Advances in Difference Equations
a SpringerOpen Journal

RESEARCH ARTICLE Open Access

On nonlocal boundary value problems of nonlinear q-difference equations

Bashir Ahmad[1*] and Juan J Nieto[1,2]

* Correspondence: bashir_qau@yahoo.com
[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

This paper studies a nonlocal boundary value problem of nonlinear third-order q-difference equations. Our results are based on Leray-Schauder degree theory and some standard fixed point theorems.

MSC 2000: 39A05; 39A13.

Keywords: q-difference equations, nonlocal boundary conditions, Leray-Schauder degree theory, fixed point theorems

1 Introduction

In this paper, we study a nonlocal nonlinear boundary value problem (BVP) of third-order q-difference equations given by

$$\begin{cases} D_q^3 u(t) = f(t, u(t)), t \in I_q, \\ u(0) = 0, \; D_q u(0) = 0, \; u(1) = \alpha u(\eta), \end{cases} \tag{1.1}$$

where $f \in C(I_q \times \mathbb{R}, \mathbb{R})$, $I_q = \{q^n : n \in \mathbb{N}\} \cup \{0,1\}$, $q \in (0,1)$ is a fixed constant, $\eta \in \{q^n : n \in \mathbb{N}\}$ and $\alpha \neq 1/\eta^2$ is a real number.

The subject of q-difference equations has evolved into a multidisciplinary subject in the last few decades. In fact, it is a truly operational subject and its operational formulas were often used with great success in the theory of classical orthogonal polynomials and Bessel functions [1,2]. For some pioneer work on q-difference equations, we refer the reader to [1,3-5], whereas the recent development of the subject can be found in [6-17] and references therein. However, the theory of boundary value problems for nonlinear q-difference equations is still in the initial stages and many aspects of this theory need to be explored. In particular, the study of nonlocal boundary value problems for nonlinear q-difference equations is yet to be initiated.

The aim of our paper is to present some existence results for the problem (1.1). The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we apply Banach's contraction principle to prove the uniqueness of the solution of the problem, while the third result is based on Krasnoselskii's fixed point theorem. The methods used are standard; however, their exposition in the framework of problem (1.1) is new. In Sect. 2, we present some basic material that we need in the sequel and Sect. 3 contains main results of the paper. Some illustrative examples are also discussed.

2 Preliminaries

Let us recall some basic concepts of q-calculus [8,9].

For $0 < q < 1$, we define the q-derivative of a real-valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in I_q - \{0\}, \quad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

Note that

$$\lim_{q \to 1^-} D_q f(t) = f'(t).$$

The higher order q-derivatives are defined inductively as

$$D_q^0 f(t) = f(t), \quad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

For example, $D_q(t^k) = [k]_q t^{k-1}$, where k is a positive integer and the q-bracket $[k]_q = (q^k - 1)/(q - 1)$. In particular, $D_q(t^2) = (1 + q)t$.

For $y \geq 0$, let us set $\mathcal{J}_y = \{yq^n : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ and define the definite q-integral of a function $f : \mathcal{J}_y \to \mathbb{R}$ by

$$I_q f(y) = \int_0^y f(s) d_q s = \sum_{n=0}^{\infty} y(1-q)q^n f(yq^n)$$

provided that the series converges. For $b_1, b_2 \in \mathcal{J}_y (b_1 = yq^{n_1}, b_2 = yq^{n_2}$ for some $n_1, n_2 \in \mathbb{N})$, we define

$$\int_{b_1}^{b_2} f(s) d_q s = I_q f(b_2) - I_q f(b_1) = (1-q) \sum_{n=0}^{\infty} q^n [b_2 f(b_2 q^n) - b_1 f(b_1 q^n)].$$

Similarly, we have

$$I_q^0 f(t) = f(t), \quad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

Observe that

$$D_q I_q f(x) = f(x), \tag{2.1}$$

and if f is continuous at $x = 0$, then

$$I_q D_q f(x) = f(x) - f(0).$$

This implies that if $D_q f(t) = \sigma(t)$, then $f(t) = I_q \sigma(t) + c$, where c is an arbitrary constant.

