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Positive solutions of a singular semipositone boundary value problems for fourth-order coupled difference equations

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Abstract

In this article, we study the boundary value problems for the fourth-order nonlinear coupled difference equations

$$\begin{cases} \Delta^4 u(i-2) = f_1(i, v(i)) + e_1(i), & i \in [2, T+2], \\ \Delta^4 v(i-2) = f_2(i, u(i)) + e_2(i), & i \in [2, T+2], \\ u(0) = u(1) = v(1) = v(0) = 0, \\ u(T+3) = u(T+4) = v(T+4) = v(T+3) = 0. \end{cases}$$

Throughout our nonlinearity may be singular. We establish the existence of positive solutions for the fourth-order coupled systems. The proof relies on Schauder's fixed point theorem.

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Introduction

In this article, we consider the following boundary value problems of difference equations

$$\begin{cases} \Delta^4 u(i-2) = f_1(i, v(i)) + e_1(i), & i \in [2, T+2], \\ \Delta^4 v(i-2) = f_2(i, u(i)) + e_2(i), & i \in [2, T+2], \\ u(0) = u(1) = v(1) = v(0) = 0, \\ u(T+3) = u(T+4) = v(T+4) = v(T+3) = 0, \end{cases} \quad (1)$$

here $[2, T+2] = \{2, 3, \dots, T+2\}$ and $u, v, e_j : [0, T+4] \rightarrow R, f_j : [0, T+4] \times R \rightarrow R^+, j = 1, 2$. We will let $[a, b]$ denote the discrete integer set $[a, b] = \{a, a+1, \dots, b\}$, $C([a, b])$ denote set of continuous function on $[a, b]$ (discrete topology) with norm $\|\cdot\| = \max_{k \in [a, b]} |\cdot|$.

Recently, many literature on the boundary value of difference equations have appeared. We refer the reader to [1–17] and the references therein, which include work on Agarwal, Eloe, Erber, O'Regan, Henderson, Merdivenci, Yu, Ma *et al.*, concerning the existence of positive solutions and the corresponding eigenvalue problems. Recently, the existence of

positive solutions of fourth-order discrete boundary value problems have been studied by several authors, for examples, see [16–18] and the references therein.

The fourth-order boundary value problems of ordinary value problems have important application in various branches of pure and applied science. As in entrepreneurial network evolution studies, the research paradigm of ‘U related to V’ was always applied, while U or V was the measurement of the attitudes of the entrepreneurs who answered the questionnaire. According to cognitive psychology studies [19, 20], questionnaire shows the change of attitude, which is the fourth-order dependent of original signal. So whether this paradigm will lead to meaningful causal outcome, which is basically depended on difference equations.

The fourth-order boundary value problems of ordinary value problems have important application in various branches of pure and applied science. They arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory [21–23]. For examples, the deformations of an elastic beam can be described by the boundary value problems of the fourth-order ordinary differential equations. There have been extensive studies on fourth-order boundary value problems with diverse boundary conditions via many methods, for example [24–27] and the references therein.

The remaining part of the article is organized as follows. In Section “Preliminaries”, some preliminary results will be given. In the remaining sections, by employing a basic application of Schauder’s fixed point theorem, we state and prove the existence results for (1). Our view point sheds some new light on problems with weak force potentials and prove that in some situations weak singularities may stimulate the existence of positive solutions.

Preliminaries

In this section, we state the preliminary information that we need to prove the main results. From [4, 5], we have the following lemma.

Lemma 1 *x(i) is a solution of equation*

$$\begin{cases} \Delta^4 x(i - 2) = h(i), & i \in [2, T + 2], \\ x(0) = x(1) = x(T + 3) = x(T + 4) = 0 \end{cases} \tag{2}$$

if only and if

$$x(i) = \sum_{j=2}^{T+2} G(i, j)h(j), \quad i \in [0, T + 4], \tag{3}$$

where

$$G(i, j) = \begin{cases} \frac{(T+4-i)^2(j-1)^2}{2} \left(\frac{i}{(T+3)^2} - \frac{(T+4+2i)(j-1)}{3(T+4)^3} \right), & 2 \leq j \leq i + 1, \\ \frac{i^2(T+4-j)^2}{6} \left(\frac{(T+3-i)(T+4+2j)}{(T+4)^3} - \frac{T+4-j}{(T+3)^2} \right), & i + 1 < j \leq T + 2. \end{cases} \tag{4}$$

Lemma 2 *The Green’s function G(i, j) defined by (4) have properties*

$$\begin{aligned} C_0 i^2(T + 4 - i)^2(j - 1)^2(T + 4 - j)^2 &\leq G(i, j) \leq (j - 1)^2(T + 4 - j)^2, \\ G(i, j) &\leq i^2(T + 4 - i)^2, \end{aligned}$$

where $C_0 = \frac{1}{3(T+4)^7}$.

For our constructions, we shall consider the Banach space $E = C([0, T + 4])$ equipped with the standard norm $\|x\| = \max_{0 \leq i \leq T+4} |x(i)|$, $x \in E$. We define a cone $P \subset E$ by

$$P = \{x \in E \mid x(i) \geq C_0 i^2 (T + 4 - i)^2 \|x\|, i \in [0, T + 4]\}.$$

Define an operator $A(u, v) = (A_1(u), A_2(v)) : P \times P \rightarrow E \times E$ by

$$\begin{aligned} A_1 u(i) &= \sum_{j=2}^{T+2} G(i, j) (f_1(i, v(i)) + e_1(i)), \quad i \in [0, T + 4], v \in P, \\ A_2 v(i) &= \sum_{j=2}^{T+2} G(i, j) (f_2(i, u(i)) + e_2(i)), \quad i \in [0, T + 4], u \in P. \end{aligned} \tag{5}$$

Notice from (5) and Lemma 2 that, for $u \in P$, we have

$$\begin{aligned} A_1 u(i) &= \sum_{j=2}^{T+2} G(i, j) (f_1(i, v(i)) + e_1(i)) \leq \sum_{j=2}^{T+2} (j - 1)^2 (T + 4 - j)^2 (f_1(i, v(i)) + e_1(i)), \\ A_2 v(i) &= \sum_{j=2}^{T+2} G(i, j) (f_2(i, u(i)) + e_2(i)) \leq \sum_{j=2}^{T+2} (j - 1)^2 (T + 4 - j)^2 (f_2(i, u(i)) + e_2(i)), \end{aligned}$$

then $\|A_1 v\| \leq \sum_{j=2}^{T+2} (j - 1)^2 (T + 4 - j)^2 (f_1(i, v(i)) + e_1(i))$ and $\|A_2 u\| \leq \sum_{j=2}^{T+2} (j - 1)^2 (T + 4 - j)^2 (f_2(i, u(i)) + e_2(i))$.

