}\theta \|x\|^r$$

for all $x \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1.

Theorem 2.4. *Let r be a positive real number with* $r < 3$, *and let* $f : X \to Y$ *be an odd mapping satisfying* (2.7). *Then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{9 + 2^r}{8 - 2^r} P(x)^r$$

for all $x \in X$.

Proof. Replacing y by x and letting $g(x) := f(2x) - 2f(x)$ in (2.11), we get

$$\left\|g(x) - \frac{1}{8}g(2x)\right\| \leq \frac{2^r + 9}{8} P(x)^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2.

3. Hyers-Ulam stability of the functional equation (1.3): an even mapping case

In this section, we prove the Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in paranormed spaces: an even mapping case.

Note that $P(2x) \leq 2P(x)$ for all $x \in Y$.

Theorem 3.1. *Let* r, θ *be positive real numbers with* $r > 2$, *and let* $f : Y \to X$ *be an even mapping satisfying* $f(0) = 0$ *and* (2.1). *Then there exists a unique quadratic mapping* $Q_2 : Y \to X$ *such that*

$$P(f(2x) - 16f(x) - Q_2(x)) \leq \frac{2^r + 9}{2^r - 4}\theta \|x\|^r$$

for all $x \in Y$.

Proof. Letting $x = y$ in (2.1), we get

$$P(f(3y) - 6f(2y) + 15f(y)) \leq 2\theta \|y\|^r \tag{3.1}$$

for all $y \in Y$.

Replacing x by $2y$ in (2.1), we get

$$P(f(4y) - 4f(3y) + 4f(2y) + 4f(y)) \leq (2^r + 1)\theta \|y\|^r \tag{3.2}$$

for all $y \in Y$.

By (3.1) and (3.2),

$$\begin{aligned} &P(f(4y) - 20f(2y) + 64f(y)) \\ &\quad \leq P(4(f(3y) - 6f(2y) + 15f(y))) + P(f(4y) - 4f(3y) + 4f(2y) + 4f(y)) \\ &\quad \leq 4P(f(3y) - 6f(2y) + 15f(y)) + P(f(4y) - 4f(3y) + 4f(2y) + 4f(y)) \\ &\quad \leq 8\theta \|y\|^r + (2^r + 1)\theta \|y\|^r = (2^p + 9)\theta \|y\|^r \end{aligned} \tag{3.3}$$

for all $y \in Y$. Replacing y by $\frac{x}{2}$ and letting $g(x) := f(2x) - 16f(x)$ in (3.3), we get

$$P\left(g(x) - 4g\left(\frac{x}{2}\right)\right) \leq \frac{2^r + 9}{2^r}\theta \|x\|^r$$

for all $x \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1.

Theorem 3.2. *Let r be a positive real number with $r < 2$, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quadratic mapping $Q_2 : X \to Y$ such that*

$$\|f(2x) - 16f(x) - Q_2(x)\| \leq \frac{9 + 2^r}{4 - 2^r}P(x)^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.7), we get

$$\|f(3y) - 6f(2y) + 15f(y)\| \leq 2P(y)^r \tag{3.5}$$

for all $y \in X$.

Replacing x by $2y$ in (2.7), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \leq (2^r + 1)P(y)^r \tag{3.6}$$

for all $y \in X$.

By (3.5) and (3.6),

$$\begin{aligned} &\|f(4y) - 20f(2y) + 64f(y)\| \\ &\quad \leq \|4(f(3y) - 6f(2y) + 15f(y))\| + \|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \\ &\quad \leq 4\|f(3y) - 6f(2y) + 15f(y)\| + \|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \\ &\quad \leq 8P(y)^r + (2^r + 1)P(y)^r = (2^p + 9)P(y)^r \end{aligned} \tag{3.7}$$

for all $y \in X$. Replacing y by x and letting $g(x) := f(2x) - 16f(x)$ in (3.7), we get

$$\left\|g(x) - \frac{1}{4}g(2x)\right\| \leq \frac{2^r + 9}{4}P(x)^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2.

Theorem 3.3. *Let r, θ be positive real numbers with $r > 4$, and let $f : Y \to X$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then there exists a unique quartic mapping $Q_4 : Y \to X$ such that*

$$P(f(2x) - 4f(x) - Q_4(x)) \leq \frac{2^r + 9}{2^r - 16}\theta \|x\|^r$$

for all $x \in Y$.

Proof. Replacing y by $\frac{x}{2}$ and letting $g(x) := f(2x) - 4f(x)$ in (3.3), we get

$$P\left(g(x) - 16g\left(\frac{x}{2}\right)\right) \leq \frac{2^r + 9}{2^r}\theta \|x\|^r$$

for all $x \in Y$.