In q-calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = D_q g(t) h(t) + g(qt) D_q h(t), \tag{2.2}$$

$$\int_0^x f(t) D_q g(t) dqt = \left[f(t) g(t)\right]_0^x - \int_0^x D_q f(t) g(qt) d_q t. \tag{2.3}$$

In the limit $q \to 1^-$, the above results correspond to their counterparts in standard calculus.

For $f, g : \mathcal{J}_\gamma \to \mathbb{R}$, it is possible to introduce an inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)d_qt$$

and the resulting Hilbert space is denoted by $L_q^2(0,1)$.

As argued in [16], we can write the solution of the third-order q-difference equation $D_q^3u(t) = v(t)$ in the following form:

$$u(t) = \int_0^t \left(\alpha_1(q)t^2 + \alpha_2(q)ts + \alpha_3(q)s^2\right)v(s)d_qs + a_0 + a_1t + a_2t^2, \tag{2.4}$$

where a_0, a_1, a_2 are arbitrary constants and $\alpha_1(q)$, $\alpha_2(q)$, $\alpha_3(q)$ can be fixed appropriately.

Choosing $\alpha_1(q) = 1/(1+q)$, $\alpha_2(q) = -q$, $\alpha_3(q) = q^3/(1+q)$ and using (2.1) and (2.2), we find that

$$D_qu(t) = \int_0^t tv(s)d_qs - \int_0^t qsv(s)d_qs,\ \ D_q^2u(t) = \int_0^t v(s)d_qs, D_q^3u(t) = v(t).$$

Thus, the solution (2.4) of $D_q^3u(t) = v(t)$ takes the form

$$u(t) = \int_0^t \left(\frac{t^2 + q^3s^2}{1+q} - qts\right)v(s)d_qs + a_0 + a_1t + a_2t^2. \tag{2.5}$$

Lemma 2.1 *The BVP (1.1) is equivalent to the integral equation*

$$\begin{aligned} u(t) = &\int_0^t \left(\frac{t^2 + q^3s^2}{1+q} - qts\right)f(s,u(s))d_qs + \frac{t_2}{1-\alpha\eta^2} \\ &\times \left[\alpha\int_0^\eta \left(\frac{\eta^2 + q^3s^2}{1+q} - q\eta s\right)f(s,u(s))d_qs - \int_0^1 \left(\frac{1 + q^3s^2}{1+q} - qs\right)f(s,u(s))d_qs\right]. \end{aligned} \tag{2.6}$$

Proof. In view of (2.5), the solution of $D_q^3u = f(t,u)$ can be written as

$$u(t) = \int_0^t \left(\frac{t^2 + q^3s^2}{1+q} - qts\right)f(s,u(s))d_qs + a_0 + a_1t + a_2t^2, \tag{2.7}$$

where a_1, a_2, a_2 are arbitrary constants. Using the boundary conditions of (1.1) in (2.7), we find that $a_0 = 0$, $a_1 = 0$ and

$$a_2 = \frac{1}{1-\alpha\eta^2}\left[\alpha\int_0^\eta \left(\frac{\eta^2 + q^3s^2}{1+q} - q\eta s\right)f(s,u(s))d_qs - \int_0^1 \left(\frac{1 + q^3s^2}{1+q} - qs\right)f(s,u(s))d_qs\right].$$

Substituting the values of a_0, a_1 and a_2 in (2.7), we obtain (2.6). This completes the proof.