On the other hand, we have

$$\begin{aligned} A_1 u(i) &\geq C_0 i^2 (T + 4 - i)^2 \sum_{j=2}^{T+2} (j - 1)^2 (T + 4 - j)^2 (f_1(i, v(i)) + e_1(i)) \\ &\geq C_0 i^2 (T + 4 - i)^2 \|A_1 u\|, \\ A_2 v(i) &\geq C_0 i^2 (T + 4 - i)^2 \sum_{j=2}^{T+2} (j - 1)^2 (T + 4 - j)^2 (f_2(i, u(i)) + e_2(i)) \\ &\geq C_0 i^2 (T + 4 - i)^2 \|A_2 v\|. \end{aligned}$$

Thus, $A(P \times P) \subset P \times P$. In addition, standard arguments show that A is completely continuous.

Now, we define the function $\gamma_k : E \rightarrow E$ by

$$\gamma_k(i) = \sum_{j=2}^{T+2} \frac{G(i, j)}{i^2 (T + 4 - i)^2} e_k(j), \quad i \in [0, T + 4], k = 1, 2,$$

which is the unique solution of

$$\begin{cases} \Delta^4 x(i - 2) = e_k(i), & i \in [2, T + 2], \\ x(0) = x(1) = x(T + 3) = x(T + 4) = 0. \end{cases}$$

Throughout this article, we use the following notations

$$\gamma_{k^*} = \min_{k,i} \gamma_k(i), \quad \gamma_k^* = \max_{k,i} \gamma_k(i).$$

The case $\gamma_{1^*} \geq 0, \gamma_{2^*} \geq 0$

Theorem 1 Assume that there exists $b_k > 0, \hat{b}_k > 0$ and $0 < \alpha_k < 1$ such that

$$0 \leq \frac{\hat{b}_k(i)}{x^{\alpha_k}} \leq f_k(i, x) \leq \frac{b_k(i)}{x^{\alpha_k}}, \quad \text{for all } x > 0, i \in [2, T + 2], k = 1, 2. \tag{H}$$

If $\gamma_{1^*} \geq 0, \gamma_{2^*} \geq 0$, then there exists a positive solution of (1).

Proof A solution of (1) is just a fixed point of the completely continuous map $A(x, y) = (A_1x, A_2y) : P \times P \rightarrow P \times P$, from (5) we have

$$\begin{aligned} (A_1u)(t) &= \sum_{j=2}^{T+2} G(i, j) f_1(i, v(j)) + \gamma_1(i), \\ (A_2v)(t) &= \sum_{j=2}^{T+2} G(i, j) f_2(i, u(j)) + \gamma_2(i). \end{aligned}$$

By a direct application of Schauder’s fixed point theorem, the proof is finished if we prove that A maps the closed convex set defined as

$$\begin{aligned} K = \{ (u, v) \in P \times P : r_1 i^2 (T + 4 - i)^2 \leq u(i) \leq R_1 i^2 (T + 4 - i)^2, \\ r_2 i^2 (T + 4 - i)^2 \leq v(i) \leq R_2 i^2 (T + 4 - i)^2, i \in [0, T + 4] \}, \end{aligned}$$

into itself, where $R_1 > r_1 > 0, R_2 > r_2 > 0$ are positive constants to be fixed properly. For convenience, we introduce the following notations

$$\beta_k(i) = \sum_{j=2}^{T+2} \frac{G(i, j)}{i^2 (T + 4 - i)^2} b_k(j), \quad \hat{\beta}_k(i) = \sum_{j=2}^{T+2} \frac{G(i, j)}{i^2 (T + 4 - i)^2} \hat{b}_k(j), \quad k = 1, 2.$$

Given $(u, v) \in K$, by the nonnegative sign of G and $f_k, k = 1, 2$, we have

$$\begin{aligned} (A_1u)(i) &= i^2 (T + 4 - i)^2 \sum_{j=2}^{T+2} \frac{G(i, j)}{i^2 (T + 4 - i)^2} f_1(j, v(j)) + \gamma_1(i) \\ &\geq i^2 (T + 4 - i)^2 \sum_{j=2}^{T+2} \frac{G(i, j)}{i^2 (T + 4 - i)^2} \frac{\hat{b}_1(j)}{v^{\alpha_1}(j)} + i^2 (T + 4 - i)^2 \gamma_{1^*} \\ &\geq i^2 (T + 4 - i)^2 \sum_{j=2}^{T+2} \frac{G(i, j)}{i^2 (T + 4 - i)^2} \frac{\hat{b}_1(j)}{R_2^{\alpha_1}} \\ &\geq i^2 (T + 4 - i)^2 \hat{\beta}_{1^*} \cdot \frac{1}{R_2^{\alpha_1}}. \end{aligned}$$

Note for every $(u, v) \in K$

$$\begin{aligned} (A_1 u)(i) &= i^2(T+4-i)^2 \sum_{j=2}^{T+2} \frac{G(i,j)}{i^2(T+4-i)^2} f_1(j, v(j)) + \gamma_1(i) \\ &\leq i^2(T+4-i)^2 \sum_{j=2}^{T+2} \frac{G(i,j)}{i^2(T+4-i)^2} \frac{b_1(j)}{r_2^{\alpha_1}} + i^2(T+4-i)^2 \gamma_1^* \\ &\leq i^2(T+4-i)^2 \sum_{j=2}^{T+2} \frac{G(i,j)}{i^2(T+4-i)^2} \frac{b_1(j)}{r_2^{\alpha_1}} + i^2(T+4-i)^2 \gamma_1^* \\ &\leq i^2(T+4-i)^2 \left(\beta_1^* \cdot \frac{1}{r_2^{\alpha_1}} + \gamma_1^* \right). \end{aligned}$$