The rest of the proof is similar to the proof of Theorem 2.1.

Theorem 3.4. *Let r be a positive real number with r* < 4*, and let* $f : X \to Y$ *be an even mapping satisfying f*(0) = 0 *and* (2.7)*. Then there exists a unique quartic mapping* $Q_4 : X \to Y$ *such that*

$$\|f(2x) - 4f(x) - Q_4(x)\| \le \frac{9 + 2^r}{16 - 2^r} P(x)^r$$

for all $x \in X$.

Proof. Replacing y by x and letting $g(x) := f(2x) - 4f(x)$ in (3.7), we get

$$\left\|g(x) - \frac{1}{16} g(2x)\right\| \le \frac{2^r + 9}{16} P(x)^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2.

Let $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$. Then f_o is odd and f_e is even. f_o, f_e satisfy the functional equation (1.3). Let $g_o(x) := f_o(2x) - 2f_o(x)$ and $h_o(x) := f_o(2x) - 8f_o(x)$. Then $f_o(x) = \frac{1}{6} g_o(x) - \frac{1}{6} h_o(x)$. Let $g_e(x) := f_e(2x) - 4f_e(x)$ and $h_e(x) := f_e(2x) - 16f_e(x)$. Then $f_e(x) = \frac{1}{12} g_e(x) - \frac{1}{12} h_e(x)$. Thus

$$f(x) = \frac{1}{6} g_o(x) - \frac{1}{6} h_o(x) + \frac{1}{12} g_e(x) - \frac{1}{12} h_e(x).$$

Theorem 3.5. *Let r, θ be positive real numbers with r* > 4*. Let* $f : Y \to X$ *be a mapping satisfying f*(0) = 0 *and* (2.1)*. Then there exist an additive mapping* $A : Y \to X$*, a quadratic mapping* $Q_2 : Y \to X$*, a cubic mapping* $C : Y \to X$ *and a quartic mapping* $Q_4 : Y \to X$ *such that*

$$\begin{aligned} &P(24f(x) - 4A(x) - 2Q_2(x) - 4C(x) - 2Q_4(x)) \\ &\quad \le \left(\frac{4(2^r + 9)}{2^r - 2} + \frac{2(2^r + 9)}{2^r - 4} + \frac{4(2^r + 9)}{2^r - 8} + \frac{2(2^r + 9)}{2^r - 16} \right) \theta \|x\|^r \end{aligned}$$

for all $x \in Y$.

Theorem 3.6. *Let r be a positive real number with r* < 1*. Let* $f : X \to Y$ *be a mapping satisfying f*(0) = 0 *and* (2.7)*. Then there exist an additive mapping* $A : X \to Y$*, a quadratic mapping* $Q_2 : X \to Y$*, a cubic mapping* $C : X \to Y$ *and a quartic mapping* $Q_4 : X \to Y$ *such that*

$$\begin{aligned} &\|24f(x) - 4A(x) - 2Q_2(x) - 4C(x) - 2Q_4(x)\| \\ &\quad \le \left(\frac{4(2^r + 9)}{2 - 2^r} + \frac{2(2^r + 9)}{4 - 2^r} + \frac{4(2^r + 9)}{8 - 2^r} + \frac{2(2^r + 9)}{16 - 2^r} \right) P(x)^r \end{aligned}$$

for all $x \in X$.

Acknowledgements
This study was supported by the Daejin University Research Grants in 2012.

Author details
[1]Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea [2]Department of Mathematics, Daejin University, Kyeonggi 487-711, Korea

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 5 March 2012 Accepted: 17 May 2012 Published: 17 May 2012