We define

$$\begin{aligned} G_1 &= \max_{t\in I_q}\left\| \int_0^t \frac{(t-s)^2}{2} d_q s + \frac{t^2}{1-\alpha\eta^2}\left[\alpha\int_0^\eta \frac{(\eta-s)^2}{2}d_q s - \int_0^1 \frac{(1-s)^2}{2} d_q s\right]\right\| \\ &= \max\left\{\frac{|\gamma|(1+q)q^2}{(1+q+q^2)^4}, \frac{|\alpha|\eta^2(1-\eta)}{|1-\alpha\eta^2|(1+q)(1+q+q^2)}\right\}, \end{aligned} \tag{2.8}$$

where

$$\gamma = \eta + \frac{1-\eta}{1-\alpha\eta^2}.$$

Remark **2.1** For $q \to 1^-$, equation (2.6) takes the form

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-s)^2}{2} f(s,u(s))ds + \frac{t^2}{1-\alpha\eta^2}\left[\alpha\int_0^\eta \frac{(\eta-s)^2}{2} f(s,u(s))ds\right. \\ &\left. - \int_0^1 \frac{(1-s)^2}{2} f(s,u(s))ds\right]. \end{aligned}$$

which is equivalent to the solution of a classical third-order nonlocal boundary value problem

$$u'''(t) = f(t,u(t)), \quad u(0) = 0, \quad u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad 0 \le t \le 1, \quad 0 < \eta < 1. \tag{2.9}$$

3 Existence results

Let $\mathcal{C}_q = C(I_q, \mathbb{R})$ denote the Banach space of all continuous functions from $I_q \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in I_q\}$.

Theorem 3.1 *Assume that there exist constants* $M_1 \ge 0$ *and* $M_2 > 0$ *such that* $M_1 G_1 < 1$ *and* $|f(t, u)| \le M_1|u| + M_2$ *for all* $t \in I_q$, $u \in \mathbb{R}$, *where* G_1 *is given by (2.8). Then the problem (1.1) has at least one solution.*

Proof. Let $B_R \subset \mathcal{C}_q$ be a suitable ball with radius $R > 0$. Define an operator $\mathcal{F} : \bar{B}_R - \mathcal{C}_q$ as

$$\begin{aligned} [\mathcal{F}u](t) = &\int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts\right) f(s,u(s)) d_q s \\ &+ \frac{t^2}{1-\alpha\eta^2}\left[\alpha\int_0^\eta \left(\frac{\eta^2+q^3s^2}{1+q} - q\eta 2\right) f(s,u(s))\, d_q s\right. \\ &\left. - \int_0^1 \left(\frac{1+q^3s^2}{1+q} - qs\right) f(s,u(s)) d_q s\right]. \end{aligned}$$

In view of Lemma 2.1, we just need to prove the existence of at least one solution $u \in \mathcal{C}_q$ such that $u = \mathcal{F}u$. Thus, it is sufficient to show that the operator $\mathcal{F}$ satisfies

$$u \neq \lambda \mathcal{F} u, \quad \forall u \in \partial B_R \text{ and } \forall \lambda \in [0,1]. \tag{3.1}$$

Let us define

$$H(\lambda, y) = \lambda \mathcal{F} u, \quad u \in \mathcal{C}_q, \quad \lambda \in [0,1].$$

Then, by Arzela-Ascoli theorem, $h_\lambda(u) = u - H(\lambda, u) = u - \lambda \mathcal{F} u$ is completely continuous. If (3.1) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\lambda, B_R, 0) &= \deg(\mathcal{I} - \lambda \mathcal{F}, B_R, 0) = \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(\mathcal{I}, B_R, 0) = 1 \neq 0, \quad 0 \in B_R,\end{aligned}$$

where $\mathcal{I}$ denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(t) = u - \lambda \mathcal{F} u = 0$ for at least one $u \in B_R$. Let us set

$$B_R = \left\{u \in \mathcal{C}_q : \|u\| < R\right\},$$

where R will be fixed later. In order to prove (3.1), we assume that $u = \lambda \mathcal{F} u$ for some $\lambda \in [0,1]$ and for all $t \in I_q$ so that