Similarly, by the same strategy, we have

$$\begin{aligned} (A_2 v)(i) &= i^2(T+4-i)^2 \sum_{j=2}^{T+2} \frac{G(i,j)}{i^2(T+4-i)^2} f_2(j, u(j)) + \gamma_2(i) \\ &\geq i^2(T+4-i)^2 \sum_{j=2}^{T+2} \frac{G(i,j)}{i^2(T+4-i)^2} \frac{\hat{b}_2(j)}{R_1^{\alpha_2}} + i^2(T+4-i)^2 \gamma_2^* \\ &\geq i^2(T+4-i)^2 \sum_{j=2}^{T+2} \frac{G(i,j)}{i^2(T+4-i)^2} \frac{\hat{b}_2(j)}{R_1^{\alpha_2}} \\ &\geq i^2(T+4-i)^2 \hat{\beta}_2^* \cdot \frac{1}{R_1^{\alpha_2}}, \\ (A_2 v)(i) &= i^2(T+4-i)^2 \sum_{j=2}^{T+2} \frac{G(i,j)}{i^2(T+4-i)^2} f_2(j, u(j)) + \gamma_2(i) \\ &\leq i^2(T+4-i)^2 \sum_{j=2}^{T+2} \frac{G(i,j)}{i^2(T+4-i)^2} \frac{b_2(j)}{r_1^{\alpha_2}} + i^2(T+4-i)^2 \gamma_2^* \\ &\leq i^2(T+4-i)^2 \sum_{j=2}^{T+2} \frac{G(i,j)}{i^2(T+4-i)^2} \frac{b_2(j)}{r_1^{\alpha_2}} + i^2(T+4-i)^2 \gamma_2^* \\ &\leq i^2(T+4-i)^2 \left(\beta_2^* \cdot \frac{1}{r_1^{\alpha_2}} + \gamma_2^* \right). \end{aligned}$$

Thus $(A_1 u, A_2 v) \in K$ if r_1, r_2, R_1 and R_2 are chosen so that

$$\begin{aligned} \hat{\beta}_1^* \cdot \frac{1}{R_2^{\alpha_1}} &\geq r_1, & \beta_1^* \cdot \frac{1}{r_2^{\alpha_1}} + \gamma_1^* &\leq R_1, \\ \hat{\beta}_2^* \cdot \frac{1}{R_1^{\alpha_2}} &\geq r_2, & \beta_2^* \cdot \frac{1}{r_1^{\alpha_2}} + \gamma_2^* &\leq R_2. \end{aligned}$$

Note that $\hat{\beta}_1^*, \beta_1^* > 0$ and taking $R = R_1 = R_2, r = r_1 = r_2, r = \frac{1}{R}$, it is sufficient to find $R > 1$ such that

$$\hat{\beta}_1^* \cdot R^{1-\alpha_1} \geq 1, \quad \beta_1^* \cdot R^{\alpha_1} + \gamma_1^* \leq R,$$

$$\hat{\beta}_2^* \cdot R^{1-\alpha_2} \geq 1, \quad \beta_2^* \cdot R^{\alpha_2} + \gamma_2^* \leq R$$

and these inequalities hold for R big enough because $\alpha_i < 1$. The proof is complete. \square

The case $\gamma_1^* \leq 0, \gamma_2^* \leq 0$

The aim of this section is to show that the presence of a weak singular nonlinearity makes it possible to find positive solutions if $\gamma_1^* \leq 0, \gamma_2^* \leq 0$.

Theorem 2 *Assume that there exist $b_k, \hat{b}_k > 0$ and $0 < \alpha_k < 1$, such that (H) is satisfied. If $\gamma_1^* \leq 0, \gamma_2^* \leq 0$, and*

$$\begin{aligned} \gamma_1^* &\geq \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_1^*}{(\beta_2^*)^{\alpha_1}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}} \left(1 - \frac{1}{\alpha_1 \alpha_2} \right), \\ \gamma_2^* &\geq \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}} \left(1 - \frac{1}{\alpha_1 \alpha_2} \right), \end{aligned} \tag{6}$$

then there exists a positive solution of (1).

Proof We follow the same strategy and notation as in the proof of ahead theorem. In this case, to prove that $A : K \rightarrow K$, it is sufficient to find $0 < r_1 < R_1, 0 < r_2 < R_2$ such that

$$\frac{\hat{\beta}_1^*}{R_2^{\alpha_1}} + \gamma_1^* \geq r_1, \quad \frac{\beta_1^*}{r_2^{\alpha_1}} \leq R_1, \tag{7}$$

$$\frac{\hat{\beta}_2^*}{R_1^{\alpha_2}} + \gamma_2^* \geq r_2, \quad \frac{\beta_2^*}{r_1^{\alpha_2}} \leq R_2. \tag{8}$$

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}}, R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}}$, then the first inequality of (8) holds if r_2 satisfies

$$\hat{\beta}_2^* (\beta_1^*)^{-\alpha_2} r_2^{\alpha_1 \alpha_2} + \gamma_2^* \geq r_2,$$

or equivalently

$$\gamma_2^* \geq g(r_2) := r_2 - \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}} r_2^{\alpha_1 \alpha_2}.$$

The function $g(r_2)$ possesses a minimum at

$$r_{20} := \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}}.$$

Taking $r_2 = r_{20}$, then (6) holds if

$$\gamma_2^* \geq g(r_{20}) = \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}} \left(1 - \frac{1}{\alpha_1 \alpha_2} \right).$$

Similarly,

$$\gamma_1^* \geq h(r_1) := r_1 - \frac{\hat{\beta}_1^*}{(\beta_2^*)^{\alpha_1}} r_1^{\alpha_1 \alpha_2},$$

$h(r_1)$ possesses a minimum at

$$r_{10} := \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_1^*}{(\beta_2^*)^{\alpha_1}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}},$$

$$\gamma_1^* \geq \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_1^*}{(\beta_2^*)^{\alpha_1}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}} \left(1 - \frac{1}{\alpha_1 \alpha_2} \right).$$

Taking $r_1 = r_{10}$, $r_2 = r_{20}$, then the first inequalities in (7) and (8) hold if $\gamma_1^* \geq g(r_1)$ and $\gamma_2^* \geq g(r_2)$, which are just condition (6). The second inequalities hold directly from the choice of R_1 and R_2 , so it remains to prove that $R_1 = \frac{\beta_1^*}{r_{20}^{\alpha_1}} > r_{10}$, $R_2 = \frac{\beta_2^*}{r_{10}^{\alpha_2}} > r_{20}$. This is easily verified through elementary computations:

$$\begin{aligned} R_1 &= \frac{\beta_1^*}{r_{20}^{\alpha_1}} = \frac{\beta_1^*}{\{[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}}]^{\frac{1}{1-\alpha_1 \alpha_2}}\}^{\alpha_1}} \\ &= \frac{\beta_1^*}{[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}}]^{\frac{\alpha_1}{1-\alpha_1 \alpha_2}}} = \frac{(\beta_1^*)^{1+\frac{\alpha_1 \alpha_2}{1-\alpha_1 \alpha_2}}}{(\alpha_1 \alpha_2 \cdot \hat{\beta}_2^*)^{\frac{\alpha_1}{1-\alpha_1 \alpha_2}}} \\ &= \frac{(\beta_1^*)^{\frac{1}{1-\alpha_1 \alpha_2}}}{[(\alpha_1 \alpha_2 \cdot \hat{\beta}_2^*)^{\alpha_1}]^{\frac{1}{1-\alpha_1 \alpha_2}}} = \left[\frac{\beta_1^*}{(\alpha_1 \alpha_2 \cdot \hat{\beta}_2^*)^{\alpha_1}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}} \\ &= \left[\frac{1}{(\alpha_1 \alpha_2)^{\alpha_1}} \cdot \frac{\beta_1^*}{(\hat{\beta}_2^*)^{\alpha_1}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}} > \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_1^*}{(\beta_2^*)^{\alpha_1}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}} = r_{10}, \end{aligned}$$

since $\hat{\beta}_k^* \leq \beta_k^*$, $k = 1, 2$. Similarly, we have $R_2 > r_{20}$. The proof is complete. □

The case $\gamma_1^* \geq 0$, $\gamma_2^* \leq 0$ ($\gamma_1^* \leq 0$, $\gamma_2^* \geq 0$)

Theorem 3 Assume (H) is satisfied. If $\gamma_1^* \geq 0$, $\gamma_2^* \leq 0$ and

$$\gamma_2^* \geq r_{21} - \hat{\beta}_2^* \cdot \frac{r_{21}^{\alpha_1 \alpha_2}}{(\beta_1^* + \gamma_1^* r_{21}^{\alpha_1})^{\alpha_2}}, \tag{9}$$

where $0 < r_{21} < +\infty$ is a unique positive solution of the equation

$$r_2^{1-\alpha_1 \alpha_2} (\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{1+\alpha_2} = \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*,$$

then there exists a positive solution of (1).

Proof In this case, to prove that $A : K \rightarrow K$, it is sufficient to find $r_1 < R_1$, $r_2 < R_2$ such that

$$\frac{\hat{\beta}_1^*}{R_2^{\alpha_1}} \geq r_1, \quad \frac{\beta_2^*}{r_1^{\alpha_2}} \leq R_2. \tag{10}$$

$$\frac{\hat{\beta}_2^*}{R_1^{\alpha_2}} + \gamma_2^* \geq r_2, \quad \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1. \tag{11}$$

If we fix $R_2 = \frac{\hat{\beta}_1^*}{r_1^{\alpha_2}}$, then the first inequality of (10) holds if r_1 satisfies

$$\frac{\hat{\beta}_1^*}{(\beta_2^*)^{\alpha_1}} \cdot r_1^{\alpha_1 \alpha_2} \geq r_1,$$

or equivalently

$$0 < r_1 \leq \left[\frac{\hat{\beta}_1^*}{(\beta_2^*)^{\alpha_1}} \right]^{\frac{1}{1-\alpha_1 \alpha_2}}. \tag{12}$$

If we chose $r_1 > 0$ small enough, then (12) holds, and R_2 is big enough.

If we fix $R_1 = \frac{\hat{\beta}_1^*}{r_2^{\alpha_1}} + \gamma_1^*$ then the first inequality of (11) holds if r_2 satisfies

$$\begin{aligned} \gamma_2^* &\geq r_2 - \frac{\hat{\beta}_2^*}{R_1^{\alpha_2}} \\ &= r_2 - \hat{\beta}_2^* \cdot \frac{1}{\left(\frac{\hat{\beta}_1^*}{r_2^{\alpha_1}} + \gamma_1^*\right)^{\alpha_2}} \\ &= r_2 - \hat{\beta}_2^* \cdot \frac{1}{\left(\frac{\hat{\beta}_1^* + \gamma_1^* r_2^{\alpha_1}}{r_2^{\alpha_1}}\right)^{\alpha_2}} \\ &= r_2 - \hat{\beta}_2^* \cdot \frac{r_2^{\alpha_1 \alpha_2}}{(\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}}, \end{aligned}$$

or equivalently

$$\gamma_2^* \geq f(r_2) := r_2 - \hat{\beta}_2^* \cdot \frac{r_2^{\alpha_1 \alpha_2}}{(\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}}.$$

According to

$$\begin{aligned} f'(r_2) &= 1 - \hat{\beta}_2^* \cdot \frac{1}{(\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}} \cdot [\alpha_1 \alpha_2 r_2^{\alpha_1 \alpha_2 - 1} (\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2} \\ &\quad - r_2^{\alpha_1 \alpha_2} \alpha_2 (\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2 - 1} \alpha_1 \gamma_1^* r_2^{\alpha_1 - 1}] \\ &= 1 - \frac{\hat{\beta}_2^* \alpha_1 \alpha_2 r_2^{\alpha_1 \alpha_2 - 1}}{(\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}} \left[1 - \frac{r_2^{\alpha_1} \gamma_1^*}{\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1}} \right] \\ &= 1 - \alpha_1 \alpha_2 \hat{\beta}_1^* \hat{\beta}_2^* r_2^{\alpha_1 \alpha_2 - 1} (\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-1 - \alpha_2}, \end{aligned}$$

we have $f'(0) = -\infty, f'(+\infty) = 1$, then there exists r_{21} such that $f'(r_{21}) = 0$, and

$$\begin{aligned} f''(r_2) &= -[\alpha_1 \alpha_2 \hat{\beta}_1^* \hat{\beta}_2^* (\alpha_1 \alpha_2 - 1) r_2^{\alpha_1 \alpha_2 - 2} (\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-1 - \alpha_2} \\ &\quad + \alpha_1 \alpha_2 \hat{\beta}_1^* \hat{\beta}_2^* r_2^{\alpha_1 \alpha_2 - 1} (-1 - \alpha_2) (\hat{\beta}_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-2 - \alpha_2} \gamma_1^* \alpha_1 r_2^{\alpha_1 - 1}] > 0. \end{aligned}$$

Then the function $f(r_2)$ possesses a minimum at r_{21} , i.e., $f(r_{21}) = \min_{r_2 \in (0, +\infty)} f(r_2)$.