References
1. Fast, H: Sur la convergence statistique. Colloq Math. **2**, 241–244 (1951)
2. Steinhaus, H: Sur la convergence ordinaire et la convergence asymptotique. Colloq Math. **2**, 73–34 (1951)
3. Fridy, JA: On statistical convergence. Analysis. **5**, 301–313 (1985)
4. Karakus, S: Statistical convergence on probabilistic normed spaces. Math Commun. **12**, 11–23 (2007)
5. Mursaleen, M: λ-statistical convergence. Math Slovaca. **50**, 111–115 (2000)
6. Mursaleen, M, Mohiuddine, SA: On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. J Comput Anal Math. **233**, 142–149 (2009). doi:10.1016/j.cam.2009.07.005
7. Šalát, T: On the statistically convergent sequences of real numbers. Math Slovaca. **30**, 139–150 (1980)
8. Kolk, E: The statistical convergence in Banach spaces. Tartu Ul Toime. **928**, 41–52 (1991)
9. Wilansky, A: Modern Methods in Topological Vector Space. McGraw-Hill International Book Co., New York (1978)
10. Ulam, SM: A Collection of the Mathematical Problems. Interscience Publication, New York (1960)
11. Hyers, DH: On the stability of the linear functional equation. Proc Natl Acad Sci USA. **27**, 222–224 (1941). doi:10.1073/pnas.27.4.222
12. Aoki, T: On the stability of the linear transformation in Banach spaces. J Math Soc Jpn. **2**, 64–66 (1950). doi:10.2969/jmsj/00210064
13. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc Am Math Soc. **72**, 297–300 (1978). doi:10.1090/S0002-9939-1978-0507327-1
14. Găvruta, P: A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J Math Anal Appl. **184**, 431–436 (1994). doi:10.1006/jmaa.1994.1211
15. Rassias, TM: Problem 16; 2, in Report of the 27th International Symposium on Functional Equations. Aequationes Math **39**, 292–293 (1990). 309
16. Gajda, Z: On stability of additive mappings. Int J Math Math Sci. **14**, 431–434 (1991). doi:10.1155/S016117129100056X
17. Rassias, TM, Šemrl, P: On the behaviour of mappings which do not satisfy Hyers-Ulam stability. Proc Am Math Soc. **114**, 989–993 (1992). doi:10.1090/S0002-9939-1992-1059634-1
18. Czerwik, P: Functional Equations and Inequalities in Several Variables. World Scientific Publishing Company, New Jersey (2002)
19. Hyers, DH, Isac, G, Rassias, TM: Stability of Functional Equations in Several Variables. Birkhäuser, Basel (1998)
20. Rassias, JM: On approximation of approximately linear mappings by linear mappings. J Funct Anal. **46**, 126–130 (1982). doi:10.1016/0022-1236(82)90048-9
21. Skof, F: Proprietà locali e approssimazione di operatori. Rend Sem Mat Fis Milano. **53**, 113–129 (1983). doi:10.1007/BF02924890
22. Cholewa, PW: Remarks on the stability of functional equations. Aequationes Math. **27**, 76–86 (1984). doi:10.1007/BF02192660
23. Czerwik, S: On the stability of the quadratic mapping in normed spaces. Abh Math Sem Univ Hamburg. **62**, 59–64 (1992). doi:10.1007/BF02941618
24. Aczel, J, Dhombres, J: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
25. Eshaghi Gordji, M, Savadkouhi, MB: Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces. Appl Math Lett. **23**, 1198–1202 (2010). doi:10.1016/j.aml.2010.05.011
26. Isac, G, Rassias, TM: On the Hyers-Ulam stability of ψ-additive mappings. J Approx Theory. **72**, 131–137 (1993). doi:10.1006/jath.1993.1010
27. Jun, K, Lee, Y: A generalization of the Hyers-Ulam-Rassias stability of the pexiderized quadratic equations. J Math Anal Appl. **297**, 70–86 (2004). doi:10.1016/j.jmaa.2004.04.009
28. Jung, S: Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis. Hadronic Press Inc., Palm Harbor, FL (2001)
29. Park, C: Homomorphisms between Poisson JC^*-algebras. Bull Braz Math Soc. **36**, 79–97 (2005). doi:10.1007/s00574-005-0029-z
30. Rassias, JM: Solution of a problem of Ulam. J Approx Theory. **57**, 268–273 (1989). doi:10.1016/0021-9045(89)90041-5
31. Jun, K, Kim, H: The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. J Math Anal Appl. **274**, 867–878 (2002). doi:10.1016/S0022-247X(02)00415-8
32. Lee, S, Im, S, Hwang, I: Quartic functional equations. J Math Anal Appl. **307**, 387–394 (2005). doi:10.1016/j.jmaa.2004.12.062
33. Eshaghi-Gordji, M, Kaboli-Gharetapeh, S, Park, C, Zolfaghari, S: Stability of an additive-cubic-quartic functional equation. Adv Diff Equ **2009** (2009). Article ID 395693, 20
34. Eshaghi-Gordji, M, Abbaszadeh, S, Park, C: On the stability of a generalized quadratic and quartic type functional equation in quasi-Banach spaces. J Inequal Appl **2009** (2009). Article ID 153084, 26

doi:10.1186/1687-1847-2012-63
Cite this article as: Park and Lee: An AQCQ-functional equation in paranormed spaces. *Advances in Difference Equations* 2012 **2012**:63.