$$\begin{aligned}
|u(t)| = |\lambda[\mathcal{F}u](t)| &\leq \Bigg| \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts \right) f(s, u(s)) d_q s \\
&\quad + \frac{t^2}{1 - \alpha\eta^2} \Bigg[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s \right) f(s, u(s))\, d_q s \\
&\quad - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs \right) f(s, u(s)) d_q s \Bigg] \Bigg| \\
&\leq \Bigg| \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts \right) \left(M_1 |u(s)| + M_2\right) d_q s \\
&\quad + \frac{t^2}{1 - \alpha\eta^2} \Bigg[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s \right) \left(M_1 |u(s)| + M_2\right) d_q s \\
&\quad - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs \right) \left(M_1 |u(s)| + M_2\right) d_q s \Bigg] \Bigg| \\
&\leq (M_1 \|u\| + M_2) \max_{t \in I_q} \Bigg| \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts \right) d_q s \\
&\quad + \frac{t^2}{1 - \alpha\eta^2} \Bigg[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s \right) d_q s - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs \right) d_q s \Bigg] \Bigg| \\
&\leq (M_1 \|u\| + M_2)\, G_1,
\end{aligned}$$

which implies that

$$\|u\| \leq \frac{M_2 G_1}{1 - M_1 G_1}.$$

Letting $R = \dfrac{M_2 G_1}{1 - M_1 G_1} + 1$, (3.1) holds. This completes the proof.

Theorem 3.2 *Let $f : I_q \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the Lipschitz condition*

$$\left|f(t,u) - f(t,v)\right| \le L\,|u - v|, \quad \forall t \in I_q,\ u, v \in \mathbb{R},$$

where L is a Lipschitz constant. Then the boundary value problem (1.1) has a unique solution provided $L < 1/G_1$, where G_1 is given by (2.8).

Proof. Let us define an operator $\mathcal{F} : \mathcal{C}_q \to \mathcal{C}_q$ by

$$\begin{aligned}[\mathcal{F}u](t) &= \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts\right) f(s, u(s))d_q s \\ &\quad + \frac{t^2}{1 - \alpha\eta^2}\left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s\right) f(s, u(s))d_q s \right.\\ &\quad \left. - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs\right) f(s, u(s))d_q s\right].\end{aligned}$$

Let us set $\max_{t \in I_q} \left|f(t,0)\right| = M$ and choose

$$r \ge \frac{MG_1}{1 - LG_1} \tag{3.2}$$

Then we show that $\mathcal{F}B_r \subset B_r$, where $B_r = \{u \in \mathcal{C}_q : \|u\| \le r\}$. For $u \in B_r$, we have

$$\begin{aligned}\|\mathcal{F}u\| &= \max_{t \in I_q}\left|\int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts\right) f(s, u(s))d_q s \right.\\ &\quad + \frac{t^2}{1 - \alpha\eta^2}\left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s\right) f(s, u(s))d_q s \right.\\ &\quad \left.\left. - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs\right) f(s, u(s))d_q s\right]\right| \\ &= \max_{t \in I_q}\left|\int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts\right) [(f(s, u(s)) - f(s,0)) + f(s,0)]d_q s \right.\\ &\quad + \frac{t^2}{1 - \alpha\eta^2}\left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s\right) [(f(s, u(s)) - f(s,0)) + f(s,0)]d_q s \right.\\ &\quad \left.\left. - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs\right) [(f(s, u(s)) - f(s,0)) + f(s,0)]d_q s\right]\right| \\ &\le (L\,\|u\| + M|) \max_{t \in I_q}\left|\int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts\right) d_q s \right.\\ &\quad \left. + \frac{t^2}{1 - \alpha\eta^2}\left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s\right) d_q s - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs\right) d_q s\right]\right| \\ &\le G_1(L_r + M) \le r.\end{aligned}$$

where we have used (3.2).

Now, for $u, v \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\|\mathcal{F}u - \mathcal{F}v\| \\
&= \max_{t\in I_q} \left|[\mathcal{F}u](t) - [\mathcal{F}v](t)\right| \\
&\le \max_{t\in I_q} \Bigg| \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts\right) f(s, u(s)) d_q s \\
&\quad + \frac{t^2}{1-\alpha\eta^2}\left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s\right) f(s, u(s)) d_q s \right. \\
&\quad \left. - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs\right) f(s, u(s)) d_q s\right]\Bigg| \\
&\le L \max_{t\in I_q} \Bigg| \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts\right) d_q s \\
&\quad + \frac{t^2}{1-\alpha\eta^2}\left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s\right) d_q s - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs\right) d_q s\right]\Bigg| \|u - v\| \\
&\le L G_1 \|u - v\| .
\end{aligned}$$

As $L < 1/G_1$, therefore $\mathcal{F}$ is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof.