Note $f'(r_{21}) = 0$, then we have

$$1 - \alpha_1 \alpha_2 \hat{\beta}_1^* \hat{\beta}_2^* r_{21}^{\alpha_1 \alpha_2 - 1} (\hat{\beta}_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1})^{-1 - \alpha_2} = 0,$$

or equivalently,

$$r_{21}^{1-\alpha_1\alpha_2} (\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*}.$$

Taking $r_2 = r_{21}$, then the first inequality in (11) holds if $\gamma_{2^*} \geq f(r_{21})$, which is just condition (9). The second inequality holds directly by the choice of R_2 , and it would remain to prove that $r_{21} < R_2$ and $r_{10} < R_1$. These inequalities hold for R_2 big enough and r_1 small enough. The proof is complete. \square

Similarly, we have the following theorem.

Theorem 4 Assume (H) are satisfied. If $\gamma_1^* \leq 0, \gamma_{2^*} \geq 0$ and

$$\gamma_1^* \geq r_{11} - \hat{\beta}_{1^*} \cdot \frac{r_{11}^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_{2^*}r_{11}^{\alpha_2})^{\alpha_1}},$$

where $0 < r_{11} < +\infty$ is a unique positive solution of the equation

$$r_1^{1-\alpha_1\alpha_2} (\beta_2^* + \gamma_{2^*} \cdot r_1^{\alpha_2})^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1^*},$$

then there exists a positive solution of (1).

The case $\gamma_{1^*} < 0 < \gamma_1^*, \gamma_{2^*} < 0 < \gamma_2^*$

Theorem 5 Assume (H) is satisfied. If $\gamma_{1^*} < 0 < \gamma_1^*, \gamma_{2^*} < 0 < \gamma_2^*$ and

$$\gamma_1^* \geq r_{10} - \hat{\beta}_{1^*} \cdot \frac{r_{10}^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_{2^*}r_{10}^{\alpha_2})^{\alpha_1}}, \tag{13}$$

$$\gamma_{2^*} \geq r_{20} - \hat{\beta}_{2^*} \cdot \frac{r_{20}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^*r_{20}^{\alpha_1})^{\alpha_2}}, \tag{14}$$

where $0 < r_{10} < +\infty$ is a unique positive solution of the equation

$$r_1^{1-\alpha_1\alpha_2} (\beta_2^* + \gamma_{2^*} \cdot r_1^{\alpha_2})^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1^*},$$

and $0 < r_{20} < +\infty$ is a unique positive solution of the equation

$$r_2^{1-\alpha_1\alpha_2} (\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*},$$

then there exists a positive solution of (1).

Proof We follow the same strategy and notation as in the proof of ahead theorem. In this case, to prove that $A : K \rightarrow K$, it is sufficient to find $r_1 < R_1, r_2 < R_2$ such that

$$\frac{\hat{\beta}_{1^*}}{R_2^{\alpha_1}} + \gamma_1^* \geq r_1, \quad \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1, \tag{15}$$

$$\frac{\hat{\beta}_{2^*}}{R_1^{\alpha_2}} + \gamma_{2^*} \geq r_2, \quad \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_{2^*} \leq R_2. \tag{16}$$

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*$, $R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^*$, then the first inequality of (15) holds if r_2 satisfies

$$\gamma_2^* \geq g(r_2) := r_2 - \hat{\beta}_2^* \cdot \frac{r_2^{\alpha_1 \alpha_2}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}}.$$

According to

$$\begin{aligned} g'(r_2) &= 1 - \hat{\beta}_2^* \cdot \frac{1}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{2\alpha_2}} \cdot [\alpha_1 \alpha_2 r_2^{\alpha_1 \alpha_2 - 1} (\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2} \\ &\quad - r_2^{\alpha_1 \alpha_2} \alpha_2 (\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2 - 1} \alpha_1 \gamma_1^* r_2^{\alpha_1 - 1}] \\ &= 1 - \frac{\hat{\beta}_2^* \alpha_1 \alpha_2 r_2^{\alpha_1 \alpha_2 - 1}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}} \left[1 - \frac{r_2^{\alpha_1} \gamma_1^*}{\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}} \right] \\ &= 1 - \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^* r_2^{\alpha_1 \alpha_2 - 1} (\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-1 - \alpha_2}, \end{aligned}$$

we have $g'(0) = -\infty$, $g'(+\infty) = 1$, then there exists r_{20} such that $g'(r_{20}) = 0$, and

$$\begin{aligned} g''(r_2) &= -[\alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^* (\alpha_1 \alpha_2 - 1) r_2^{\alpha_1 \alpha_2 - 2} (\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-1 - \alpha_2} \\ &\quad + \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^* r_2^{\alpha_1 \alpha_2 - 1} (-1 - \alpha_2) (\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-2 - \alpha_2} \gamma_1^* \alpha_1 r_2^{\alpha_1 - 1}] > 0. \end{aligned}$$

Then the function $g(r_2)$ possesses a minimum at r_{20} , i.e., $g(r_{20}) = \min_{r_2 \in (0, +\infty)} g(r_2)$.

Note $g'(r_{20}) = 0$, then we have

$$1 - \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^* r_{21}^{\alpha_1 \alpha_2 - 1} (\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1})^{-1 - \alpha_2} = 0,$$

or equivalently,

$$r_{20}^{1 - \alpha_1 \alpha_2} (\beta_1^* + \gamma_1^* \cdot r_{20}^{\alpha_1})^{1 + \alpha_2} = \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*.$$

Similarly,

$$\gamma_1^* \geq g(r_1) := r_1 - \hat{\beta}_1^* \cdot \frac{r_1^{\alpha_1 \alpha_2}}{(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2})^{\alpha_1}}.$$

$g(r_{10}) = \min_{r_1 \in (0, +\infty)} g(r_1)$, and

$$r_{10}^{1 - \alpha_1 \alpha_2} (\beta_2^* + \gamma_2^* \cdot r_{10}^{\alpha_2})^{1 + \alpha_1} = \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*.$$

Taking $r_1 = r_{10}$ and $r_2 = r_{20}$, then the first inequality in (15) and (16) hold if $\gamma_1^* \geq g(r_{10})$, $\gamma_2^* \geq g(r_{20})$, which are just condition (13) and (14). The second inequalities hold directly by the choice of R_1 and R_2 , and it would remain to prove that $r_{10} < R_1$ and $r_{20} < R_2$. This is easily verified through elementary computations.

$$\begin{aligned} R_1 &= \frac{\beta_1^*}{r_{20}^{\alpha_1}} + \gamma_1^* \\ &= \frac{\beta_1^* + \gamma_1^* \cdot r_{20}^{\alpha_1}}{r_{20}^{\alpha_1}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{(\alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*)^{\frac{1}{1+\alpha_2}} \cdot r_{20}^{\frac{\alpha_1 \alpha_2 - 1}{1+\alpha_2}}}{r_{20}^{\alpha_1}} \\
 &= (\alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*)^{\frac{1}{1+\alpha_2}} \cdot r_{20}^{-\frac{1+\alpha_1}{1+\alpha_2}}.
 \end{aligned}$$

The proof is the same as that in $R_1, R_2 = (\alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*)^{\frac{1}{1+\alpha_1}} \cdot r_{10}^{-\frac{1+\alpha_2}{1+\alpha_1}}$.

Next, we will prove $r_{10} < R_1, r_{20} < R_2$, or equivalently,

$$\begin{aligned}
 r_{10} r_{20}^{\frac{1+\alpha_1}{1+\alpha_2}} &< (\alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*)^{\frac{1}{1+\alpha_2}}, \\
 r_{20} r_{10}^{\frac{1+\alpha_2}{1+\alpha_1}} &< (\alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*)^{\frac{1}{1+\alpha_1}}.
 \end{aligned}$$

Namely,

$$r_{10}^{1+\alpha_2} r_{20}^{1+\alpha_1} < \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*, r_{20}^{1+\alpha_1} r_{10}^{1+\alpha_2} < \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*.$$

On the other hand,

$$r_{20}^{1-\alpha_1 \alpha_2} (\beta_1^*)^{1+\alpha_2} \leq \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*.$$

Then

$$r_{20} \leq (\alpha_1 \alpha_2 (\beta_1^*)^{-\alpha_2} \hat{\beta}_2^*)^{\frac{1}{1-\alpha_1 \alpha_2}}.$$

Similarly

$$r_{10} \leq (\alpha_1 \alpha_2 (\beta_2^*)^{-\alpha_1} \hat{\beta}_1^*)^{\frac{1}{1-\alpha_1 \alpha_2}}.$$

By (13) and (14),

$$r_{10}^{1+\alpha_2} r_{20}^{1+\alpha_1} \leq (\alpha_1 \alpha_2 (\beta_2^*)^{-\alpha_1} \hat{\beta}_1^*)^{\frac{1+\alpha_2}{1-\alpha_1 \alpha_2}} (\alpha_1 \alpha_2 (\beta_1^*)^{-\alpha_2} \hat{\beta}_2^*)^{\frac{1+\alpha_1}{1-\alpha_1 \alpha_2}}.$$

Now if we can prove

$$(\alpha_1 \alpha_2 (\beta_2^*)^{-\alpha_1} \hat{\beta}_1^*)^{\frac{1+\alpha_2}{1-\alpha_1 \alpha_2}} (\alpha_1 \alpha_2 (\beta_1^*)^{-\alpha_2} \hat{\beta}_2^*)^{\frac{1+\alpha_1}{1-\alpha_1 \alpha_2}} < \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*$$

then

$$r_{10}^{1+\alpha_2} r_{20}^{1+\alpha_1} < \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*.$$

In fact,

$$(\alpha_1 \alpha_2)^{\frac{2+\alpha_2+\alpha_1-1}{1-\alpha_1 \alpha_2}} \cdot \left(\frac{\hat{\beta}_1^*}{\beta_1^*}\right)^{\frac{1+\alpha_2}{1-\alpha_1 \alpha_2}} \cdot \left(\frac{\hat{\beta}_2^*}{\beta_2^*}\right)^{\frac{\alpha_1(1+\alpha_2)}{1-\alpha_1 \alpha_2}} < 1,$$

since $\hat{\beta}_i^* \leq \beta_i^*, i = 1, 2$. Similarly, we have $r_{20}^{1+\alpha_1} r_{10}^{1+\alpha_2} < \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*$, we omit the details. Now we can obtain $r_{10} < R_1, r_{20} < R_2$. The proof is complete. \square

The case $\gamma_1^* \leq 0, \gamma_2^* < 0 < \gamma_2^* (\gamma_2^* \leq 0, \gamma_1^* < 0 < \gamma_1^*)$

Theorem 6 Assume (H) are satisfied. If $\gamma_1^* \leq 0, \gamma_2^* < 0 < \gamma_2^*$ and

$$\gamma_2^* \geq \left(1 - \frac{1}{\alpha_1 \alpha_2}\right) \left[\alpha_1 \alpha_2 \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}}\right]^{\frac{1}{1-\alpha_1 \alpha_2}}, \tag{17}$$

$$\gamma_1^* \geq r_{11} - \hat{\beta}_1^* \cdot \frac{r_{11}^{\alpha_1 \alpha_2}}{(\beta_2^* + \gamma_2^* r_{11}^{\alpha_2})^{\alpha_1}}, \tag{18}$$

where $0 < r_{11} < +\infty$ is a unique positive solution of the equation

$$r_{11}^{1-\alpha_1 \alpha_2} (\beta_2^* + \gamma_2^* \cdot r_{11}^{\alpha_2})^{1+\alpha_1} = \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*, \tag{19}$$

then there exists a positive solution of (1).

Proof In this case, to prove that $A : K \rightarrow K$, it is sufficient to find $r_1 < R_1, r_2 < R_2$ such that

$$\frac{\hat{\beta}_1^*}{R_2^{\alpha_1}} + \gamma_1^* \geq r_1, \quad \frac{\beta_1^*}{r_2^{\alpha_1}} \leq R_1, \tag{20}$$

$$\frac{\hat{\beta}_2^*}{R_1^{\alpha_2}} + \gamma_2^* \geq r_2, \quad \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^* \leq R_2. \tag{21}$$

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}}, R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^*$, then the first inequality of (21) holds if r_2 satisfies

$$\gamma_2^* \geq r_2 - \frac{\hat{\beta}_2^*}{R_1^{\alpha_2}} = r_2 - \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}} \cdot r_2^{\alpha_1 \alpha_2}, \tag{22}$$

or equivalently

$$\gamma_2^* \geq f(r_2) := r_2 - \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}} \cdot r_2^{\alpha_1 \alpha_2}. \tag{23}$$

Then the function $f(r_2)$ possesses a minimum at

$$r_{21} = \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}}\right]^{\frac{1}{1-\alpha_1 \alpha_2}}, \tag{24}$$

i.e., $f(r_{21}) = \min_{r_2 \in (0, +\infty)} f(r_2)$.