To prove the next existence result, we need the following known fixed point theorem due to Krasnoselskii [18].

Theorem 3.3 *Let $\mathcal{M}$ be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (i) $Ax + By \in \mathcal{M}$ *whenever* $x, y \in \mathcal{M}$; (ii) *A is compact and continuous;* (iii) *B is a contraction mapping. Then there exists* $z \in \mathcal{M}$ *such that* $z = Az + Bz$.

Theorem 3.4 *Assume that* $f : I_q \times \mathbb{R} \to \mathbb{R}$ *is a continuous function such that*

$$\left|f(t, u) - f(t, v)\right| \le L\,|u - v|\,, \quad \forall t \in I_q, \quad u, v \in \mathbb{R}. \tag{3.3}$$

Furthermore, $|f(t, u)| \le \mu(t)$, $\forall (t, u) \in I_q \times \mathbb{R}$, *with* $\mu \in C(I_q, \mathbb{R}^+)$. *Then the boundary value problem (1.1) has at least one solution on* I_q *if*

$$\frac{\left|1 - \alpha\eta^3\right|}{\left|1 - \alpha\eta^2\right|(1+q)(1+q+q^2)} < 1. \tag{3.4}$$

Proof. Letting $\sup_{t\in I_q} \left|\mu(t)\right| = \|\mu\|$, we fix $\bar{r} \ge \|\mu\|\, G_1$ (G_1 is given by (2.8) and consider $B_{\bar{r}} = \{u \in \mathbb{R} : \|u\| \le \bar{r}\}$. We define the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ on $B_{\bar{r}}$ as

$$[\mathcal{P}_1 u](t) = \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} - qts\right) f(s, u(s)) d_q s,$$

$$\begin{aligned}
[\mathcal{P}_2 u](t) = \frac{t^2}{1-\alpha\eta^2}&\left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s\right) f(s, u(s)) d_q s \right. \\
&\left. - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs\right) f(s, u(s)) d_q s\right].
\end{aligned}$$

For $u, v \in B_{\bar{r}}$, we find that

$$\begin{aligned}
&\|\mathcal{P}_1 u + \mathcal{P}_2 v\| \\
&\le \|\mu\| \max_{t\in I} \left| \int_0^t \left(\frac{t^2 + q^3 s^2}{1+q} + qts \right) d_q s \right. \\
&\left. + \frac{t^2}{1-\alpha\eta^2} \left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s \right) d_q s - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs \right) d_q s \right] \right| \\
&= \|\mu\| \, G_1 \le \bar{r}.
\end{aligned}$$

Thus, $\mathcal{P}_1 u + \mathcal{P}_2 v \in B_{\bar{r}}$. It follows from (3.3) and (3.4) that $\mathcal{P}_2$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}_1$ is continuous. Also, $\mathcal{P}_1$ is uniformly bounded on $B_{\bar{r}}$ as

$$\|\mathcal{P}_1 u\| \le \frac{1}{(1+q)(1+q+q^2)}.$$

Now we prove the compactness of the operator $\mathcal{P}_1$.

In view of (H_1), we define $\sup_{(t,u)\in I_q \times B_r} |f(t,u)| = \bar{f}$, and consequently we have