On the analogy of (22), we obtain

$$\gamma_1^* \geq r_1 - \hat{\beta}_1^* \cdot \frac{r_1^{\alpha_1 \alpha_2}}{(\beta_2^* + \gamma_2^* r_1^{\alpha_2})^{\alpha_1}}, \tag{25}$$

or equivalently,

$$\gamma_1^* \geq h(r_1) := r_1 - \hat{\beta}_1^* \cdot \frac{r_1^{\alpha_1 \alpha_2}}{(\beta_2^* + \gamma_2^* r_1^{\alpha_2})^{\alpha_1}}. \tag{26}$$

According to

$$h'(r_1) := 1 - \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^* r_1^{\alpha_1 \alpha_2 - 1} (\beta_2^* + \gamma_2^* r_1^{\alpha_2})^{-1 - \alpha_1}, \tag{27}$$

we have $h'(0) = -\infty$, $h'(+\infty) = 1$, then there exists r_{11} such that $h'(r_{11}) = 0$, and

$$h''(r_1) = -[\alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^* (\alpha_1 \alpha_2 - 1) r_1^{\alpha_1 \alpha_2 - 2} (\beta_2^* + \gamma_2^* r_1^{\alpha_2})^{-1 - \alpha_1} + \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^* r_1^{\alpha_1 \alpha_2 - 1} (-1 - \alpha_1) (\beta_2^* + \gamma_2^* r_1^{\alpha_2})^{-2 - \alpha_1} \gamma_2^* \alpha_2 r_1^{\alpha_2 - 1}] > 0. \tag{28}$$

Then the function $h(r_1)$ possesses a minimum at r_{11} , i.e., $h(r_{11}) = \min_{r_1 \in (0, +\infty)} f(r_1)$.

Note $h'(r_{11}) = 0$, then we have

$$1 - \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^* r_{11}^{\alpha_1 \alpha_2 - 1} (\beta_2^* + \gamma_2^* r_{11}^{\alpha_2})^{-1 - \alpha_1} = 0.$$

Namely,

$$r_{11}^{1 - \alpha_1 \alpha_2} (\beta_2^* + \gamma_2^* r_{11}^{\alpha_2})^{1 + \alpha_1} = \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*.$$

Taking $r_2 = r_{21}$ and $r_1 = r_{11}$ then the first inequality in (20) and (21) hold if $\gamma_2^* \geq h(r_{21})$ and $\gamma_1^* \geq h(r_{11})$ which are just condition (17) and (18). The second inequalities hold directly by the choice of R_2 and R_1 , so it would remain to prove that $R_1 = \frac{\beta_1^*}{r_{21}^{\alpha_1}} > r_{11}$, $R_2 = \frac{\beta_2^*}{r_{11}^{\alpha_2}} + \gamma_2^* > r_{21}$. Now we turn to prove that $R_1 > r_{11}$, $R_2 > r_{21}$.

First,

$$\begin{aligned} R_1 &= \frac{\beta_1^*}{r_{21}^{\alpha_1}} = \frac{\beta_1^*}{\{[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}}]^{1 - \alpha_1 \alpha_2}\}^{\alpha_1}} \\ &= \frac{\beta_1^*}{[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_2^*}{(\beta_1^*)^{\alpha_2}}]^{1 - \alpha_1 \alpha_2}} = \frac{(\beta_1^*)^{1 + \frac{\alpha_1 \alpha_2}{1 - \alpha_1 \alpha_2}}}{(\alpha_1 \alpha_2 \cdot \hat{\beta}_2^*)^{1 - \alpha_1 \alpha_2}} \\ &= \frac{(\beta_1^*)^{\frac{1}{1 - \alpha_1 \alpha_2}}}{[(\alpha_1 \alpha_2 \cdot \hat{\beta}_2^*)^{\alpha_1}]^{\frac{1}{1 - \alpha_1 \alpha_2}}} = \left[\frac{\beta_1^*}{(\alpha_1 \alpha_2 \cdot \hat{\beta}_2^*)^{\alpha_1}} \right]^{\frac{1}{1 - \alpha_1 \alpha_2}} \\ &= \left[\frac{1}{(\alpha_1 \alpha_2)^{\alpha_1}} \cdot \frac{\beta_1^*}{(\hat{\beta}_2^*)^{\alpha_1}} \right]^{\frac{1}{1 - \alpha_1 \alpha_2}} > \left[\alpha_1 \alpha_2 \cdot \frac{\hat{\beta}_1^*}{(\beta_2^*)^{\alpha_1}} \right]^{\frac{1}{1 - \alpha_1 \alpha_2}} = r_{11}, \end{aligned}$$

since $\hat{\beta}_i^* \leq \beta_i^*$, $i = 1, 2$.

On the other hand,

$$R_2 = \frac{\beta_2^*}{r_{11}^{\alpha_2}} + \gamma_2^* = \frac{\beta_2^* + \gamma_2^* r_{11}^{\alpha_2}}{r_{11}^{\alpha_2}}. \tag{29}$$

By (18), we have

$$\beta_2^* + \gamma_2^* r_{11}^{\alpha_2} = (\alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*)^{\frac{1}{1 + \alpha_1}} r_{11}^{\frac{\alpha_1 \alpha_2 - 1}{1 + \alpha_1}}. \tag{30}$$

Combing (29) and (30),

$$R_2 = (\alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*)^{\frac{1}{1+\alpha_1}} r_{11}^{-\frac{1+\alpha_2}{1+\alpha_1}}. \tag{31}$$

In what follows, we will verify that $R_2 > r_{21}$. In fact,

$$(\alpha_1 \alpha_2)^{\frac{2+\alpha_2+\alpha_1}{1-\alpha_1 \alpha_2} - 1} \cdot \left(\frac{\hat{\beta}_2^*}{\beta_2^*}\right)^{\frac{1+\alpha_1}{1-\alpha_1 \alpha_2}} \cdot \left(\frac{\hat{\beta}_1^*}{\beta_1^*}\right)^{\frac{\alpha_2(1+\alpha_1)}{1-\alpha_1 \alpha_2}} < 1,$$

since $\hat{\beta}_i^* \leq \beta_i^*$, $i = 1, 2$. Thus

$$(\alpha_1 \alpha_2 \beta_1^{*(-\alpha_2)} \hat{\beta}_2^*)^{\frac{1+\alpha_1}{1-\alpha_1 \alpha_2}} \cdot (\alpha_1 \alpha_2 \beta_2^{*(-\alpha_1)} \hat{\beta}_1^*)^{\frac{1+\alpha_2}{1-\alpha_1 \alpha_2}} < \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*. \tag{32}$$

On the other hand,

$$r_{21}^{1-\alpha_1 \alpha_2} \beta_1^{*(1+\alpha_2)} \leq \alpha_1 \alpha_2 \beta_1^* \hat{\beta}_2^*,$$

$$r_{11}^{1-\alpha_1 \alpha_2} \beta_2^{*(1+\alpha_1)} \leq \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*.$$