$$\begin{aligned}
&\left| [\mathcal{P}_1 u](t_1) - [\mathcal{P}_1 u](t_2) \right| \\
&= \left| \int_0^{t_1} \left(\frac{t_1^2 + q^3 s^2}{1+q} - qt_1 s \right) f(s,u(s)) d_q s - \int_0^{t_2} \left(\frac{t_2^2 + q^3 s^2}{1+q} - qt_2 s \right) f(s,u(s)) d_q s \right. \\
&\left. + \frac{(t_1^2 + t_2^2)}{1-\alpha\eta^2} \left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta 2 \right) f(s,u(s)) d_q s - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs \right) f(s,u(s)) d_q s \right] \right| \\
&\le \bar{f} \left| \int_0^{t_1} \frac{(t_1 - t_2)[t_1 + t_2 - q(1+q)s]}{1+q} d_q s - \int_{t_1}^{t_2} \left(\frac{t_2^2 + q^3 s^2}{1+q} - qt_2 s \right) d_q s \right. \\
&\left. + \frac{(t_1^2 - t_2^2)}{1-\alpha\eta^2} \left[\alpha \int_0^\eta \left(\frac{\eta^2 + q^3 s^2}{1+q} - q\eta s \right) d_q s - \int_0^1 \left(\frac{1 + q^3 s^2}{1+q} - qs \right) d_q s \right] \right|,
\end{aligned}$$

which is independent of u and tends to zero as $t_2 \to t_1$. So $\mathcal{P}_1$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}_1$ is compact on $B_{\bar{r}}$. Thus all the assump tions of Theorem 3.3 are satisfied. So the conclusion of Theorem 3.3 implies that (1.1) has at least one solution on I_q. This completes the proof.

Remark **3.1** In the limit $q \to 1^-$, our results reduce to the ones for a classical third-order nonlocal nonlinear boundary value problem (2.9).

Example 3.1. Consider the following problem

$$\begin{cases} D_{\frac{1}{2}}^3 u(t) = \frac{M_1}{(2\pi)} \sin(2\pi u) + \frac{|u|}{1+|u|} + t^2, \quad t \in [0,1]_{1/2}, \\ u(0) = 0, \quad D_{\frac{1}{2}} u(0) = 0, \quad u(1) = 2u(1/2). \end{cases} \tag{3.5}$$

Here $q = 1/2$ and M_1 will be fixed later. Observe that

$$|f(t,u)| = \left| \frac{M_1}{(2\pi)} \sin(2\pi u) + \frac{|u|}{1+|u|} + t^2 \right| \le M_1 |u| + 2,$$

and

$$\begin{aligned} G_1 &= \max\left\{\frac{|\gamma|\,(1+q)q^2}{(1+q+q^2)^4}, \frac{|\alpha|\,\eta^2(1-\eta)}{\left|1-\alpha\eta^2\right|(1+q)(1+q+q^2)}\right\} \\ &= \frac{|\alpha|\,\eta^2(1-\eta)}{\left|1-\alpha\eta^2\right|(1+q)(1+q+q^2)} = 4/21. \end{aligned}$$

Clearly $M_2 = 2$ and we can choose $M_1 < \frac{1}{G_1} = 21/4$. Thus, Theorem 3.1 applies to the problem (3.5).

Example 3.2. Consider the following problem with unbounded nonlinearity

$$\begin{cases} D^3_{\frac{1}{2}}u(t) = 5u + cosu + (u^2/(1+u^2)), \quad t \in [0,1]_{1/2}, \\ u(0) = 0, \quad D_{\frac{1}{2}}u(0) = 0, \quad u(1) = 2u(1/2). \end{cases} \tag{3.6}$$

Clearly

$$\left|f(t,u)\right| = \left|5u + cosu + (u^2/(1+u^2))\right| \le 5\,|u| + 2,$$

with $M_1 = 5 < 1/G_1 = 21/4$ (G_1 is given in Example 3.1) and $M_2 = 2$. Thus, by the conclusion of Theorem 3.1, the problem (3.6) has a solution.

Example 3.3. Consider

$$\begin{cases} D^3_{\frac{3}{4}}u(t) = L\left(\cos\ t + \tan^{-1}u\right), \quad t \in [0,1]_{3/4}, \\ u(0) = 0,\ D_{\frac{3}{4}}u(0) = 0, \quad u(1) = u(1/4). \end{cases} \tag{3.7}$$

With $f(t, u) = L$ (cos t + $\tan^{-1}$ u), we find that

$$\left|f(t,u) - f(t,v)\right| \le L\left|\tan^{-1}u - \tan^{-1}v\right| \le L\,|u-v|$$

and

$$\begin{aligned} G_1 &= \max\left\{\frac{|\gamma|\,(1+q)q^2}{(1+q+q^2)^4}, \frac{|\alpha|\,\eta^2(1-\eta)}{\left|1-\alpha\eta^2\right|(1+q)(1+q+q^2)}\right\} \\ &= \frac{|\gamma|\,(1+q)q^2}{(1+q+q^2)^4} = \frac{86704128}{2398926080}. \end{aligned}$$

Fixing $L < \frac{1}{G_1} \approx 27.668$, it follows by Theorem 3.2 that the problem (3.7) has a unique solution.

Acknowledgements

The research of Bashir Ahmad was supported by Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia. The authors thank the reviewers for their useful comments.

Author details

[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia [2]Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782 Santiago, Spain

Authors' contributions

Each of the authors, BA and JJN contributed to each part of this study equally and read and approved the final version of the manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 6 March 2012 Accepted: 13 June 2012 Published: 13 June 2012

References

1. Jackson, FH: On q-difference equations. Am J Math. **32**, 305–314 (1910). doi:10.2307/2370183
2. Ernst, T: q-Calculus as operational algebra. Proc Estonian Acad Sci. **58**, 73–97 (2009). doi:10.3176/proc.2009.2.01
3. Carmichael, RD: The general theory of linear q-difference equations. Am J Math. **34**, 147–168 (1912). doi:10.2307/2369887
4. Mason, TE: On properties of the solutions of linear q-difference equations with entire function coefficients. Am J Math. **37**, 439–444 (1915). doi:10.2307/2370216
5. Adams, CR: On the linear ordinary q-difference equation. Ann Math. **30**, 195–205 (1928). doi:10.2307/1968274
6. Floreanini, R, Vinet, L: Quantum symmetries of q-difference equations. J Math
7. Ernst, T: A new notation for q-calculus and a new q-Taylor formula, U.U.D.M. Department of Mathematics, Uppsala University (1999) Report 1999:25, ISSN 1101-3591
8. Gasper, G, Rahman, M: Basic Hypergeometric Series. Cambridge University Press, Cambridge (1990)
9. Kac, V, Cheung, P: Quantum Calculus. Springer, New York (2002)
10. Bangerezako, G: Variational q-calculus. J Math Anal Appl. **289**, 650–665 (2004). doi:10.1016/j.jmaa.2003.09.004
11. Ismail, MEH, Simeonov, P: q-difference operators for orthogonal polynomials. J Comput Appl Math. **233**, 749–761 (2009). doi:10.1016/j.cam.2009.02.044
12. Bekker, MB, Bohner, MJ, Herega, AN, Voulov, H: Spectral analysis of a q-difference operator. J Phys A **43**(14), 15 (2010). art. no. 145207
13. Bohner, M, Guseinov, GSh: The h-Laplace and q-Laplace transforms. J Math Anal Appl. **365**, 75–92 (2010). doi:10.1016/j.jmaa.2009.09.061
14. Ahmad, B, Ntouyas, SK: Boundary value problems for q-difference inclusions. Abstr Appl Anal**2011**, 15. article ID 292860
15. Abdeljawad, T, Baleanu, D: Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function. Commun. Nonlinear Sci Numer Simul. **16**, 4682–4688 (2011). doi:10.1016/j.cnsns.2011.01.026
16. Ahmad, B: Boundary value problems for nonlinear third-order *q*-difference equations. Electron. J Differ Equ. **2011**(94), 1–7 (2011)
17. Ahmad, B, Alsaedi, A, Ntouyas, SK: A study of second-order *q*-difference equations with boundary conditions. Adv Differ Equ **35** (2012). in press, doi:10.1186/1687-1847-2012-35
18. Smart, DR: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)

doi:10.1186/1687-1847-2012-81
Cite this article as: Ahmad and Nieto: **On nonlocal boundary value problems of nonlinear *q*-difference equations.** *Advances in Difference Equations* 2012 **2012**:81.