Thus one can see easily note that

$$r_{21} \leq (\alpha_1 \alpha_2 \beta_1^{*(-\alpha_2)} \hat{\beta}_2^*)^{\frac{1}{1-\alpha_1 \alpha_2}}, \tag{33}$$

$$r_{11} \leq (\alpha_1 \alpha_2 \beta_2^{*(-\alpha_1)} \hat{\beta}_1^*)^{\frac{1}{1-\alpha_1 \alpha_2}}. \tag{34}$$

From (33) and (34),

$$r_{11}^{1+\alpha_2} r_{21}^{1+\alpha_1} \leq (\alpha_1 \alpha_2 \beta_2^{*(-\alpha_1)} \hat{\beta}_1^*)^{\frac{1+\alpha_2}{1-\alpha_1 \alpha_2}} (\alpha_1 \alpha_2 \beta_1^{*(-\alpha_2)} \hat{\beta}_2^*)^{\frac{1+\alpha_1}{1-\alpha_1 \alpha_2}}. \tag{35}$$

Combing (32) and (35),

$$r_{11}^{1+\alpha_2} r_{21}^{1+\alpha_1} < \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*.$$

Therefore,

$$r_{21} r_{11}^{\frac{1+\alpha_2}{1+\alpha_1}} < (\alpha_1 \alpha_2 \beta_2^* \hat{\beta}_1^*)^{\frac{1}{1+\alpha_1}}.$$

Recall (31), we obtain $r_{21} < R_2$ immediately. The proof is complete. □

Similarly, we have the following theorem.

Theorem 7 Assume (H) are satisfied. If $\gamma_2^* \leq 0$, $\gamma_1^* < 0 < \gamma_1^*$ and

$$\gamma_1^* \geq \left(1 - \frac{1}{\alpha_1 \alpha_2}\right) \cdot \left[\alpha_1 \alpha_2 \frac{\hat{\beta}_1^*}{(\beta_2^*)^{\alpha_1}}\right]^{\frac{1}{1-\alpha_1 \alpha_2}},$$

$$\gamma_2^* \geq r_{21} - \hat{\beta}_2^* \cdot \frac{r_{21}^{\alpha_1 \alpha_2}}{(\beta_1^* + \gamma_1^* r_{21}^{\alpha_1})^{\alpha_2}},$$

where $0 < r_{21} < +\infty$ is a unique positive solution of the equation

$$r_2^{1-\alpha_1\alpha_2} (\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*},$$

then there exists a positive solution of (1).

The case $\gamma_{1^*} \geq 0, \gamma_{2^*} < 0 < \gamma_2^*$ ($\gamma_{2^*} \geq 0, \gamma_{1^*} < 0 < \gamma_1^*$)

Theorem 8 Assume (H) are satisfied. If $\gamma_{1^*} \geq 0, \gamma_{2^*} < 0 < \gamma_2^*$ and

$$\gamma_{2^*} \geq r_{22} - \hat{\beta}_{2^*} \cdot \frac{r_{22}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* r_{22}^{\alpha_1})^{\alpha_2}},$$

where $0 < r_{22} < +\infty$ is a unique positive solution of the equation

$$r_2^{1-\alpha_1\alpha_2} (\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*},$$

then there exists a positive T -periodic solution of (1).

Proof In this case, to prove that $A : K \rightarrow K$, it is sufficient to find $r_1 < R_1, r_2 < R_2$ such that

$$\frac{\hat{\beta}_{1^*}}{R_2^{\alpha_1}} \geq r_1, \quad \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \leq R_1. \tag{36}$$

$$\frac{\hat{\beta}_{2^*}}{R_1^{\alpha_2}} + \gamma_{2^*} \geq r_2, \quad \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^* \leq R_2. \tag{37}$$

If we fix $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*, R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}} + \gamma_2^*$, then the first inequality of (37) satisfies

$$\hat{\beta}_{2^*} \cdot \left(\frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \right)^{-\alpha_2} + \gamma_{2^*} \geq r_2,$$

or equivalently

$$\gamma_{2^*} \geq l(r_2) := r_2 - \frac{\hat{\beta}_{2^*}}{(\beta_1^* + \gamma_1^* r_2^{\alpha_1})^{\alpha_2}} \cdot r_2^{\alpha_1\alpha_2}.$$

Then the function $l(r_2)$ possesses a minimum at r_{22} , i.e., $l(r_{22}) = \min_{r_2 \in (0, +\infty)} l(r_2)$.

Note $l'(r_{22}) = 0$, then we have

$$1 - \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*}r_{22}^{\alpha_1\alpha_2-1}(\beta_1^* + \gamma_1^* \cdot r_{22}^{\alpha_1})^{-1-\alpha_2} = 0.$$

Therefore,

$$r_{22}^{1-\alpha_1\alpha_2} (\beta_1^* + \gamma_1^* \cdot r_{22}^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2^*}.$$

Note that $\hat{\beta}_i^*, \beta_i^* > 0, i = 1, 2$. And taking $r_2 = r_{22}, R_1 = \frac{\beta_1^*}{r_{22}^{\alpha_1}} + \gamma_1^*, r_1 = \frac{1}{R_2}$, it is sufficient to find $r_1 < R_1, r_2 < R_2$ such that

$$R_2^{\alpha_1-1} \leq \beta_1^*, \quad R_2^{\alpha_2}\beta_2^* + \gamma_2^* \leq R_2$$

and these inequalities hold for R_2 big enough because $\alpha_i < 1$. The proof is completed. \square

Similarly, we have the following theorem.

Theorem 9 Assume (H) are satisfied. If $\gamma_2^* \geq 0$, $\gamma_1^* < 0 < \gamma_1^*$ and

$$\gamma_1^* \geq r_{12} - \hat{\beta}_{1^*} \cdot \frac{r_{12}^{\alpha_1 \alpha_2}}{(\beta_2^* + \gamma_2^* r_{12}^{\alpha_2})^{\alpha_1}},$$

where $0 < r_{12} < +\infty$ is a unique positive solution of the equation

$$r_1^{1-\alpha_1 \alpha_2} (\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2})^{1+\alpha_1} = \alpha_1 \alpha_2 \beta_2^* \hat{\beta}_{1^*},$$

then there exists a positive solution of (1).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

WL, YF, BS, and YP have critically revised the manuscript and have made substantial contributions to conception. WL designed the research and drafted the manuscript. YF, BS and YP revised it critically for important intellectual content. All authors read and approved the final manuscript.

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